Knowing How to Understand (Generalized) Tensor Disjunction

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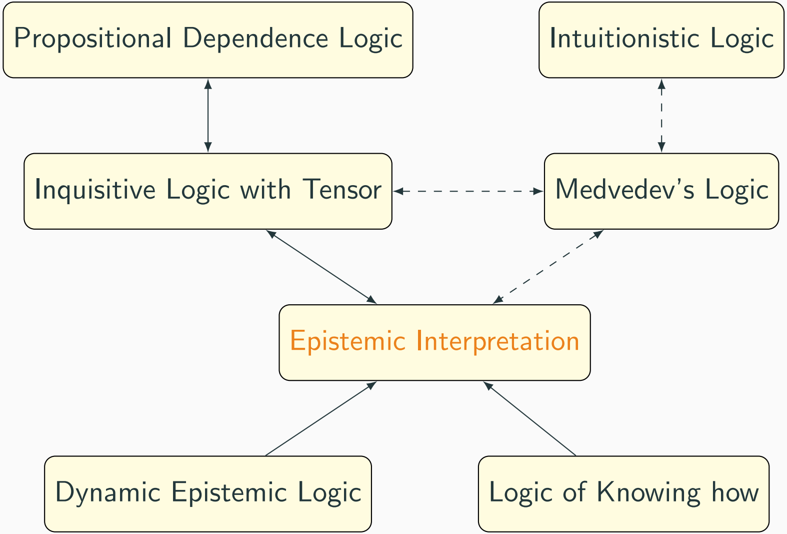
The big picture



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How to understand $\otimes$?

Disjunctions in team semantics (Semantics of $\mathsf{InqB}^{\otimes}$) [CB19]

Given $\mathbf{P}$, an (information) model is a pair $\mathcal{M} = \langle W, V \rangle$ where:

- W is a non-empty set of possible worlds;
- $V : \mathbf{P} \to \wp(W)$ is a valuation function.

Support is a relation between (information) states or teams $s \subseteq W$ and formulas (written as $\mathcal{M}, s \Vdash \alpha$):

- $\mathcal{M}, s \Vdash p$ iff $\forall w \in s, w \in V(p)$.
- $\mathcal{M}, s \Vdash \bot$ iff $s = \varnothing$.
- $\mathcal{M}, s \Vdash (\alpha \wedge \beta)$ iff $\mathcal{M}, s \Vdash \alpha$ and $\mathcal{M}, s \Vdash \beta$.
- $\mathcal{M}, s \Vdash (\alpha \to \beta)$ iff $\forall t \subseteq s$: if $\mathcal{M}, t \Vdash \alpha$ then $\mathcal{M}, t \Vdash \beta$.
- $\mathcal{M}, s \Vdash (\alpha \vee \beta)$ iff $\mathcal{M}, s \Vdash \alpha$ or $\mathcal{M}, s \Vdash \beta$.
- $\mathcal{M}, s \Vdash (\alpha \otimes \beta)$ iff there exist $t, t' \subseteq s$, s.t. $\mathcal{M}, t \Vdash \alpha$, $\mathcal{M}, t' \Vdash \beta$, and $t \cup t' = s$.
- $\mathcal{M}, s \Vdash (\alpha \veedot \beta)$ iff $\mathcal{M}, \{w\} \Vdash \alpha$ or $\mathcal{M}, \{w\} \Vdash \beta$ for all $w \in s$.

Disjunctions in team semantics (Proof system of InqB$^{\otimes}$)[CIY20]

$$\frac{\varphi}{\varphi \vee \psi}\,(\vee i_1) \qquad \frac{\psi}{\varphi \vee \psi}\,(\vee i_2) \qquad \frac{\varphi}{\varphi \otimes \psi}\,(\otimes i_1) \qquad \frac{\psi}{\varphi \otimes \psi}\,(\otimes i_2)$$

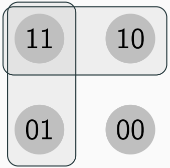
$$\frac{\begin{matrix}[\varphi] \\ \vdots \\ \chi\end{matrix} \quad \begin{matrix}[\psi] \\ \vdots \\ \chi\end{matrix} \quad \varphi \vee \psi}{\chi}\,(\vee e) \qquad \frac{\begin{matrix}[\varphi] \\ \vdots \\ \alpha\end{matrix} \quad \begin{matrix}[\psi] \\ \vdots \\ \alpha\end{matrix} \quad \varphi \otimes \psi}{\alpha}\,(\otimes e) \qquad \frac{\begin{matrix}[\varphi] \\ \vdots \\ \varphi'\end{matrix} \quad \begin{matrix}[\psi] \\ \vdots \\ \psi'\end{matrix} \quad \varphi \otimes \psi}{\varphi' \otimes \psi'}\,(\otimes r)$$

$$\frac{\alpha \to (\varphi \vee \psi)}{(\alpha \to \varphi) \vee (\alpha \to \psi)}\,(s) \qquad \frac{\varphi \otimes (\psi \vee \chi)}{(\varphi \otimes \psi) \vee (\varphi \otimes \chi)}\,(\otimes d) \qquad \frac{\varphi \otimes \psi}{\psi \otimes \varphi}\,(\otimes c) \qquad (\alpha \text{ is } \vee\text{-free})$$

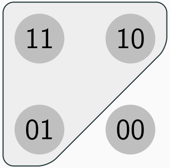
The epistemic interpretation

Different disjunctions in team semantics and the epistemic perspective

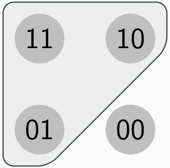
- $\mathcal{M}, s \Vdash (\alpha \vee \beta)$ iff $\mathcal{M}, s \Vdash \alpha$ or $\mathcal{M}, s \Vdash \beta$.
- $\mathcal{M}, s \Vdash (\alpha \otimes \beta)$ iff there exist $t, t' \subseteq s$, s.t. $\mathcal{M}, t \Vdash \alpha$, $\mathcal{M}, t' \Vdash \beta$, and $t \cup t' = s$.
- $\mathcal{M}, s \Vdash (\alpha \veebar \beta)$ iff $\mathcal{M}, \{w\} \Vdash \alpha$ or $\mathcal{M}, \{w\} \Vdash \beta$ for all $w \in s$.



