# Constructivism: Views from relevance logic

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# Constructivism, Informally

This is a talk about non-classical logic and, specifically, intuitionistic logic (**J**) and logics in its vicinity. Intuitionistic logic is perhaps the best known of all non-classical logics today. It arose in the early 20th century out of *intuitionism*, a movement in the foundations of mathematics founded by L. E. J. Brouwer.

The specifics of intuitionism are beyond the scope of this talk, but the important thing, for our purposes, is that it is a species of *constructivism*. Hallmarks of constructivism are the assimilation of the notions of truth and proof and the rejection of proofs which rely on non-constructive principles (e.g., excluded middle).

# Constructivism, Informally

The following is a classic example of a non-constructive proof:

**Theorem**

*There are irrational numbers $a$ and $b$ such that $a^b$ is rational.*

**Proof.**

Either $\sqrt{2}^{\sqrt{2}}$ is rational or it is not. If it is, put $a = b = \sqrt{2}$. If it is not, put $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$, which suffices. $\square$

# Constructivism, Informally

What is problematic about this proof, from a constructivist viewpoint, is its use of excluded middle. We are not told which of the two cases obtain—$\sqrt{2}^{\sqrt{2}}$ being rational or not—and it might even be, for all we know, impossible to say which.[1] Relying on excluded middle, the proof doesn't construct witnesses to the theorem, that is, it doesn't say which of two possible pairs are the witnesses.

Intuitionistic logic is supposed to give a formalization of constructively acceptable reasoning.

[1] In this particular case it *is* possible to say which ($\sqrt{2}^{\sqrt{2}}$ is irrational), but that doesn't change the philosophical point.

# The BHK Semantics, Informally

There is an informal semantics for the language of intuitionistic logic, the so-called *BHK semantics*, which brings out the proof-centric approach to truth. Here are the conditions for the propositional fragment:[2]

(i) A proof of $\varphi \wedge \psi$ is a combination of a proof of $\varphi$ and a proof of $\psi$;

(ii) A proof of $\varphi \vee \psi$ is given by giving either a proof of $\varphi$ or a proof of $\psi$;

(iii) A proof of $\varphi \to \psi$ is a construction which, given a proof of $\varphi$, returns a proof of $\psi$;

(iv) There is no proof of $\perp$.

[2] There is some variation in how these conditions are presented; consult, for example, Artemov [1, pp. 1–2] and Fine [7, p. 550].

# The BHK Semantics, Informally

This semantics is, I emphasize, *informal* (I will look at some formal versions shortly). Nevertheless, one can already see intuitively how excluded middle fails. For if neither $\varphi$ nor $\neg\varphi$ ($\neg\varphi := \varphi \to \bot$) has a proof, by the BHK condition for disjunction, $\varphi \vee \neg\varphi$ doesn't either. Thus, it is not generally guaranteed that $\varphi \vee \neg\varphi$ holds.

# The Plan of the Talk

In this talk I will consider different ways in which one might give a formal semantics (i.e., model-theory) for intuitionistic logic, all of which have an affinity to the informal BHK semantics and all of which yield, by philosophically well-motivated and natural modifications, heterodox broadly relevant logics.

# The Semilattice Semantics

The first semantic framework I examine for intuitionistic logic—the semilattice semantics—is technically the simplest. I present and interpret different pieces of the model-theory in succession.

I note that the formal machinery deployed here is largely due to Urquhart [18, 19, 20] (for an overview, consult Standefer [17]). However, his primary interest was not intuitionistic logic, but relevance logic. Furthermore, my own interpretation of the semantics does not align with his interpretation of it (though see Urquhart [22]). A detailed examination of the interpretation and application of this semantics to intuitionistic logic can be found in my [27].

# The Semilattice Semantics: Machinery

## DEFINITION (Semilattice Frame)

A *semilattice frame* is a structure $\mathfrak{F} = \langle S, 0, \sqcup \rangle$ where $\langle S, \sqcup \rangle$ is a join-semilattice and $0 \in S$ is lattice bottom.

The elements of the set $S$ of a semilattice frame are to be thought of as proofs (constructions), 0 is to be thought of as the null proof, and $x \sqcup y$ is to be thought of as the proof resulting from the combination of the proofs $x$ and $y$. Recall that $\sqcup$ satisfies the properties of associativity, commutativity, and idempotence.

# The Semilattice Semantics: Machinery

Does proof combination satisfy such properties? Consider the sort of proof combination exemplified by combining proofs of lemmata or sub-claims into proofs of theorems or propositions.

