Assignment One

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Direction:

Please answer all the questions below and hand in your answers before the due day. All work, must be handed in **on time**.

Due day:

April. 12, 2021

Please hand it in by the class time.

Questions:

1. For each of the following functions, indicate how much the function's value will change if its argument is increased fourfold.

   a. $log_2 n$ b. $\sqrt{n}$ c. n d. n^2 e. n^3 f. 2^n

解：a. $\log_2 4n - \log_2 n = \log_2 4 = 2$

b、$\frac{\sqrt{4n}}{\sqrt{n}} = 2$

c、$\frac{4n}{n} = 4$

d、$\frac{16n^2}{n^2} = 16$

e、$\frac{64n^3}{n^3} = 64$

f、$\frac{2^{4n}}{2^n} = (2^n)^3$

2. Prove (by using the definitions of the notations involved) or disprove (by giving a specific counterexample) the following assertions.

   a. If $t(n) \in O(g(n))$, then $g(n) \in \Omega(t(n))$.

   b. $\Theta(\alpha g(n)) = \Theta(g(n))$, where $\alpha > 0$.

   c. $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$.

   d. For any two nonnegative functions $t(n)$ and $g(n)$ defined on the set of nonnegative integers, either $t(n) \in O(g(n))$, or $t(n) \in \Omega(g(n))$, or both.

解：a、正确 证：由 $t(n) \in O(g(n))$ 知 $t(n) \leq cg(n)$ $n \geq n_0$ $\exists C > 0$

$\Rightarrow \frac{1}{C} t(n) \leq g(n)$ 对 $n \geq n_0$ $\therefore g(n) \in \Omega(t(n))$

b、正确 证：设 $f(n) \in \Theta(\alpha g(n))$ 下证 $f(n) \in \Theta(g(n))$

$b\alpha g(n) \leq f(n) \leq C\alpha g(n)$ 令 $C_1 = \alpha C$ 得 $b_1 g(n) \leq f(n) \leq C_1 g(n)$ 对 $n > n_0$

$b_1 = \alpha b$

$\therefore f(n) \in \Theta(g(n))$

设 $f(n) \in \Theta(g(n))$ 下证 $f(n) \in \Theta(\alpha g(n))$

对 $n \geq n_0$ $\exists b, C$ $b g(n) \leq f(n) \leq C g(n)$ 使 $\frac{b}{\alpha}\alpha g(n) \leq f(n) \leq \frac{C}{\alpha}\alpha g(n)$ 令 $C_1 = \frac{C}{\alpha}$

$b_1 = \frac{b}{\alpha}$

$\therefore$ 对 $n > n_0$ $\exists \begin{matrix} C_1 > 0 \\ b_1 > 0 \end{matrix}$ $b_1 g(n) \leq f(n) \leq C_1 \alpha g(n)$ $f(n) \in \Theta(\alpha g(n))$

综上所述 $\Theta(\alpha g(n)) = \Theta(g(n))$

c、正确 证：易证 $f(n) \in \Theta(g(n))$ 时 $f(n) \in O(g(n)) \cap \Omega(g(n))$

下证 $f(n) \in O(g(n)) \cap \Omega(g(n))$ 时 $f(n) \in \Theta(g(n))$

证：$f(n) \in O(g(n)) \cap \Omega(g(n)) \Rightarrow n > n_0$ $\exists b, C$ 使 $b(g(n)) \leq f(n) \leq C(g(n))$

由 Θ 定义得 $f(n) \in \Theta(g(n))$ $\therefore \Theta(g(n)) \subseteq O(g(n)) \cap \Omega(g(n))$

$\therefore \Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$ $O(g(n)) \cap \Omega(g(n)) \subseteq \Theta(g(n))$

d、不正确 $t(n) = \cos n + 1$ $g(n) = \sin n + 1$

3. Solve the following recurrence relations.

a. $x(n) = 3x(n-1)$ for $n > 1$, $x(1) = 4$

b. $x(n) = x(n-1) + n$ for $n > 0$, $x(0) = 0$

c. $x(n) = x(n/2) + n$ for $n > 1$, $x(1) = 1$ (solve for $n = 2^k$)

解：a、$x(n) = 3x(n-1) = 3^2 x(n-1) = \cdots = 3^{n-1} x(1) = 4 \cdot 3^{n-1}$

b、$n=1$ 时 $x(1)=1$　　$n=2$ 时 $x(2)=3$

猜测 $x(n) = \frac{n(n+1)}{2}$　下用数学归纳法证明

$n=0,1,2$ 时成立　设对 k 成立 $x(k) = \frac{k(k+1)}{2}$.

$x(k+1) = x(k) + k+1 = (\frac{k}{2}+1)(k+1) = \frac{(k+1)[(k+1)+1]}{2}$

对 $k+1$ 也成立　∴ $x(n) = \frac{n(n+1)}{2}$ 成立.

c、$x(2^k) = x(2^{k-1}) + 2^k$

$= x(2^{k-2}) + 2^k + 2^{k-1}$

$= \cdots = 2^k + 2^{k-1} + \cdots + 2^1 + x(2^{k-k})$

$= 2^{k+1} - 1 = 2 \cdot 2^k - 1 = 2n - 1$