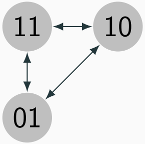
(a) $p \vee q$



(b) $p \otimes q$



(c) $p \veebar q$

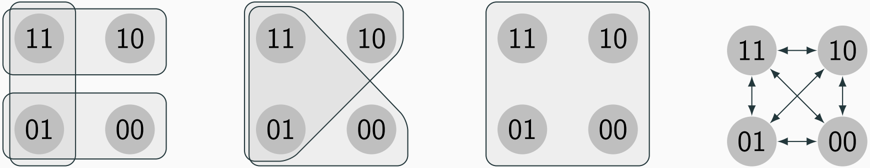


(d) S5 model

Kripke semantics: $\mathcal{M}_s, w \vDash \mathcal{K}(p \vee q)$ iff $\forall v \in \mathcal{M}_s$, (if wRv, then) $\mathcal{M}_s, v \vDash p \vee q$

$s \Vdash (\alpha \veebar \beta)$ iff $\mathcal{M}_s, w \vDash \mathcal{K}(\alpha \vee \beta)$ where $w \in \mathcal{M}_s$???

The epistemic perspective



(e) $?p \vee q$ (f) $?p \otimes q$ (g) $?p \mathbin{\dot{\vee}} q$ (h) S5 model

- $(\alpha \otimes \alpha) \leftrightarrow \alpha$ is not valid in general.
- Non-classical logics often have hidden modalities, e.g., $p \strictif q := \Box(p \to q)$
- Making the implicit explicit
- Intuitionistic-like logics have a strong epistemic flavour to many...
- Heyting [Hey30]: **Intuitionistic truth of α = knowing how to prove α**
- Martin-Löf and Hintikka had also similar informal ideas.

Missing tool: we need to know how to formalize know-how

Epistemic logic of know-wh can help

Know-how can be formalized via epistemic logic of know-wh (how, why, what and so on) [Wan18] featuring the so-called *bundled modalities* such as $\exists x\Box$ [Wan17].

Knowing how to prove α ($\mathcal{K}h\alpha$)

iff

there is a proof such that you know that it is a proof of α: $\exists x\mathcal{K}(x \in \mathit{Proof}(\alpha))$

The meaning of *Proof* is given by BHK-interpretation (a constructive interpretation of intuitionist logic).

- a proof of $\alpha \wedge \beta$ is given by a proof of α and a proof of β,
- a proof of $\alpha \vee \beta$ is given by a proof of α or a proof of β,
- a proof of $\alpha \to \beta$ is a function which converts each proof of α to a proof of β.

From Intuitionistic logic to various intermediate logics

For *Inquisitive Logic* we can give an alternative interpretation:

Inquisitive truth of α = knowing how to resolve α

Similar idea appeared as early as in [Cia09].

For each α in $\mathbf{InqB}^{\otimes}$ we associate a know-how formula $\mathcal{K}h\alpha$ with it, and view each non-empty *team* as an S5 epistemic model $\mathcal{M}$ such that:

$$\boxed{\mathcal{M}, w \vDash \mathcal{K}h\alpha \quad \Leftrightarrow \quad \exists r\ \mathcal{K}(r \text{ is a resolution of } \alpha)}$$

With proper BHK definitions of resolutions on each world, we can show $\{\alpha \mid \mathcal{K}h\alpha \text{ is valid}\}$ is exactly $\mathbf{InqB}^{\otimes}$.

Knowing how to interpret $\mathrm{InqB}^{\otimes}$

The language of propositional logic with tensor disjunction ($\mathbf{PL}^{\otimes}$) [CB19]

$$\alpha ::= p \mid \bot \mid (\alpha \wedge \alpha) \mid (\alpha \vee \alpha) \mid (\alpha \to \alpha) \mid (\alpha \otimes \alpha)$$

where $p \in \mathbf{P}$. Note that $\mathbf{PL}^{\otimes}$ is the language for $\mathbf{InqB}^{\otimes}$.

The language of *Public Announcement Logic with Know-how Operator and Propositional Quantifier* (**PALKhΠ**)

$$\varphi ::= p \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \otimes \varphi) \mid (\varphi \to \varphi) \mid \mathcal{K}\varphi \mid \mathcal{K}h\alpha \mid \forall p\varphi \mid [\varphi]\varphi$$

where $p \in \mathbf{P}$ and $\alpha \in \mathbf{PL}^{\otimes}$. We write $\neg\varphi$ for $\varphi \to \bot$, $\widehat{\mathcal{K}}$ for $\neg\mathcal{K}\neg$, and $\exists p$ for $\neg\forall p\neg$ for all $p \in \mathbf{P}$ for all $\varphi \in$ **PALKhΠ**. Intuitively, $\mathcal{K}\varphi$ expresses that "the agent *knows that* φ", $\mathcal{K}h\alpha$ says that "the agent *knows how* to resolve α", $\forall p\varphi$ says that "for any proposition p, φ holds" and $[\varphi]\psi$ means that "after announcing φ, ψ holds".

Epistemic model

Each epistemic model corresponds to a team/state in team/inquisitive semantics.

An (epistemic) model of **PALKhΠ** is a pair $\mathcal{M} = \langle W, V \rangle$ where:

- W is a non-empty set of possible worlds;
- $V : \mathbf{P} \to \wp(W)$ is a valuation function.

Note that the epistemic model is by default all-connected. Therefore all epistemic models are $S5$ models.