As a concrete example, consider the claim that if $3 \nmid n$, then $3|n^2 - 1$. The proof of this conditional can be regarded as a combination of the proofs that the conditional holds in each of the three exhaustive cases that (i) $n \mod 3 = 0$, (ii) $n \mod 3 = 1$, and (iii) $n \mod 3 = 2$.[3] The proof of the main claim is naturally regarded as the same regardless of how the proofs of the sub-claims are arranged (associativity and commutativity) or if they are, for whatever reason—say, emphasis—duplicated (idempotence).

[3] In case (i), the conditional holds vacuously.

# The Semilattice Semantics: Machinery

Note that there is *no* completeness requirement imposed on semilattice frames (i.e., no requirement that the combination of any arbitrary collection of proofs be a proof). Such a requirement is arguably anathema to the finitistic nature of the constructions under consideration: there is no reason to think that a combination of infinitely many proofs is itself, in general, a proof.

# The Semilattice Semantics: Machinery

## DEFINITION (J Semilattice Model)

A (propositional) **J** *semilattice model* is a structure $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ where $\mathfrak{F} = \langle S, 0, \sqcup \rangle$ is a frame and $V : \Pi \cup \{\bot\} \to \mathcal{P}(S)$ is subject to the conditions that:

1. $x \in V(p)$ implies $x \sqcup y \in V(p)$;
2. $x \in V(\bot)$ implies $x \sqcup y \in V(\bot)$;
3. $x \in V(\bot)$ implies $x \in V(p)$ (for all $p$).

# The Semilattice Semantics: Machinery

Where $\mathfrak{M} = \langle S, 0, \sqcup, V\rangle$ is a **J** semilattice model and $x \in S$, the relation $\models_x^{\mathfrak{M}}$ is defined as follows:

(a) $\models_x^{\mathfrak{M}} \alpha$ if and only if $x \in V(\alpha)$, for $\alpha \in \Pi \cup \{\perp\}$;

(b) $\models_x^{\mathfrak{M}} \varphi \wedge \psi$ if and only if $\exists y, z \in S$ such that $x = y \sqcup z$, $\models_y^{\mathfrak{M}} \varphi$ and $\models_z^{\mathfrak{M}} \psi$;[4]

(c) $\models_x^{\mathfrak{M}} \varphi \vee \psi$ if and only if $\models_x^{\mathfrak{M}} \varphi$ or $\models_x^{\mathfrak{M}} \psi$;

(d) $\models_x^{\mathfrak{M}} \varphi \to \psi$ if and only if for all $y \in S$, $\not\models_y^{\mathfrak{M}} \varphi$ or $\models_{x\sqcup y}^{\mathfrak{M}} \psi$.

[4]This is the truth condition Urquhart [19, p. 164] assigns not to extensional conjunction ($\wedge$), but rather to *intensional conjunction* or *fusion* ($\circ$). Compare with van Fraassen [9, p. 484].

# The Semilattice Semantics: Machinery

The connections between these truth conditions and the aforementioned BHK conditions should be more-or-less transparent, though the case of $\perp$ is a little delicate (cf. Jankov [12], Veldman [23]). In the BHK spirit, $\models_x^{\mathfrak{M}} \varphi$ is to be read as: $x$ is a proof of (i.e., construction establishing) $\varphi$ (in $\mathfrak{M}$). That this relation is *inexact*, in the sense of Fine [8, p. 558], is indicated by the following:

**LEMMA (Heredity Lemma)**

*For any* $\mathbf{J}$ *semilattice model* $\mathfrak{M} = \langle S, 0, \sqcup, V \rangle$, *any formula* $\varphi$, *and any* $x, y \in S$, $\models_x^{\mathfrak{M}} \varphi$ *implies* $\models_{x \sqcup y}^{\mathfrak{M}} \varphi$.

More on how to "exactify" or "relevantize" the semantics anon.

# The Semilattice Semantics: Machinery

It is natural enough to define validity in terms of what is verified by the null proof (i.e., 0) in all proof spaces (i.e., semilattice frames):

## DEFINITION (Validity)

Where $\mathfrak{M} = \langle S, 0, \sqcup, V \rangle$ is a **J** semilattice model, $\varphi$ is valid in $\mathfrak{M}$ ($\models^{\mathfrak{M}} \varphi$) if $\models_0^{\mathfrak{M}} \varphi$. Where $\mathfrak{F} = \langle S, 0, \sqcup \rangle$ is a semilattice frame, $\varphi$ is valid in $\mathfrak{F}$ ($\models^{\mathfrak{F}} \varphi$) if for all **J** semilattice models $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, $\models^{\mathfrak{M}} \varphi$. Finally, $\varphi$ is *valid* in **J** ($\models_{\mathbf{J}} \varphi$) if for all semilattice frames $\mathfrak{F}$, $\models^{\mathfrak{F}} \varphi$.

