Relational Colour Refinement for Non-Relational Signatures

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Introduction

- Colour Refinement is an important and interesting algorithm
- It is applied in modern isomorphism solvers
- It can be characterised logically and combinatorially
- Extension to more than graphs seems desirable
- Recently, Scheidt and Schweikardt bibliography introduced Relational Colour Refinement
- Conceptually similar to classical Colour Refinement
- Also has a logical and a combinatorial characterisation

Contents of this presentation

Classical Colour Refinement

Colour Refinement

- Also called CR or 1-dimensional Weisfeiler-Leman algorithm
- Iterative graph algorithm
- Constructs colour for every vertex, based on colours of neighbours

Definition (Colour Refinement)

For graph $G = (V, E)$, for every $v \in V$ and $i \in \mathbb{N}$:

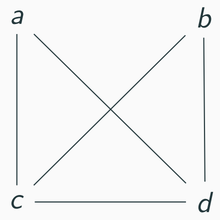
$$C_0(v) := 0$$

and

$$C_{i+1}(v) := (C_i(v), \{\!\{ C_i(u) : \{v, u\} \in E \}\!\}).$$

Example for CR

G:



- $C_0(a) = C_0(b) = C_0(c) = C_0(d) = 0$

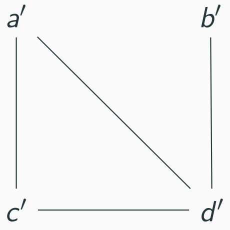
- $C_1(a) = (C_0(a), \{\!\{0, 0\}\!\}) = C_1(b)$
- $C_1(c) = (C_0(c), \{\!\{0, 0, 0\}\!\}) = C_1(d)$

- $C_2(a) = (C_1(a), \{\!\{C_1(c), C_1(c)\}\!\}) = C_2(b)$
- $C_2(c) = (C_1(c), \{\!\{C_1(a), C_1(a), C_1(c)\}\!\}) = C_2(d)$

Distinguished graphs

- CR distinguishes two graphs G and H, if
- there exists $C_i(v)$ in colouring of G or H, such that number of vertices with colour $C_i(v)$ is different in G than in H

H:



- Colours in first round equal
- $C_1(b') = (C_0(b'), \{\!\{ C_0(d') \}\!\}) = (0, \{\!\{0\}\!\})$ does not appear in G

$\Rightarrow$ Colour Refinement distinguishes G and H.

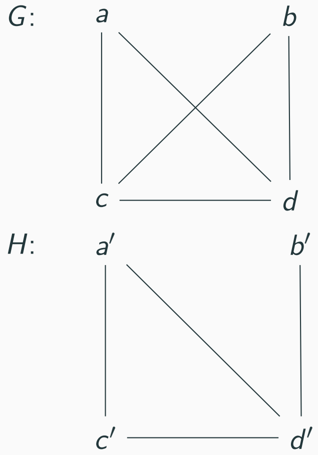
Characterisations of CR

- There are equivalent characterisations for CR
- Due to bibliography:
  CR distinguishes G and H if, and only if, there exists $\varphi \in \mathsf{C}_2$, such that $G \models \varphi$ and $H \not\models \varphi$
- Due to bibliography:
  CR distinguishes G and H if, and only if, there exists tree T, such that $\hom(T, G) \neq \hom(T, H)$

Examples for G and H:

- $\varphi := \exists^{\geq 1} x. \neg \exists^{\geq 2} y. E(x, y)$
- $T := (\{v, u\}, \{\{v, u\}\})$

Application of Characterisations to Example



- Used existence of colour $(0, \{\!\{0\}\!\})$ in colouring of H to distinguish G and H
- From colour it follows that vertex with degree 1 exists
- $\exists^{\geq 1} x \,.\, \neg \exists^{\geq 2} y \,.\, E(x, y)$ distinguishes G and H