Semantics

For $\varphi, \psi \in$ **PALKhΠ**, $\alpha \in \mathbf{PL}^{\otimes}$ and $\mathcal{M}, w$ where $\mathcal{M} = \langle W, V \rangle$:

$\mathcal{M}, w \vDash p$	$\Leftrightarrow$	$w \in V(p) \qquad (\mathcal{M}, w \nvDash \bot)$
$\mathcal{M}, w \vDash (\varphi \bigcirc \psi)$	$\Leftrightarrow$	$\mathcal{M}, w \vDash \varphi$ or $\mathcal{M}, w \vDash \psi, \bigcirc \in \{\vee, \otimes\}$
$\mathcal{M}, w \vDash (\varphi \wedge \psi)$	$\Leftrightarrow$	$\mathcal{M}, w \vDash \varphi$ and $\mathcal{M}, w \vDash \psi$
$\mathcal{M}, w \vDash (\varphi \to \psi)$	$\Leftrightarrow$	$\mathcal{M}, w \vDash \varphi$ implies $\mathcal{M}, w \vDash \psi$
$\mathcal{M}, w \vDash \mathcal{K}\varphi$	$\Leftrightarrow$	for any $v \in \mathcal{M}, \mathcal{M}, v \vDash \varphi$
$\mathcal{M}, w \vDash \forall p \varphi$	$\Leftrightarrow$	for any $U \in \wp(W), \mathcal{M}(p \mapsto U), w \vDash \varphi$
$\mathcal{M}, w \vDash [\psi]\varphi$	$\Leftrightarrow$	$\mathcal{M}, w \vDash \psi$ implies $\mathcal{M}_{\vert\psi}, w \vDash \varphi$

$\mathcal{M}, w \vDash \forall p[p]\varphi$

$$\mathcal{M}, w \vDash \mathcal{K}h\alpha \quad \Leftrightarrow \quad ???$$

there exists an x s.t. for all $v \in \mathcal{M}, P_\alpha(x)$

Definition (Resolution Space)

In order to give a *BHK*-like semantics, we define resolution space S and world-dependent resolution R.

S is a function assigning each $\alpha \in \mathbf{PL}^{\otimes}$ its set of potential resolutions:

$$
\begin{aligned}
S(p) &= \{p\}, \text{ for } p \in \mathbf{P} \\
S(\bot) &= \{\bot\} \\
S(\alpha \vee \beta) &= (S(\alpha) \times \{0\}) \cup (S(\beta) \times \{1\}) \\
S(\alpha \wedge \beta) &= S(\alpha) \times S(\beta) \\
S(\alpha \to \beta) &= S(\beta)^{S(\alpha)} \\
S(\alpha \otimes \beta) &= S(\alpha) \times S(\beta)
\end{aligned}
$$

It follows immediately that $S(\alpha)$ is **finite** and **non-empty** for $\alpha \in \mathbf{PL}^{\otimes}$.

Definition (Resolution)

Given $\mathcal{M}$, $R : W \times \mathbf{PL}^{\otimes} \to \bigcup_{\alpha \in \mathbf{PL}^{\otimes}} S(\alpha)$ is defined as follows:

$$R(w, \bot) = \varnothing$$

$$R(w, p) = \begin{cases} \{p\} & \text{if } w \in V_{\mathcal{M}}(p) \\ \varnothing & \text{otherwise} \end{cases}$$

$$R(w, \alpha \vee \beta) = (R(w, \alpha) \times \{0\}) \cup (R(w, \beta) \times \{1\})$$

$$R(w, \alpha \wedge \beta) = R(w, \alpha) \times R(w, \beta)$$

$$R(w, \alpha \to \beta) = \{f \in S(\beta)^{S(\alpha)} \mid f[R(w, \alpha)] \subseteq R(w, \beta)\}$$

$$R(w, \alpha \otimes \beta) = (R(w, \alpha) \times S(\beta)) \cup (S(\alpha) \times R(w, \beta))$$

It follows immediately that $R(w, \alpha) \subseteq S(\alpha)$ for $w \in \mathcal{M}$ and $\alpha \in \mathbf{PL}^{\otimes}$.

A surprising link

There is a surprising link between $\otimes$ and weak disjunction proposed by Medvedev [Med66] on the logic of finite problems!

$\langle a, b \rangle$ is a resolution to the weak disjunction $\alpha \sqcup \beta$ iff either "a resolves α" or "b resolves β". These two resolutions correspond to the two-part covering of the whole team in the team semantics.

Epistemically, $\alpha \otimes \beta$ is true iff there are two resolutions a and b such that you know that either "a resolves α" or "b resolves β", i.e., know how to resolve one of α and β, but may not know which.

Semantics

For $\varphi, \psi \in$ **PALKhΠ**, $\alpha \in$ **PL**$^{\otimes}$ and $\mathcal{M}, w$ where $\mathcal{M} = \langle W, V \rangle$:

$$
\boxed{
\begin{array}{lcl}
\mathcal{M}, w \vDash p & \Leftrightarrow & w \in V(p) \qquad (\mathcal{M}, w \nvDash \bot) \\
\mathcal{M}, w \vDash (\varphi \bigcirc \psi) & \Leftrightarrow & \mathcal{M}, w \vDash \varphi \text{ or } \mathcal{M}, w \vDash \psi, \bigcirc \in \{\vee, \otimes\} \\
\mathcal{M}, w \vDash (\varphi \wedge \psi) & \Leftrightarrow & \mathcal{M}, w \vDash \varphi \text{ and } \mathcal{M}, w \vDash \psi \\
\mathcal{M}, w \vDash (\varphi \to \psi) & \Leftrightarrow & \mathcal{M}, w \vDash \varphi \text{ implies } \mathcal{M}, w \vDash \psi \\
\mathcal{M}, w \vDash \mathcal{K}\varphi & \Leftrightarrow & \text{for any } v \in \mathcal{M}, \mathcal{M}, v \vDash \varphi \\
\mathcal{M}, w \vDash \mathcal{K}h\alpha & \Leftrightarrow & \text{there exists an } x \in S(\alpha) \\
 & & \text{s.t. for any } v \in \mathcal{M}, x \in R(v, \alpha) \\
\mathcal{M}, w \vDash \forall p \varphi & \Leftrightarrow & \text{for any } U \in \wp(W), \mathcal{M}(p \mapsto U), w \vDash \varphi \\
\mathcal{M}, w \vDash [\psi]\varphi & \Leftrightarrow & \mathcal{M}, w \vDash \psi \text{ implies } \mathcal{M}_{|\psi}, w \vDash \varphi
\end{array}
}
$$