# Technical Results

Having laid out the basic semilattice semantic framework, I will canvass some technical results concerning **J** and this semantics. Most of these results are given in detail in my [26, 27]; I remark that the core completeness results piggy-back on arguments from Fine [7] (and, less directly, Kripke [15]).

# Basic Semilattice Completeness

Fix your favorite axiomatization of **J**. We have:

**THEOREM (Semilattice Completeness)**

$\vdash_{\mathbf{J}} \varphi$ *if and only if* $\models_{\mathbf{J}} \varphi$.

# Basic Semilattice FMP

A logic **L** has the *finite model property* (FMP) with respect to a given semantics if $\nvdash_{\mathbf{L}} \varphi$ implies that $\varphi$ fails in some finite model of that semantics for **L**. Then:

**THEOREM (Semilattice FMP)**

**J** *has the finite model property with respect to its semilattice semantics.*

The result is worth noting, because of the negative results of Weiss [25] and, very recently, Knudstorp [14].

# Basic Semilattice FMP

## Remark

As a technical aside, the tree construction used by Fine [7, p. 570] does not generally yield a finite tree model (or, downstream, finite semilattice model) from a finite model, and so does not (together with the usual fmp result for **J**) yield the fmp for **J** with respect to its semilattice semantics; instead, use the tree construction from Kripke [15, p. 110] (Theorem 1, part 2).

# Atomistic Semilattice Completeness

An *atom* in a semilattice with a least element 0 is an element $x$ such that $x$ covers 0 (i.e., $0 < x$ and for any $y$ such that $0 \leq y \leq x$, $y = 0$ or $y = x$). A semilattice frame $\mathfrak{F}$ is *atomic* if every nonzero element contains an atom and *atomistic* if every element is a join of atoms.

**THEOREM (Atomistic Semilattice Completeness)**

$\vdash_{\mathbf{J}} \varphi$ *if and only if* $\varphi$ *is valid in all atomistic* $\mathbf{J}$ *semilattice models.*

This has (what is arguably) a nice constructivist philosophical upshot: $\mathbf{J}$ is characterized by those spaces in which every proof is a combination of certain basic proofs.

# Subscripted Proof-Theory

Elsewhere I've examined labelled or subscripted sequent systems for **J** which are closely related to this semantics (again, largely following pre-existing work concerned with relevance logic; see, e.g., Urquhart [20] and Giambrone and Urquhart [10]).

In the interest of doing something slightly novel (but see Charlwood [3]), here I'll just briefly present *both* subscripted sequent *and* natural deduction systems for the $\rightarrow, \bot$-fragment of **J**.

# Subscripted Natural Deduction

The natural deduction system $\mathcal{N}\mathbf{J}_{\bot,\to}$ is given by the rules ($\alpha \in \Pi \cup \{\bot\}$):[5]

$$(\mathsf{I}{\to})\ \dfrac{\begin{array}{c}[\varphi_{\{k\}}]\\ \mathcal{D}_1\\ \psi_{x\cup\{k\}}\end{array}}{\varphi \to \psi_{x\setminus\{k\}}} \qquad (\mathsf{E}{\to})\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \varphi \to \psi_x\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \varphi_y\end{array}}{\psi_{x\cup y}}$$

$$(\alpha\mathsf{H})\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \alpha_x\end{array}}{\alpha_{x\cup y}} \qquad (\mathsf{E}\bot)\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \bot_x\end{array}}{p_x}$$

[5] $[\varphi_{\{k\}}]$ is the (possibly empty) set of *open assumptions* of the noted form; (I$\to$) has the restriction that every assumption above $\psi_{x\cup\{k\}}$ with the subscript $\{k\}$ is of the form $\varphi_{\{k\}}$.