- There are 5 edges in G but only 4 in H
- Tree $T := (\{v, u\}, \{\{v, u\}\})$ has 10 homomorphisms to G and 8 to H

Relational Colour Refinement

Relational Colour Refinement

- Called RCR for short
- Introduced by Scheidt and Schweikardt bibliography
- Applies variant of classical Colour Refinement on tuples of structure
- Uses atomic type (set of relations that contain tuple) as part of initial colouring
- Uses pairs of indices as edges to define shared elements of tuples
- Formally:

$$\mathrm{atp}(\mathbf{a}) = \{R \in \sigma : \mathbf{a} \in R\}$$

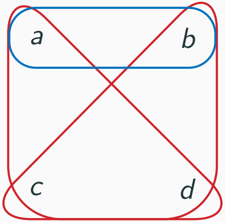
and

$$\mathrm{stp}(\mathbf{a}, \mathbf{b}) = \{(i, j) \in [n] \times [m] : a_i = b_j\}$$

The Algorithm

- For relational structure $\mathfrak{A}$ and all tuples $\mathbf{a} \in \mathbf{A}$:
- Initial colour: $\varrho_0(\mathbf{a}) = (\mathrm{atp}(\mathbf{a}), \mathrm{stp}(\mathbf{a}, \mathbf{a}))$
- For the next rounds: $\varrho_{i+1}(\mathbf{a}) = (\varrho_i(\mathbf{a}), \{\{(\mathrm{stp}(\mathbf{a}, \mathbf{b}), \varrho_i(\mathbf{b})) : \mathrm{stp}(\mathbf{a}, \mathbf{b}) \neq \emptyset\}\})$

An Example for RCR



- Structure $\mathfrak{A} = (A, R^{\mathfrak{A}}, T^{\mathfrak{A}})$
- $A = \{a, b, c, d\},\ R^{\mathfrak{A}} = \{(a, b)\},\ T^{\mathfrak{A}} = \{(a, c, d), (b, c, d)\}$

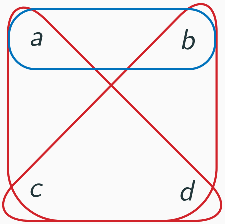
- $\varrho_0((a, b)) = (\{R\}, \{(1, 1), (2, 2)\})$ and
  $\varrho_0((a, c, d)) = \varrho_0((b, c, d)) = (\{T\}, \{(1, 1), (2, 2), (3, 3)\})$
- $$\varrho_1((a, c, d)) = (\varrho_0((a, c, d)), \{\!\{(\{(1, 1)\}, \varrho_0((a, b))), \ldots\}\!\})$$

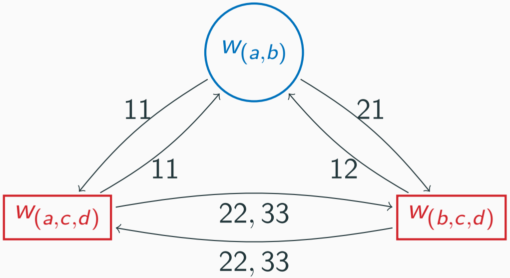
  and

  $$\varrho_1((b, c, d)) = (\varrho_0((b, c, d)), \{\!\{(\{(1, 2)\}, \varrho_0((a, b))), \ldots\}\!\})$$

Equivalent formulation of RCR

- RCR can be equivalently defined as colour refinement on coloured multigraphs (graph with vertex and edge colouring)
- Create vertex for every tuple
- Colour vertices using atomic type
- Define edge relation for every pair of indices
    - Connect vertices if elements of index-pair are the same



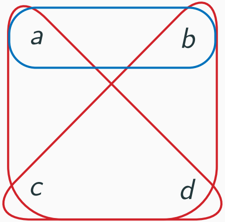


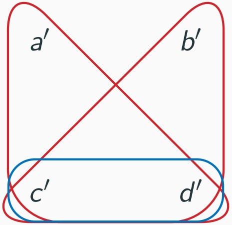
Colour Refinement for Coloured Multigraphs