Let $R(\mathcal{M}, \alpha) = (\bigcap_{v \in \mathcal{M}} R(v, \alpha))$. The semantics of $\mathcal{K}h$ can be reformulated as below for notational brevity.

$$\boxed{\mathcal{M}, w \vDash \mathcal{K}h\alpha \quad \Leftrightarrow \quad R(\mathcal{M}, \alpha) \neq \varnothing}$$

Remarks on the semantics

Obervation ($\mathcal{K}$ and $\mathcal{K}h$)

Note that the truth conditions of $\mathcal{K}$ and $\mathcal{K}h$ do not depend on the designated world, therefore for any model $\mathcal{M}, w$:

- $\mathcal{M}, w \vDash \mathcal{K}h\alpha \Leftrightarrow \mathcal{M} \vDash \mathcal{K}h\alpha$, and $\mathcal{M}, w \vDash \neg\mathcal{K}h\varphi \Leftrightarrow \mathcal{M} \vDash \neg\mathcal{K}h\varphi$;
- $\mathcal{M}, w \vDash \mathcal{K}\alpha \Leftrightarrow \mathcal{M} \vDash \mathcal{K}\alpha$, and $\mathcal{M}, w \vDash \neg\mathcal{K}\alpha \Leftrightarrow \mathcal{M} \vDash \neg\mathcal{K}\alpha$.

As a consequence, the introspection axioms $\mathcal{K}h\alpha \leftrightarrow \mathcal{K}\mathcal{K}h\alpha$ and $\neg\mathcal{K}h\alpha \leftrightarrow \mathcal{K}\neg\mathcal{K}h\alpha$ are valid.

Classical semantics of α

For any $\alpha \in \mathbf{PL}^{\otimes}$ and $\mathcal{M}, w$, $\mathcal{M}, w \vDash \alpha \Leftrightarrow R(w, \alpha) \neq \varnothing$, where $\alpha \in \mathbf{PL}^{\otimes}$.

Proof. Induction on the structure of α:

$$\mathcal{M}, w \vDash p \Leftrightarrow p \in V_{\mathcal{M}}(w) \Leftrightarrow R(w, p) = \{p\} \Leftrightarrow R(w, p) \neq \varnothing$$

$$\mathcal{M}, w \nvDash \bot \Leftrightarrow R(w, \bot) = \varnothing$$

$$\begin{aligned}
\mathcal{M}, w \vDash (\alpha \vee \beta) &\Leftrightarrow \mathcal{M}, w \vDash \alpha \text{ or } \mathcal{M}, w \vDash \beta \Leftrightarrow R(w, \alpha) \neq \varnothing \text{ or } R(w, \beta) \neq \varnothing \\
&\Leftrightarrow \text{ there exists an } x \in R(w, \alpha) \text{ or there exists a } y \in R(w, \beta) \\
&\Leftrightarrow \text{ there exists a pair } \langle x, 0 \rangle \text{ or } \langle y, 1 \rangle \text{ in } R(w, \alpha \vee \beta) \\
&\Leftrightarrow R(w, \alpha \vee \beta) \neq \varnothing
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}, w \vDash (\alpha \otimes \beta) &\Leftrightarrow \mathcal{M}, w \vDash \alpha \text{ or } \mathcal{M}, w \vDash \beta \Leftrightarrow R(w, \alpha) \neq \varnothing \text{ or } R(w, \beta) \neq \varnothing \\
&\Leftrightarrow \text{ there exists an } x \in R(w, \alpha) \text{ or there exists a } y \in R(w, \beta) \\
&\Leftrightarrow \text{ there exists a pair } \langle x, x' \rangle \text{ or } \langle y', y \rangle \text{ in } R(w, \alpha \otimes \beta) \\
&\qquad \text{such that } y' \in S(\alpha) \text{ and } x' \in S(\beta) \quad \Leftrightarrow R(w, \alpha \otimes \beta) \neq \varnothing
\end{aligned}$$

Reduction

Now we have an alternative semantics for $\mathcal{K}\alpha$:

$$\boxed{\mathcal{M}, w \vDash \mathcal{K}\alpha \quad \Leftrightarrow \quad \textbf{for any } v \in \mathcal{M}, \ \textbf{there exists} \text{ an } x \in R(v, \alpha)}$$

Comparing with the following semantics of $\mathcal{K}h$, it now becomes clear that the distinction between $\mathcal{K}h$ and $\mathcal{K}$ is exactly the distinction between de re and de dicto, i.e., knowing α is resolvable vs. know how α is resolved.

$$\boxed{\mathcal{M}, w \vDash \mathcal{K}h\alpha \quad \Leftrightarrow \quad \textbf{there exists} \text{ an } x \text{ s.t. } \textbf{for any } v \in \mathcal{M}, x \in R(v, \alpha)}$$

Based on this distinction, $\mathcal{K}h\alpha$ is clearly stronger than $\mathcal{K}\alpha$:

$\mathcal{K}h\alpha \to \mathcal{K}\alpha$ is valid for all $\alpha \in \mathbf{PL}^{\otimes}$.

The distinction disappears, if we consider the atomic propositions since there can be at most one fixed resolution p for each $p \in \mathbf{PL}^{\otimes}$.