# Subscripted Natural Deduction

The following generalizations of ($\alpha$H) and (E$\perp$) are *admissible* in $\mathcal{N}\mathbf{J}_{\perp,\to}$:

$$(\varphi\mathrm{H})\ \dfrac{\begin{matrix}\mathcal{D}_1\\ \varphi_x\end{matrix}}{\varphi_{x\cup y}}\qquad (\mathrm{E}\perp')\ \dfrac{\begin{matrix}\mathcal{D}_1\\ \perp_x\end{matrix}}{\varphi_x}$$

## DEFINITION (Deduction)

A *deduction* of $\varphi$ is a tree with no open assumptions constructed according to the rules whose root is $\varphi_\emptyset$. If $\varphi$ has a deduction, $\varphi$ is a *theorem* and we write $\vdash_{\mathcal{N}\mathbf{J}_{\perp,\to}} \varphi$.

# Subscripted Natural Deduction

## Example

$\vdash_{\mathcal{N}\mathbf{J}_{\perp,\rightarrow}} (\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \rightarrow (\varphi \rightarrow \psi)_{\{1\}} \qquad \varphi_{\{2\}}}{\varphi \rightarrow \psi_{\{1,2\}}} \qquad \varphi_{\{2\}}}{\psi_{\{1,2\}}}}{\varphi \rightarrow \psi_{\{1\}}}}{(\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)_{\emptyset}}$$

Note the use of multiple discharge above!

# Subscripted Sequent Calculi

The sequent system $\mathcal{G}\mathbf{J}_{\bot,\to}$ is given by the rules $(\alpha \in \Pi \cup \{\bot\})$:[6]

$$(\mathrm{Ax}) \quad \varphi_x, \Gamma \;\Rightarrow\; \Delta, \varphi_x$$

$$(\mathrm{L}{\to})\ \frac{\varphi \to \psi_x, \Gamma \;\Rightarrow\; \Delta, \varphi_y \qquad \psi_{x\cup y}, \varphi \to \psi_x, \Gamma \;\Rightarrow\; \Delta}{\varphi \to \psi_x, \Gamma \;\Rightarrow\; \Delta}$$

$$(\mathrm{R}{\to})\ \frac{\varphi_{\{k\}}, \Gamma \;\Rightarrow\; \Delta, \varphi \to \psi_x, \psi_{x\cup\{k\}}}{\Gamma \;\Rightarrow\; \Delta, \varphi \to \psi_x}$$

$$(\mathrm{p}\bot)\ \frac{\bot_x, p_x, \Gamma \;\Rightarrow\; \Delta}{\bot_x, \Gamma \;\Rightarrow\; \Delta}$$

$$(\alpha\mathrm{K})\ \frac{\alpha_x, \alpha_{x\cup y}, \Gamma \;\Rightarrow\; \Delta}{\alpha_x, \Gamma \;\Rightarrow\; \Delta}$$

[6] For $(\mathrm{R}{\to})$, $k$ does not occur in the conclusion.

# Subscripted Sequent Calculi

## DEFINITION (Derivation)

A derivation is a tree constructed using the rules whose leaves are axioms and whose root is the sequent being derived. If $\Rightarrow \varphi_{\emptyset}$ has a derivation, $\varphi$ is a *theorem* and we write $\vdash_{\mathcal{G}\mathbf{J}_{\bot,\to}} \varphi$.

## Example

$\vdash_{\mathcal{G}\mathbf{J}_{\bot,\to}} \varphi \to (\psi \to \varphi)$.

$$\dfrac{\dfrac{\dfrac{\varphi_{\{1\}}, \varphi_{\{1,2\}}, \psi_{\{2\}} \;\Rightarrow\; \varphi \to (\psi \to \varphi)_{\emptyset}, (\psi \to \varphi)_{\{1\}}, \varphi_{\{1,2\}}}{\varphi_{\{1\}}, \psi_{\{2\}} \;\Rightarrow\; \varphi \to (\psi \to \varphi)_{\emptyset}, (\psi \to \varphi)_{\{1\}}, \varphi_{\{1,2\}}}}{\varphi_{\{1\}} \;\Rightarrow\; \varphi \to (\psi \to \varphi)_{\emptyset}, (\psi \to \varphi)_{\{1\}}}}{\Rightarrow\; \varphi \to (\psi \to \varphi)_{\emptyset}}$$

# Subscripted Proof-Theory

**THEOREM (Natural Deduction Completeness)**

$\vdash_{\mathcal{N}\mathbf{J}_{\bot,\to}} \varphi$ *if and only if* $\varphi$ *is valid in all semilattice models.*

**THEOREM (Sequent Calculus Completeness)**

$\vdash_{\mathcal{G}\mathbf{J}_{\bot,\to}} \varphi$ *if and only if* $\varphi$ *is valid in all semilattice models.*

# Exact and Inexact Proof

As I noted above, the relation $\models_x^{\mathfrak{M}}$ is inexact in the semantics for **J** as evidenced by its satisfying a sort of heredity condition: if a construction or proof $x$ establishes $\varphi$, so does any construction or proof extending it. But there is a perfectly respectable notion of establishing a result that requires that the construction establishing it be wholly relevant to the result being established.