- Simple variant of classical CR
- Use vertex colouring in initial colour
- Instead of only considering colours of neighbours, consider the colour together with the colours of edges connecting them
- This on encoding of relational structure is equivalent to RCR

Distinguishing Relational Structures with RCR

- RCR distinguishes, if some colour appears differently often in the structures





- $\gamma_1((a, c, d)) = ((\{T\}, \{\dots\}), \{\!\{(\{1,1\}, (\{R\}, \{\dots\})), \dots\}\!\})$ appears in colouring of left structure but not in right
  - There is no triple in T where its first element is in a tuple in R

Relational Colour Refinement

Logical Characterisation of RCR

Guarded Fragment of Counting Logic

- We have seen how C_2 characterises CR on graphs
- Analogously: Guarded fragment of counting logic GF(C) characterises RCR
- Guarded fragment drops bound on number of variables, but introduces restriction that quantifiers need to be relativised by atomic formula

Guarded Fragment of Counting Logic

- Everything except for quantifies defined as in classical counting logic
- For atomic formula $\Delta \in \mathrm{GF(C)}$ and formula $\varphi \in \mathrm{GF(C)}$, we call Δ a guard for φ, if $\mathrm{free}(\Delta) \supseteq \mathrm{free}(\varphi)$
- Quantifiers appear only in form $\exists^{\geq i}\mathbf{v}\,.(\Delta \wedge \varphi)$, where Δ is guard for φ and $\mathrm{set}(\mathbf{v}) \subseteq \mathrm{free}(\Delta)$

- Examples: $\exists^{\geq 2}(x, y)\,.\,E(x, y) \wedge T(y) \in \mathrm{GF(C)}$, but $\exists^{\geq 3}(x, y, z)\,.\,E(x, y) \wedge E(y, z) \wedge E(z, x) \notin \mathrm{GF(C)}$

Characterising RCR Using Logic

Theorem B from bibliography

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two relational structures. Then the two following statements are equivalent.

1. RCR distinguishes $\mathfrak{A}$ and $\mathfrak{B}$
2. There exists a sentence in GF(C) that is satisfied by $\mathfrak{A}$, but not by $\mathfrak{B}$

Example for Logical Characterisation of RCR

- We used existence of $\gamma_1((a,c,d)) = ((\{T\},\{\dots\}), \{\!\{(\{1,1\},(\{R\},\{\dots\})),\dots\}\!\})$ in left structure to distinguish them
- Formula $\exists^{\geq 1}(x,y,z).\,\big(T(x,y,z) \wedge \exists^{\geq 1}(y).\,(R(x,y))\big)$ satisfied by left and not by right structure

Relational Colour Refinement

Combinatorial Characterisation of RCR

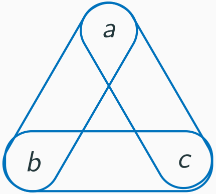
Acyclic Structures

- Counting homomorphisms from trees characterises CR on graphs
- Abstraction from trees to relational structures is needed: α-acyclic structures (in the following only acyclic structures)

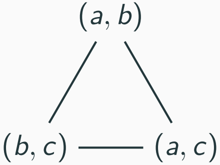
Acyclic Structures

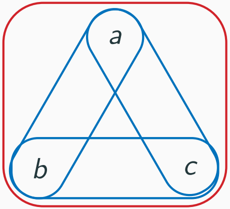
- Let $\mathfrak{C}$ be relational structure
- Join tree J for $\mathfrak{C}$ is tree with $V(J) = \mathbf{C}$ and fulfils join-tree-property:
  - For every $v \in C$, the set $\{\mathbf{x} \in V(J) : v \in \mathbf{x}\}$ induces a connected subtree
- We call $\mathfrak{C}$ acyclic, if it has a join tree