$\mathcal{K}hp \leftrightarrow \mathcal{K}p$ is valid for all $p \in \mathbf{P}$.

Reduction axioms

The following formulas and schemata are valid reflecting the team semantics:

$\mathtt{KKhp}$: $\mathcal{K}p \leftrightarrow \mathcal{K}hp$

$\mathtt{Kh}_\perp$: $\mathcal{K}h\perp \leftrightarrow \perp$

$\mathtt{Kh}_\vee$: $\mathcal{K}h(\alpha \vee \beta) \leftrightarrow \mathcal{K}h\alpha \vee \mathcal{K}h\beta$

$\mathtt{Kh}_\wedge$: $\mathcal{K}h(\alpha \wedge \beta) \leftrightarrow \mathcal{K}h\alpha \wedge \mathcal{K}h\beta$

$\mathtt{Kh}_\to$: $\mathcal{K}h(\alpha \to \beta) \leftrightarrow \mathcal{K}\forall p[p](\mathcal{K}h\alpha \to \mathcal{K}h\beta)$, where p does not occur in α or β

$\mathtt{Kh}_\otimes$: $\mathcal{K}h(\alpha \otimes \beta) \leftrightarrow \exists p_1 \exists p_2 \mathcal{K}((p_1 \otimes p_2) \wedge [p_1]\mathcal{K}h\alpha \wedge [p_2]\mathcal{K}h\beta)$,

where p_1 and p_2 do not occur in α and β respectively.

Reduction axioms

Proposition

The following formulas and schemata are valid:

$$
\begin{array}{ll}
[\,]_{\mathrm{p}} & [\chi]p \leftrightarrow (\chi \to p),\ \ p \in \mathbf{P} \cup \{\bot\}. \\
[\,]_{\bigcirc} & [\chi](\varphi \bigcirc \psi) \leftrightarrow [\chi]\varphi \bigcirc [\chi]\psi, \bigcirc \in \{\wedge, \vee, \otimes, \to\}. \\
[\,]_{\mathcal{K}} & [\chi]\mathcal{K}\varphi \leftrightarrow (\chi \to \mathcal{K}([\chi]\varphi)). \\
[\,]_{\forall} & [\chi]\forall p\varphi \leftrightarrow \forall p[\chi]\varphi, p \text{ is not in } \chi.
\end{array}
$$

Theorem

PALKhΠ *is equally expressive as its $\mathcal{K}h$-free and $[\varphi]$-free fragment over all models.*

Remarks on the semantics

Theorem

Let $\mathbf{InqB}^{\otimes}$ *be inquisitve logic with tensor defined in [CB19], it follows that* $\mathbf{Inq}^{\otimes}\mathbf{Kh} = \{\alpha \in \mathbf{PL}^{\otimes} \mid \vDash \mathcal{K}h\alpha\}$ *is exactly* $\mathbf{InqB}^{\otimes}$.

state-based team semantics	world-based Kripke semantics
A simple propositional language	A powerful modal language
support relation ($\Vdash$)	satisfaction relation ($\vDash$)
$\mathbf{InqB}^{\otimes} = \{\alpha \mid \ \Vdash \alpha\}$	$\mathbf{Inq}^{\otimes}\mathbf{Kh} = \{\alpha \mid \ \vDash \mathcal{K}h\alpha\}$
implicit	explicit

SPALKhΠ$^+$ (where $\alpha, \beta \in \mathbf{PL}^{\otimes}$, $p \in \mathbf{P}$, $\varphi, \psi, \chi \in$ **PALKhΠ**)

	S5 axioms/rules for $\mathcal{K}$;	**Rd**$\otimes$	$(\varphi \otimes \psi) \leftrightarrow (\varphi \vee \psi)$;
rRE	$\dfrac{\vdash \varphi \leftrightarrow \psi}{\vdash \chi[\varphi/\psi] \leftrightarrow \chi}$, the substitution happens outside the scope of $\mathcal{K}h$;		
DIST$_\forall$	$\forall p(\varphi \to \psi) \to (\forall p\varphi \to \forall p\psi)$;	SUB$_\forall$	$\forall p\varphi \to \varphi[\psi/p], \psi$ is free for p in φ;
SU	$\exists p(p \wedge \forall q(q \to \mathcal{K}(p \to q)))$;	GEN$_\forall$	$\dfrac{\vdash \varphi \to \psi}{\vdash \varphi \to \forall p\psi}$, p not free in φ;
KKhp	$\mathcal{K}p \leftrightarrow \mathcal{K}hp,\ \ p \in \mathbf{P} \cup \{\bot\}$;	Kh$_\vee$	$\mathcal{K}h(\alpha \vee \beta) \leftrightarrow \mathcal{K}h\alpha \vee \mathcal{K}h\beta$;
Kh$_\wedge$	$\mathcal{K}h(\alpha \wedge \beta) \leftrightarrow \mathcal{K}h\alpha \wedge \mathcal{K}h\beta$;	Kh$_\to$	$\mathcal{K}h(\alpha \to \beta) \leftrightarrow \mathcal{K}\forall p[p](\mathcal{K}h\alpha \to \mathcal{K}h\beta)$;
Kh$_\otimes$	$\mathcal{K}h(\alpha \otimes \beta) \leftrightarrow \exists p_1 \exists p_2 \mathcal{K}((p_1 \otimes p_2) \wedge [p_1]\mathcal{K}h\alpha \wedge [p_2]\mathcal{K}h\beta), p_1, p_2$ not in α, β;		
$[\,]_p$	$[\chi]p \leftrightarrow (\chi \to p),\ \ p \in \mathbf{P} \cup \{\bot\}$;		
$[\,]_\bigcirc$	$[\chi](\varphi \bigcirc \psi) \leftrightarrow [\chi]\varphi \bigcirc [\chi]\psi, \bigcirc \in \{\wedge, \vee, \otimes, \to\}$;		
$[\,]_\mathcal{K}$	$[\chi]\mathcal{K}\varphi \leftrightarrow \chi \to \mathcal{K}[\chi]\varphi$;	$[\,]_\forall$	$[\chi]\forall p\varphi \leftrightarrow \forall p[\chi]\varphi$, p is not in χ.