# Exact and Inexact Proof

Think about the following example: the usual proof of the Fundamental Theorem of Arithmetic (that every positive integer greater than 1 has a unique product of primes factorization) really has two component proofs, one establishing the existence of a unique factorization, and the other establishing its uniqueness. The proof may fairly be regarded as a combination of those two proofs. Is the proof of the theorem also a proof of the existence claim? Of course it *implies* the existence claim, but the proof contains much that is extraneous to establishing that claim. In other words, it's not an exact proof of that result, and is, so to speak, largely irrelevant to it.

# Relevant Neighbors

So here's a perfectly natural idea: take the same frames and truth conditions as given above for **J**, but focus on the exact or relevant notion of proof. That is, throw out the heredity conditions from the definition of the models, but otherwise proceed as above. There are a few variations on this idea, depending on how thoroughgoing you want to be:

**SM** Drop all special constraints on **J** semilattice models (this is sort of a relevant analogue of the system of Johansson [13]);

**SJ**$''$ Retain only the constraint that $x \in V(\bot)$ implies $x \in V(p)$ (for all $p$);

**SJ**$'$ In addition to the constraint of **SJ**$''$, require heredity only for $\bot$ (cf. Weiss [27, §5]).

# Relevant Neighbors

All of these systems are, in a natural semantic sense, relevant (though becoming less 'strictly relevant' as the list goes on); in fact, all of them satisfy the *Variable Sharing Property* ($\models_{\mathbf{L}} \varphi \rightarrow \psi \Rightarrow \Pi(\varphi) \cap \Pi(\psi) \neq \emptyset$) in their positive fragments (modify the argument in Weiss [24]).

They can also all lay claim to embodying some natural constructivist motivations; indeed, by an easy extension of the techniques of West and Weiss [29], all can be proved to satisfy the *Disjunction Property* ($\models_{\mathbf{L}} \varphi \vee \psi \Rightarrow \models_{\mathbf{L}} \varphi$ or $\models_{\mathbf{L}} \psi$).

# Relevant Neighbors

The aforementioned systems (**SM**, **SJ**$''$, **SJ**$'$) are quite heterodox in many respects. In none of them does conjunction simplify ($\not\models_{\mathbf{L}} \varphi \wedge \psi \to \varphi$).[7] **SJ**$''$ is not even closed under substitution. But for all that, these systems are, I think, philosophically well-motivated and interesting.

[7] Basically, this is because $\wedge$ (extensional conjunction) becomes $\circ$ (fusion) once heredity is dropped.

# Truthmaker Semantics

The next semantic framework for **J** that I will examine is the truthmaker semantics developed by Fine [7]. It will readily be apparent that this has much in common with the semilattice semantics although the latter is, in a technical sense, more general.

# Truthmaker Semantics: Machinery

The basic formal apparatus of truthmaker semantics is the state space. At least for most applications, this is taken to be a *complete* join-semilattice [8, p. 560]. For characterizing intuitionistic logic, however, Fine requires an additional property. I will follow Fine [7, p. 565] in describing the relevant kind of state spaces as *exact frames*.

# Truthmaker Semantics: Machinery

## DEFINITION (Exact Frame)

An *exact frame* (state space) is a structure $\mathfrak{F} = \langle S, \sqsubseteq \rangle$ where $\langle S, \sqsubseteq \rangle$ is a complete, residuated lattice, that is, a lattice satisfying the following conditions (for $x, y, z \in S$):

**COMP** For any $T \subseteq S$, $\bigsqcup T \in S$

**RESID** $y \sqsubseteq x \sqcup (x \hookrightarrow y)$, where $x \hookrightarrow y := \bigsqcap \{z : y \sqsubseteq x \sqcup z\}$

A note on $\hookrightarrow$: for any states $x$ and $y$, $x \hookrightarrow y$ is the state of $x$'s leading to $y$ and represents what Fine calls a "conditional connection", the presence of which indicates that $y$ will be present if $x$ is added in [7, pp. 554–555].