Examples for Acyclic Structures

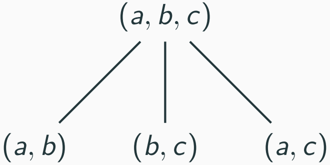


No:





Yes:

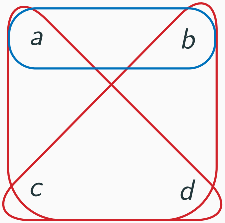


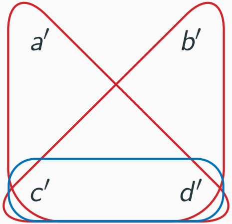
Theorem A from bibliography

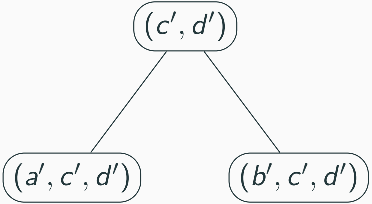
Let $\mathfrak{A}$ and $\mathfrak{B}$ be relational structures. Then the two following statements are equivalent.

1. RCR distinguishes $\mathfrak{A}$ and $\mathfrak{B}$
2. There exists an acyclic relational structure $\mathfrak{C}$, such that it has a different number of homomorphisms to $\mathfrak{A}$ than to $\mathfrak{B}$

Example for Combinatorial Characterisation of RCR







- Right tree is join tree for middle structure, therefore middle structure is acyclic
- Identity is homomorphism, so middle structure has at least one homomorphism to itself
- Middle structure has no homomorphisms to left structure

Relational Colour Refinement for Structures With Functions

Relational Colour Refinement for Structures With Functions

- Many interesting structures use functions
- Colour Refinement algorithm for such structures seems desirable
- Will use the results of Scheidt and Schweikardt bibliography and investigate how robust they are
- Following structure:
  1. Presentation of two approaches for Colour Refinement for non-relational signatures
  2. Logical characterisation of both approaches
  3. Discussion on combinatorial characterisation

Naive RCR

- Goal: Encode non-relational structures and signatures as relational ones
- Functions can directly be interpreted as relations:

$$(\mathbf{x}y) \in R_f \iff f(\mathbf{x}) = y$$

- For non-relational signature σ define relational signature σ':
    - Relation symbol $R \in \sigma$ of arity $n \to$ introduce $R \in \sigma'$ of arity n
    - Function symbol $f \in \sigma$ of arity $n \to$ introduce $R_f \in \sigma'$ of arity $n+1$
- Encode σ-structure $\mathfrak{A}$ as σ'-structure $\mathfrak{A}'$:
    - For relation symbol $R \in \sigma$: $R^{\mathfrak{A}'} := R^{\mathfrak{A}}$
    - For function symbol $f \in \sigma$: $R_f^{\mathfrak{A}'} := \{(\mathbf{x}y) : f^{\mathfrak{A}}(\mathbf{x}) = y\}$

Idea of the Transitive Expansion

- Approach is only defined for unary function symbols
- Encoding emulates the nesting of function applications
- Encode function f as family of relations $R_{f^1}, R_{f^2}, \ldots$, where $(x, y) \in R_{f^i}$, if $f^i(x) = y$
- For multiple functions, also encode alternations, for example $R_{f\,g}$ or $R_{g^2 f^3}$

Transitive Expansion i

Alternations of Function Applications

- Let σ be signature with unary function symbols
- Define set of all allowed function application alternations Alters_n^k as $\mathrm{Alters}_n^0(\sigma) = \{\mathrm{id}\}$ and

$$\begin{aligned}\mathrm{Alters}_n^k(\sigma) := \mathrm{Alters}_n^{k-1}(\sigma) \cup \{ f_1^{m_1} f_2^{m_2} \dots f_k^{m_k} : & f_1, f_2, \dots, f_k \in \sigma_{\mathrm{Func}} \\ & \wedge \forall i \in [k] \,.\, m_i \in [n] \\ & \wedge \forall i \in [k-1] \,.\, f_i \neq f_{i+1} \}.\end{aligned}$$