Completeness

Theorem (Completeness of SPALKhΠ$^+$)

System SPALKhΠ$^+$ *is complete over the class of all models.*

Proof [sketch]

$$\begin{tikzcd}
\boxed{\bot, \vee, \wedge, \to, \otimes, \mathcal{K}, \forall, [\ ], \mathcal{K}h} \arrow[r] & \boxed{\bot, \vee, \wedge, \to, \otimes, \mathcal{K}, \forall, [\ ]} \arrow[dl] \\
\boxed{\bot, \vee, \wedge, \to, \otimes, \mathcal{K}, \forall} \arrow[r] & \boxed{\bot, \vee, \wedge, \to, \mathcal{K}, \forall \text{ [Fin70]}}
\end{tikzcd}$$

$\blacksquare$

From $\otimes$ to $\otimes_n^k$

- We interpreted $\mathbf{InqB}^{\otimes}$ using a powerful language.
- Tensor is about at least one correct resolution in two, we will generalize it to at least k correct ones in n and show it does not increase the expressivity of inquisitive/dependence logic.

Generalization of Tensor Disjunction

Generalizing the tensor operator

$\alpha \otimes \beta$ is true $\Leftrightarrow$ there is a pair $\langle a, b \rangle$ such that you know that a resolves α or b resolves β. Why and how we generalize it?

Example

You completed an exam with n questions, for which you need at least k correct answers to pass. Now you only know you have passed the exam. What is your epistemic state?

For any $n \geq 2$ and $1 \leq k \leq n$, we define an n-ary connective $\otimes_n^k$ capturing that you are sure at least k of your n answers must be correct, but may not know which ones are correct.

It is not hard to see that $\otimes = \otimes_2^1$.

Semantic of $\otimes_n^k$ in the epistemic setting

Definition (Resolution space S and Resolution R for $\otimes_n^k$)

$S(\otimes_n^k(\alpha_1, \cdots, \alpha_n)) = S(\alpha_1) \times \cdots \times S(\alpha_n)$

$R(w, \otimes_n^k(\alpha_1, \cdots, \alpha_n)) = \{(r_1, \cdots, r_n) \mid k \leq |\{i \in [1, n] \mid r_i \in R(w, \alpha_i)\}|\}$

Proposition ($\otimes_n^k$ in $\mathcal{K}h$)

$\mathcal{M}, w \vDash \mathcal{K}h \otimes_n^k (\alpha_1, \cdots, \alpha_n) \Leftrightarrow$ there is an n-tuple $\langle r_1, \cdots, r_n \rangle$ such that for any $v \in \mathcal{M}$, there are at least k integers $1 \leq i \leq n$ such that $r_i \in R(v, \alpha_i)$.

Definition ($\otimes_n^k$ outside of $\mathcal{K}h$)

$\mathcal{M}, w \vDash \otimes_n^k(\varphi_1, \cdots, \varphi_n) \Leftrightarrow R(w, \otimes_n^k(\varphi_1, \cdots, \varphi_n)) \neq \varnothing$.

Proposition

$\mathcal{M}, w \vDash \otimes_n^k(\varphi_1, \cdots, \varphi_n) \Leftrightarrow \mathcal{M}, w \vDash \bigvee_{\substack{I \subseteq \{1,2,\cdots,n\} \\ |I| = k}} \bigwedge_{i \in I} \varphi_i$.

Expressive power in the epistemic setting

$Rd\otimes : (\varphi \otimes \psi) \leftrightarrow (\varphi \vee \psi)$

$\mathtt{Kh}_{\otimes} : \mathcal{K}h(\alpha \otimes \beta) \leftrightarrow \exists p_1 \exists p_2 \mathcal{K}((p_1 \otimes p_2) \wedge [p_1]\mathcal{K}h\alpha \wedge [p_2]\mathcal{K}h\beta)$

Proposition ($\mathtt{Rd}\otimes^{\mathtt{k}}_{\mathtt{n}}$ and $\mathtt{Kh}_{\otimes^{\mathtt{k}}_{\mathtt{n}}}$: Reduction axioms for $\otimes^k_n$)

The following two axioms are valid in $\mathbf{PALKh\Pi}_G$*:*

$\mathtt{Rd}\otimes^{\mathtt{k}}_{\mathtt{n}} : \otimes^k_n(\varphi_1, \cdots, \varphi_n) \leftrightarrow \bigvee_{\substack{I \subseteq \{1,2,\cdots,n\} \\ |I|=m}} \bigwedge_{i \in I} \varphi_i$

$\mathtt{Kh}_{\otimes^{\mathtt{k}}_{\mathtt{n}}} : \mathcal{K}h \otimes^k_n (\alpha_1, \cdots, \alpha_n) \leftrightarrow \exists p_1 \cdots \exists p_n \mathcal{K}(\otimes^k_n(p_1, \cdots, p_n) \wedge \bigwedge^n_{i=1} [p_i]\mathcal{K}h\alpha_i)$

Theorem

For language $\mathbf{PALKh\Pi}_G$*, system* SPALKhΠ$^+$ *plus* $\mathtt{Rd}\otimes^{\mathtt{k}}_{\mathtt{n}}$ *and* $\mathtt{Kh}_{\otimes^{\mathtt{k}}_{\mathtt{n}}}$ *is sound and complete over the class of all epistemic models.*

Support Semantics for $\otimes_n^k$

Definition (Support for $\otimes_n^k$)