# Truthmaker Semantics: Machinery

So much for frames—what are the *models*? An exact model is just like a **J** semilattice model except, predictably, we drop the heredity requirements:[8]

## DEFINITION (Exact Model)

An *exact model* is a structure $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ where $\mathfrak{F} = \langle S, \sqsubseteq \rangle$ is an exact frame and $V : \Pi \cup \{\perp\} \to \mathcal{P}(S)$ satisfying the requirement that $x \in V(\perp)$ implies $x \in V(p)$ for all $p \in \Pi$.

[8] I am here imposing Fine's "Strict Falsum Condition" rather than his "Falsum Condition" (consult Fine [7, p. 565]) but it makes no difference to validity.

# Truthmaker Semantics: Machinery

The truth conditions for exact models are the same as for **J** semi-lattice models except for the condition for $\rightarrow$:

(d′) $\models_x^{\mathfrak{M}} \varphi \rightarrow \psi$ if and only if there is a function $f : [\varphi]^{\mathfrak{M}} \rightarrow [\psi]^{\mathfrak{M}}$ such that $x = \bigsqcup\{y \hookrightarrow f(y) : \models_y^{\mathfrak{M}} \varphi\}$.

Thus, a verifier of a conditional $\varphi \rightarrow \psi$ is a fusion of conditional connections connecting verifiers of the antecedent to verifiers of the consequent. As Fine puts it, "a verifier for the conditional will tell us how to pass from any verifier of the antecedent to a verifier of the consequent" [7, p. 555]. Such a state $x$ is naturally taken to encode the function $f$ [7, p. 558].

# Truthmaker Semantics: Machinery

In Fine's semantics, $\models_x^{\mathfrak{M}} \varphi$ is taken to represent $x$'s being an exact truthmaker of $\varphi$. However, there is a corresponding inexact relation: $\blacktriangleright_x^{\mathfrak{M}} \varphi$ if and only if for some $y \sqsubseteq x$, $\models_y^{\mathfrak{M}} \varphi$ [7, p. 566]. And it is in terms of *this* relation that Fine defines the validity relation that characterizes **J**.[9]

[9] Actually, Fine [7, p. 569] defines several different consequence relations, all using the inexact relation, but his $\models_{i3}$ coincides with $\models$ as defined for the semilattice semantics.

# Exact Frames: Another Characterization

Fine's own characterization of exact frames uses the aforementioned residuation condition, which while philosophically perspicuous is perhaps technically less so. The following result yields another characterization:[10]

**PROPOSITION**

*Let $\langle S, \sqsubseteq \rangle$ be a complete lattice. Then $\langle S, \sqsubseteq \rangle$ is residuated if and only if $\langle S, \sqsubseteq \rangle$ satisfies the infinite meet-distributive law, that is,* $x \sqcup \prod_i y_i = \prod_i (x \sqcup y_i)$.

[10] Put in yet another way: exact frames are complete co-Heyting algebras [2, p. 423].

# Exact Frames: An Example

The foregoing characterization helps bring out that examples of exact frames are actually rather easy to come by.

**Example**

The structure $\langle \mathrm{sqf}(\mathbb{N})_0, | \rangle$ consisting of the squarefree non-negative integers (including 0) ordered by divisibility is a complete infinite meet-distributive lattice, and so an exact frame.

Note that in $\langle \mathrm{sqf}(\mathbb{N})_0, | \rangle$, $j \hookrightarrow k = \frac{k}{\gcd(j,k)}$ (stipulating that anything divided by 0 is 1).

# Model Equivalence

**THEOREM**

*Let $\mathfrak{F}$ be an exact frame. For any $\mathbf{J}$ semilattice model $\mathfrak{M}$ over $\mathfrak{F}$ there is an equivalent exact model $\mathfrak{M}'$ over $\mathfrak{F}$ and conversely.*

# A Relevant Neighbor?

In the context of the semilattice semantics, I suggested that the move to relevance amounted to focusing on the exact notion of verification instead of the inexact notion of verification, but keeping the definition of validity the same. In the context of truthmaker semantics, Fine [7, p. 556] has already proposed an exact notion of consequence, which similarly makes use of the exact notion of verification, but also tweaks the notion of validity: $\models_e \varphi$ if $\models_x^{\mathfrak{M}} \varphi$ for every state $x$ and every exact model $\mathfrak{M}$.

# A Relevant Neighbor?

Does $\models_e$ yield a relevance logic?