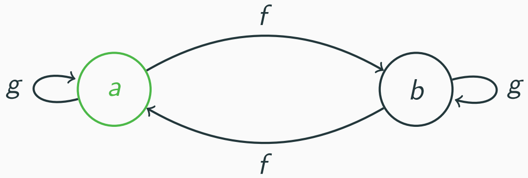
- Example:
  - $\sigma = \{R/1, f/1, g/1\}$
  - $\mathrm{Alters}_2^1(\sigma) = \{\mathrm{id}, f, f^2, g, g^2\}$

Transitive Expansion

- For alternation depth k and σ-structure $\mathfrak{A}$ with $|\mathfrak{A}| = n$ define transitive expansion $\widetilde{\mathfrak{A}}$ as a $\widetilde{\sigma}$-structure
- For $\alpha, \beta, \alpha_1, \dots, \alpha_\ell \in \mathsf{Alters}_n^k(\sigma)$ and relation symbol $R \in \sigma$ of arity ℓ, insert relation symbol $\mathsf{Eq}_{\alpha,\beta}$ of arity 2 and relation symbol $R_{\alpha_1,\dots,\alpha_\ell}$ of arity ℓ to $\widetilde{\sigma}$
- Define $\mathsf{Eq}_{\alpha,\beta}^{\widetilde{\mathfrak{A}}} := \{(x, y) : \alpha^{\mathfrak{A}}(x) = \beta^{\mathfrak{A}}(y)\}$ and $R_{\alpha_1,\dots,\alpha_\ell}^{\widetilde{\mathfrak{A}}} := \{(x_1, \dots, x_\ell) : (\alpha_1^{\mathfrak{A}}(x_1), \dots, \alpha_\ell^{\mathfrak{A}}(x_\ell)) \in R^{\mathfrak{A}}\}$

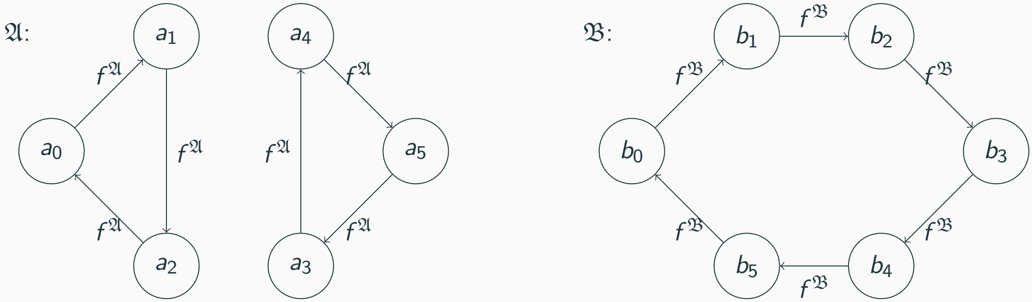
- For $k \in \mathbb{N}$ we say that RCR_k distinguishes structures $\mathfrak{A}$ and $\mathfrak{B}$, if RCR distinguishes the transitive expansions with alternation depth k

Example for the Transitive Expansion



- Structure $\mathfrak{A} = (A, R^{\mathfrak{A}}, f^{\mathfrak{A}}, g^{\mathfrak{A}})$
- $k = 1$ and $n = 2$: $\text{Alters}_2^1(\sigma) = \{\text{id}, f, f^2, g, g^2\}$
- $\widetilde{\sigma} = \{R_{\text{id}}, R_f, R_{f^2}, R_g, R_{g^2}, \text{Eq}_{\text{id},\text{id}}, \text{Eq}_{\text{id},f}, \text{Eq}_{\text{id},f^2}, \ldots, \text{Eq}_{g^2,g^2}\}$
- Examples:
    - $R_f^{\widetilde{\mathfrak{A}}} = \{b\}$
    - $\text{Eq}_{f^2,\text{id}}^{\widetilde{\mathfrak{A}}} = \{(a,a),(b,b)\}$
    - $\text{Eq}_{g,f}^{\widetilde{\mathfrak{A}}} = \{(a,b),(b,a)\}$