$\mathcal{M}, s \Vdash \otimes_n^k(\alpha_1, \cdots, \alpha_n) \Leftrightarrow$ *there exist* n *subsets* $t_1, \cdots, t_n$ *of* s *such that for any* $i \in [1, n], \mathcal{M}, t_i \Vdash \alpha_i$ *and for any* $w \in s$*, there are at least* k *indexes* $i \in [1, n]$ *such that* $w \in t_i$*.*

We can show that $\mathbf{KhLGT} = \{\alpha \in \mathbf{PLGT} | \vDash \mathcal{K}h\alpha\}$ is exactly $\mathbf{InqBGT} = \{\alpha | \Vdash \alpha\}$

An alternative proof via resolution formulas and normal forms

Definition (Resolution formulas)

- $RL(p) = \{p\}$ for $p \in \mathbf{P}$
- $RL(\bot) = \{\bot\}$
- $RL(\alpha \vee \beta) = RL(\alpha) \cup RL(\beta)$
- $RL(\alpha \wedge \beta) = \{\rho \wedge \sigma \mid \rho \in RL(\alpha) \text{ and } \sigma \in RL(\beta)\}$
- $RL(\alpha \to \beta) = \{\bigwedge_{\rho \in RL(\alpha)}(\rho \to f(\rho)) \mid f : RL(\alpha) \to RL(\beta)\}$
- $RL(\otimes_n^k(\alpha_1, \cdots, \alpha_n)) = \{\neg \bigwedge_{\substack{I \subseteq \{1,2,\cdots,n\} \\ |I|=k}} \neg \bigwedge_{i \in I} \rho_i \mid \text{ for all } i, \rho_i \in RL(\alpha_i)\}$

Proposition (Normal form)

For any $\alpha \in \mathbf{PLGT}$ and any state s, $s \Vdash \alpha$ iff $s \Vdash \bigvee_{\rho \in RL(\alpha)} \rho$.

The same expressive power

Since each formula is equivalent to its $\vee$-free normal form,

Theorem

Adding general tensors to **InqB** *does not increase the expressive power.*

Since **PD**, $\mathbf{PD}^{\vee}$, **PID**, and **InqB** are all equally expressive [YV16]. Adding general tensors to other logics will also not increase the expressive power.

Logic	Atoms	Connectives
Propositional dependence logic (**PD**)	$p_i, \neg p_i, \bot, =(p_{i1}, \ldots, p_{ik}, p_j)$	$\wedge, \otimes$
Propositional dependence logic with intuitionistic disjunction ($\mathbf{PD}^{\vee}$)	$p_i, \neg p_i, \bot$	$\wedge, \otimes, \vee$
Propositional intuitionistic dependence logic (**PID**)	$p_i, \bot, =(p_{i1}, \ldots, p_{ik}, p_j)$	$\wedge, \vee, \to$
Propositional inquisitive logic (**InqB**)	$p_i, \bot$	$\wedge, \vee, \to$

Axiomatization of inquisitive logic with generalized tensors (based on [BY22])

System SInqBGT

Axioms

`INTU`	Intuitionistic validities
`DN`	$\neg\neg p \to p$ for all $p \in \mathbf{P}$
`KP`	$(\neg\alpha \to \beta \vee \gamma) \to (\neg\alpha \to \beta) \vee (\neg\alpha \to \gamma)$
`MONO`	$\otimes_n^k(\alpha_1, \ldots, \alpha_n) \wedge \bigwedge_{i \in [1,n]}(\alpha_i \to \beta_i) \to \otimes_n^k(\beta_1, \ldots, \beta_n)$
$\mathtt{GT}_{\mathtt{Rd}}$	$\otimes_n^k(\rho_1, \ldots, \rho_n) \leftrightarrow \neg \bigwedge_{\substack{I \subseteq \{1,2,\cdots,n\} \\ \lvert I \rvert = k}} \neg \bigwedge_{i \in I} \rho_i$ where for $i \in [1, n]$, ρ_i is $\vee$-free
$\mathtt{GT}_{\vee}$	$\otimes_n^k(\alpha_1 \vee \beta_1, \ldots, \alpha_n \vee \beta_n) \to \bigvee_{\substack{\gamma_i \in \{\alpha_i, \beta_i\} \\ \text{for } i \in [1,n]}} \otimes_n^k(\gamma_1, \ldots, \gamma_n)$
where	$n \geq 2$ and $1 \leq k \leq n$

Rules

`MP` $\dfrac{\alpha, \alpha \to \beta}{\beta}$

Theorem (Completeness of SInqBGT)

For any **PLGT**-*formula* α, $\Vdash \alpha \implies \vdash \alpha$.

Proof

$$\Vdash \alpha \iff \Vdash \bigvee_{\rho \in RL(\alpha)} \rho \iff \Vdash \rho \text{ for some } \rho \in RL(\alpha)$$
$$\iff \vdash \rho \text{ for some } \rho \in RL(\alpha) \iff \vdash \bigvee\nolimits_{\rho \in RL(\alpha)} \rho \iff \vdash \alpha. \blacksquare$$

The strong completeness can be obtained based on the compactness of SInqBGT, which is equivalent to the compactness of **InqB** shown in [Cia09], since these two logics are equally expressive.

Strong completeness can also be proved directly by constructing canonical models (c.f. Chapter 3.3 of [Cia16]).

Uniform definability

In a logic with uniform substitution property, if a connective does not increase the expressive power, it must be uniformly definable from the other connectives.

$$p \bigcirc q \leftrightarrow \varphi(p, q) \stackrel{USP}{\Longrightarrow} \alpha \bigcirc \beta \leftrightarrow \varphi(\alpha, \beta)$$

In **InqB**, we only have axiom $\neg\neg p \to p$ for atomic p. So there is no uniform substitution property.

In [CB19], it is proved that $\otimes^1_2$ is not uniformly definable from $\{\bot, \wedge, \to, \vee\}$ in **InqB**.