In a fairly natural, semantically motivated sense, I think the answer is clearly 'yes'. However, it seems to be very hard to get a clear picture of what this logic is like. I reckon it has no valid formulae (e.g., $\varphi \to \varphi$ and $\bot \to \varphi$ are invalid) though it does validate at least some inferences (e.g., $\frac{\varphi \to \psi}{\varphi \to \psi \vee \theta}$). Assuming there are indeed no valid formulae, the logic does satisfy the Variable Sharing Property albeit in an uninteresting and vacuous way.

# Informal Motivation

The semantic focus heretofore has been on constructions and proofs which may be taken to establish statements. In this last portion of the talk, I want to take a somewhat wider perspective on verifiers and combination, and simultaneously move (if only just a little) beyond the BHK conditions as articulated above.

# Informal Motivation: Disjunction in Focus

To help motivate this move, it is instructive to scrutinize the truth condition for disjunction given above a bit more closely:

(c) $\models_x^{\mathfrak{M}} \varphi \vee \psi$ if and only if $\models_x^{\mathfrak{M}} \varphi$ or $\models_x^{\mathfrak{M}} \psi$.

The condition is fine paired with a suitable notion of constructions or proofs, but if we shift our focus to what is verified by pieces of information (à la Urquhart [19]) or what theories commit us to (à la Fine [6]), it begins to look more suspect.

# Informal Motivation: Disjunction in Focus

After all, it's a seemingly mundane fact that a piece of information, or a theory, can support a disjunction without thereby supporting, or determining, either disjunct (cf. Copeland [4, p. 408] and Humberstone [11, §3]).

This suggests that we should perhaps countenance another form of combination, on which a verifier of a disjunction can be regarded as something which, roughly speaking, pares down and fuses verifiers of the disjuncts into something which *merely* suffices to establish that one or the other disjunct holds.

# Bi-operational Machinery

The machinery I sketch below is mostly adapted from my [28], which in turn is directly inspired by Humberstone [11]. However, many of these ideas seem to have recurred, often independently, in a number of different places (e.g., Fine [6], Došen [5], Punčochář [16]).

# Bi-operational Machinery

**DEFINITION (Bi-operational Intuitionistic Frame)**

A *bi-operational intuitionistic frame* is just a bounded distributive lattice $\mathfrak{F} = \langle S, 0, 1, \sqcup, \sqcap \rangle$.

**DEFINITION (Bi-operational Intuitionistic Model)**

A *bi-operational intuitionistic model* is a structure $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ where $\mathfrak{F}$ is a bi-operational frame and $V : \Pi \rightarrow \mathcal{F}(\mathfrak{F})$, where $\mathcal{F}(\mathfrak{F})$ is the set of all filters over $\mathfrak{F}$.

# Bi-operational Machinery

Here's how I interpret this machinery (cf. Fine [6]). Given a frame $\mathfrak{F} = \langle S, 0, 1, \sqcup, \sqcap \rangle$, we think of the elements of $S$ as theories, 0 and 1 as special distinguished theories (the base and trivial theory, respectively), and $\sqcup$ and $\sqcap$ as certain ways of "actively" and "passively" combining theories, respectively.

Roughly, the idea is that we can think of $x, y \in S$ as collections of inert propositions, and combine them passively to get the collection of propositions *in* both $(x \sqcap y)$; or we can think of them as collections of applicable propositions, and combine them actively to get the theory *generated* by both $(x \sqcup y)$.

# Bi-operational Semantics

The truth conditions are those used with the semilattice semantics, except for $\bot$, $\wedge$, and $\vee$:[11]

(a′) $\models_x^{\mathfrak{M}} \bot$ if and only if $x = 1$;

(b′) $\models_x^{\mathfrak{M}} \varphi \wedge \psi$ if and only if $\models_x^{\mathfrak{M}} \varphi$ and $\models_x^{\mathfrak{M}} \psi$;

(c′) $\models_x^{\mathfrak{M}} \varphi \vee \psi$ if and only if $\exists y, z \in S$ such that $y \sqcap z = x$, $\models_y^{\mathfrak{M}} \varphi$ and $\models_z^{\mathfrak{M}} \psi$.

Validity is again determined by what holds at 0, and it can be shown that every formula expresses a proposition (a filter).

[11] Observe that here we've taken the extensional truth condition for $\wedge$. This is partly for simplicity, partly because we're moving beyond the BHK semantics, and partly because the appropriate formulation of the intensional condition in this setting is less transparently related to the corresponding BHK condition anyway.

# Completeness

The following result is easy enough to prove:

**THEOREM (Bi-operational Completeness)**

$\vdash_{\mathbf{J}} \varphi$ *if and only if* $\models \varphi$.