Naive Encoding versus Transitive Expansion



- Cannot be distinguishes by naive RCR: Encodings result in regular graphs
- But: Distinguished by Transitive Expansion Encoding
  - We find that $\mathrm{Eq}^{\widetilde{\mathfrak{A}}}_{f^1,\mathrm{id}} = \mathrm{Eq}^{\widetilde{\mathfrak{A}}}_{f^4,\mathrm{id}}$, not for $\widetilde{\mathfrak{B}}$
  - Sentence $\exists^{\geq 6}(x, y)\,.\,\big(\mathrm{Eq}_{f^1,\mathrm{id}}(x, y) \wedge \mathrm{Eq}_{f^4,\mathrm{id}}(x, y)\big) \in \mathrm{GF(C)}$ distinguishes encodings

Relational Colour Refinement for Structures With Functions

Logical Characterisation of Naive RCR

Nesting-Free Guarded Fragment of Counting Logic

nfGF(C)

- Extends given definition of GF(C) for non-relational signatures
- Allow atomics of the following forms
  - For relation symbol R of arity ℓ and variables $x_1, \ldots, x_\ell$: $R(x_1, \ldots, x_\ell) \in \mathrm{nfGF(C)}$
  - For variables x and y: $x = y \in \mathrm{nfGF(C)}$
  - For function symbol f of arity ℓ and variables $x_1, \ldots, x_\ell, y$: $f(x_1, \ldots, x_\ell) = y \in \mathrm{nfGF(C)}$

- Forbid nesting of terms, for example $f(g(x), y) = z$
- Informally: Usage of function symbols like relation symbols

Characterising Naive RCR Logically

Logical Characterisation of Naive RCR

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures. Then the two following statements are equivalent.

1. Naive RCR distinguishes $\mathfrak{A}$ and $\mathfrak{B}$
2. There exists a sentence $\varphi \in \mathrm{nfGF(C)}$ which is fulfilled by $\mathfrak{A}$, but not by $\mathfrak{B}$

Proof idea:

- Naive RCR distinguishes structures iff. RCR distinguishes encodings iff. there exists a sentence in GF(C) that distinguishes the encodings
- Define translation of sentences in GF(C) over signature σ' to and from sentences in nfGF(C) over signature σ
  - $R_f(\mathbf{x}y) \leftrightarrow f(\mathbf{x}) = y$

Relational Colour Refinement for Structures With Functions

Logical Characterisation of RCR_k

GF(C) **with alternation depth** k

GF(C)$_k$

- Fixate $k \in \mathbb{N}$
- Natural extension of GF(C) to non-relational signatures w.r.t. allowed atomic formulae with one restriction
- For every formula in GF(C)$_k$ and every term t that appears in it, there must exist a $n \in \mathbb{N}$, such that $t = \alpha$ for a $\alpha \in \mathrm{Alters}_n^k(\sigma)$

- Restrict number of alternations of function applications to k
- No restriction of number of application of same function in series
- Examples:
    - $\exists^{\geq 1}(x, y)\,.(f^2(g(h^3(x))) = y \wedge T(y)) \notin \mathrm{GF(C)}_2$, but in $\mathrm{GF(C)}_3$
    - $\exists^{\geq 1}(x, y)\,.(f^i(x) = y \wedge T(y)) \in \mathrm{GF(C)}_1$ for all $i \in \mathbb{N}$

Characterising RCR_k Logically i

Hinges on three lemmas:

1. Formulae $f^m(x) = y \in \mathrm{GF(C)}_1$ can be translated to formula in $\mathrm{GF(C)}_1$ that is equivalent for structures with n elements and only f^i with $i \leq n$ appears
2. Formulae $g^m(s(x)) = y \in \mathrm{GF(C)}_d$ can be translated to formula in $\mathrm{GF(C)}_d$ that is equivalent for structure with n elements and only f^i with $i \leq n$ appears
3. Formulae $R(t_1(x_1), \ldots, t_\ell(x_\ell)) \in \mathrm{GF(C)}_d$ can be translated to formula in $\mathrm{GF(C)}_d$ that is equivalent for structure with n elements and only f^i with $i \leq n$ appears

Characterising RCR_k Logically ii

Logical Characterisation of RCR_k

Let $k \in \mathbb{N}$ and let $\mathfrak{A}$ and $\mathfrak{B}$ be two structures. Then the two following statements are equivalent.

1. RCR_k distinguishes $\mathfrak{A}$ and $\mathfrak{B}$
2. There exists a sentence in $\mathsf{GF(C)}_k$ that is fulfilled by $\mathfrak{A}$, but not by $\mathfrak{B}$

Characterising RCR_k Logically iii

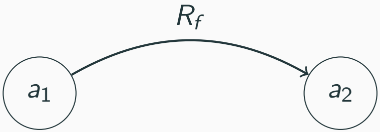
Proof idea

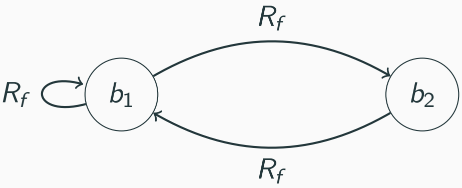
- 1. to 2.: Like, before sentence in GF(C) over signature $\widetilde{\sigma}$ can easily be translated into sentence in $\mathrm{GF(C)}_k$ over signature σ
- 2. to 1.:
    - Replace atomic subformulae by translations from lemmas
    - Rearrange resulting formula to get valid $\mathrm{GF(C)}_k$-sentence
    - Results in equivalent formula for structure with $n = |\mathfrak{A}|$ elements and for every term t there exists an $\alpha \in \mathrm{Alters}_n^k(\sigma)$, such that $t = \alpha$
    - Can easily be translated into sentence in GF(C) of signature $\widetilde{\sigma}$

Combinatorial Characterisation of RCR

Total and Functional Structures

- Let σ be a signature and $\mathfrak{A}$ a σ-structure and σ' and $\mathfrak{A}'$ the respective naive encodings
- We call $\mathfrak{A}$ total if for every n-ary function symbol $f \in \sigma$ and every n-tuple $\mathbf{x}$ there is a y, such that $(\mathbf{x}y) \in R_f^{\mathfrak{A}'}$
- We call $\mathfrak{A}$ function if for every n-ary function symbol f there are no two $n+1$-tuples $(\mathbf{x}y), (\mathbf{x}z) \in R_f^{\mathfrak{A}'}$





Non-Relational Acyclic Structures

- If we want to count homomorphisms to non-relational structures we need to determine what an non-relational, acyclic structure would look like
- Will define acyclicity w.r.t. the naive encoding

Non-Relational Acyclic Structures

- Let $\mathfrak{A}$ be a non-relational structure
- We call $\mathfrak{A}$ acyclic, if its naive encoding $\mathfrak{A}'$ is acyclic

Total and Functional Structures as Encodings

- Desired equivalence:

Non-relational, acyclic structure distinguishes $\mathfrak{A}$ and $\mathfrak{B}$ by homomorphism count

iff.?

Naive RCR distinguishes $\mathfrak{A}$ and $\mathfrak{B}$

- Result: Forward direction holds, backwards does not
- First step: Reformulate first statement:

Non-relational, acyclic structure distinguishes $\mathfrak{A}$ and $\mathfrak{B}$ by homomorphism count

iff.