Are $\otimes^k_n$ uniformly definable from $\mathbf{InqB}^{\otimes}$?

Uniform definability

Proposition

*Some positive results For any $\alpha_1, \cdots, \alpha_n \in$ **PLGT**, the following hold.*

(1) *For any $n \geq 2$, $\otimes_n^n(\alpha_1, \cdots, \alpha_n) \equiv \bigwedge_{i=1}^n \alpha_i$.*

(2) *For any $n \geq 3$, $\otimes_n^1(\alpha_1, \cdots, \alpha_n) \equiv \otimes_2^1(\otimes_{n-1}^1(\alpha_1, \cdots, \alpha_{n-1}), \alpha_n)$.*

(3) *For any $n \geq 2$, $1 \leq k \leq n$, $\otimes_n^k(\alpha_1, \cdots, \alpha_n) \equiv \otimes_{n+1}^{k+1}(\alpha_1, \cdots, \alpha_n, \top)$.*

(4) *For any $n \geq 2$, $1 \leq k \leq n$, $\otimes_n^k(\alpha_1, \cdots, \alpha_n) \equiv \otimes_{n+1}^{k}(\alpha_1, \cdots, \alpha_n, \bot)$.*

Theorem

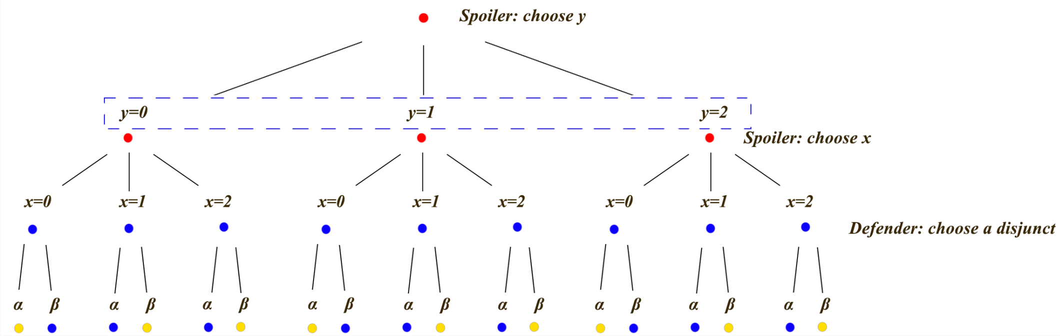
$\otimes_n^1$ and $\otimes_n^n$ can be trivially uniformly defined from other connectives $\{\bot, \wedge, \rightarrow, \vee, \otimes\}$ in **InqB**$^\otimes$.

All the other $\otimes_n^k$ are **not uniformly definable** *in* **InqB**$^\otimes$*, i.e., for any $2 \leq k \leq n-1$, $\otimes_n^k$ is not uniformly definable.*

Back to game semantics

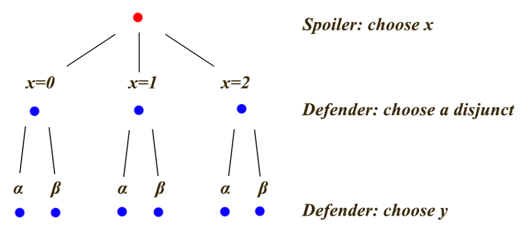
Hodges's compositional semantics for games[Hen61, HS97, Hod97a]

- Given a structure, a sentence φ is true iff $\exists$ a winning strategy for Defender.
- Let φ be $(\alpha(\vee/W)\beta)$. Then: $(A, S) \Vdash \varphi$ iff $S \neq \varnothing$ and there are T, T' such that each nonempty $\simeq_W$-set $\subseteq S$ lies in either T or T', $(A, T) \Vdash \alpha$, and $(A, T') \Vdash \beta$.
- e.g. $A = \{0, 1, 2\}$,$\forall y \forall x(\exists z\ x + y \leq z(\otimes/y)\exists z\ x + y > z)$.



A simplified version [Hod97b]

- $(D,S) \Vdash \varphi(\bar{v}) \otimes \psi(\bar{v})$ iff S is a nonempty n-ary relation on the domain D, such that the sentence $\tau_{\varphi\otimes\psi}(S) : \exists T \exists T'((S \subseteq T \cup T') \wedge \tau_\varphi(T) \wedge \tau_\psi(T'))$ is true.
- e.g. $\forall x((\exists y/x)Rxy \otimes (\exists y/x)Rxy)$, $A = \{0,1,2\}$, $R = \{(0,0),(1,0),(2,1)\}$



WS($\alpha \otimes \beta$)=WS(α) (in some cases) + WS(β) (in the other cases)

Knowing how to win the game of $\alpha \otimes \beta$ as having a winning strategy for $\alpha \otimes \beta$

Conclusions

Conclusions

- We give an epistemic interpretation of inquisitive logic with tensor using a dynamic epistemic logic of knowing how
- We show the know-how validity is exactly $\mathbf{InqB}^{\otimes}$
- We give a complete axiomitization of the full logic
- $\otimes$ is generalized to $\otimes_n^k$
- The generalized tensors do not increasing the expressive power of inquisitive logic with tensor, but they are not uniformly definable in inquisitive logic with tensor.

$\alpha \otimes \beta$ captures you have a pair of answers $\langle a, b \rangle$ and you are sure at least one of them must be correct, but may not know which ones are correct. These general tensors may have applications, such as in cryptography.

Related work and future work

- Based on our previous work *An Epistemic Interpretation of Tensor Disjunction* [WWW22a].
- We provided an alternative interpretation of $\wedge$, $\vee$, and $\rightarrow$ in **InqB**, and gave a complete axiomatization of the logic and showed that it has exactly the same expressive power as S5 modal logic [WWW22b].
- We also plan to study first-order intuitionist logic and non-classical modalities with the epistemic interpretation.
- Can we study logics without downward closure properties in this way?

Thank You for Listening!

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