# A Relevant Neighbor

Now, what might "going relevant" come to in this context? All of the properties of bounded distributive lattices sound alright given the foregoing gloss of the semantics except, I want to suggest, the requirement that $0 \sqcap x = 0$.

Thinking of 0 as the theory of logic itself (as indeed it is, in the canonical model construction for the foregoing result), you would only think that $0 \sqcap x = 0$, that is, $0 \sqsubseteq_{\sqcap} x$, held invariably if you thought that *every* theory was about logic. But this is not a very relevant line of thought. Theories can be logically well-structured without being about logic—without even commenting on logic at all.

# A Relevant Neighbor

My preferred approach to rejecting $0 \sqsubseteq_{\sqcap} x$ detours through disentangling the $\sqcap$- and $\sqcup$-orders, and move from the setting of *lattices* to the more general setting of *bisemilattices*.

A conservative approach recommends keeping as much as possible from the semantics for **J** while still honoring the central relevant intuition.

# Bi-operational Machinery

## DEFINITION (Bisemilattice)

A *bisemilattice* is a structure $\langle S, \sqcup, \sqcap \rangle$ consisting of two cohabitating semilattices.

A bisemilattice $\langle S, \sqcup, \sqcap \rangle$ is said to be *join-distributive* if its operations satisfy the equation $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$. A bisemilattice $\langle S, \sqcup, \sqcap \rangle$ is called *meet-decomposable* if the underlying meet-semilattice satisfies the property that $x \sqcap y \sqsubseteq_{\sqcap} z$ implies $\exists x', y' \in S$ such that $x \sqsubseteq_{\sqcap} x'$, $y \sqsubseteq_{\sqcap} y'$, and $z = x' \sqcap y'$.

# Bi-operational Machinery

A *bounded bisemilattice* is a structure $\langle S, 0, 1, \sqcup, \sqcap \rangle$ where $\langle S, 1, \sqcap \rangle$ is a meet-semilattice with a greatest element and $\langle S, 0, \sqcup \rangle$ is a join-semilattice with a least element.

Observe that what is greatest (least) in one order need not be greatest (least) in the other. A bounded bisemilattice in which $x \sqcup 1 = 1$ ($x \sqcap 0 = 0$) holds will be called *top respecting* (*bottom respecting*).

# Bi-operational Machinery

**DEFINITION (Bi-operational Mingle Frame)**

A *bi-operational mingle frame* $\mathfrak{F} = \langle S, 0, 1, \sqcup, \sqcap \rangle$ is a bounded, top respecting, join-distributive, meet-decomposable bisemilattice.

Now then, an intuitionistic frame turns out just to be a special case of a bi-operational mingle frame in which the absorption laws hold, or (as it turns out, equivalently) which is bottom respecting.

# The Logic **RM0**

The logic **RM0** is (speaking rather inexactly, but intuitively) like **J**, but with the weakening schema (K), $\varphi \to (\psi \to \varphi)$, weakened to the mingle schema (M), $\varphi \to (\varphi \to \varphi)$. (Do not confuse **RM0** with its better-known cousin **RM**!)

**RM0** is the logic of bi-operational mingle frames using the same truth conditions and account of validity as given above:

**THEOREM (Bi-operational Completeness)**

$\vdash_{\mathbf{RM0}} \varphi$ *if and only if* $\models \varphi$.

# The Logic **RM0**

**RM0** is an interesting logic, and it is considerably better behaved than some of the systems I mentioned earlier. It has a tidy axiomatization, it satisfies the Variable Sharing Property in its positive fragment, and it is easy to embed **J** into its extension with the Ackermann truth constant $t$ [28].

# The Logic **RM0**

Since **RM0** does not appear to fall under the general undecidability results proved by Urquhart [21] or Knudstorp [14], the problem of its (un)decidability remains open.

**RM0** is algebraized by pleasant structures that I have elsewhere called *Dunn semilattices* [28], and Francesco Paoli and I previously tried to show the *finite embeddability property* for this class, from which the finite model property (and decidability) would follow. The argument, modeled on the analogous result for Heyting algebras, was subtly fallacious. Perhaps the argument can be repaired, or else be made to work for certain interesting fragments of the logic.

# Concluding Remarks

So, in summary, I hope to have conveyed that there are many semantically natural ways of getting to relevance logic from intuitionistic logic, that very little is known about many of these constructively flavored relevance logics despite their naturalness and proximity to intuitionistic logic, and that it would be profitable to do much more work in this area.

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