Some total, functional and acyclic structure distinguishes encodings $\mathfrak{A}'$ and $\mathfrak{B}'$ by homomorphism count

Enforcing Functionality

- We can show:

$$\text{Acyclic } \sigma'\text{-structure dist. } \mathfrak{A}' \text{ and } \mathfrak{B}' \text{ by hom. count}$$
$$\text{iff.}$$
$$\text{Functional and acyclic } \sigma'\text{-structure dist. } \mathfrak{A}' \text{ and } \mathfrak{B}' \text{ by hom. count}$$

Proof idea:

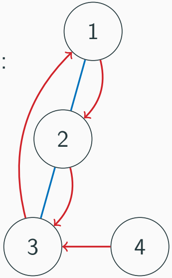
- Backwards direction is obvious
- Forwards direction eliminates collisions of the form $(\mathbf{x}y), (\mathbf{x}z) \in R_f$ by contracting y and z
- This can be done while maintaining the homomorphisms and acyclicity and can be repeated until no collisions remain

Non-Enforceability of Totality

- There are structures that are distinguished by naive RCR, but there is no acyclic and total structure that distinguishes the encodings by homomorphism count
- Two families of structures $(\mathfrak{A}_i)_{i \in \mathbb{N}_{\geq 4}}$ and $(\mathfrak{B}_i)_{i \in \mathbb{N}_{\geq 4}}$
- For all $i \in \mathbb{N}_{\geq 4}$: Naive RCR distinguishes $\mathfrak{A}_i$ and $\mathfrak{B}_i$, but no acyclic and total structure can distinguish the encodings by hom. count

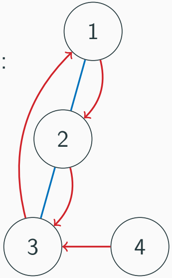
Structures that are distinguished by nRCR but not total structures i

$\mathfrak{A}_4$:

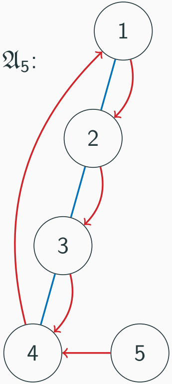


Structures that are distinguished by nRCR but not total structures i

$\mathfrak{A}_4$:

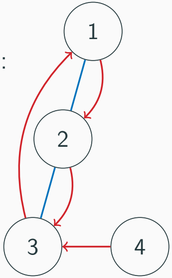


$\mathfrak{A}_5$:

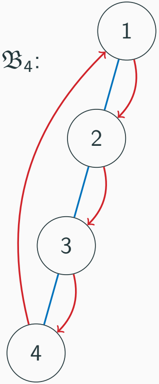


Structures that are distinguished by nRCR but not total structures i

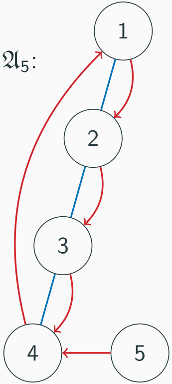
$\mathfrak{A}_4$:



$\mathfrak{B}_4$:

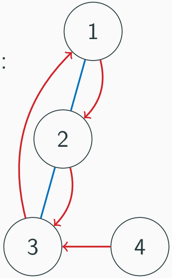


$\mathfrak{A}_5$:

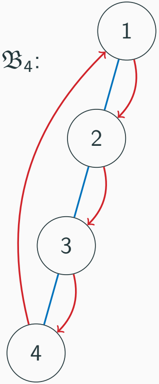


Structures that are distinguished by nRCR but not total structures i

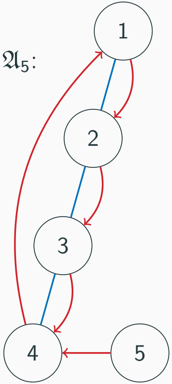
$\mathfrak{A}_4$:



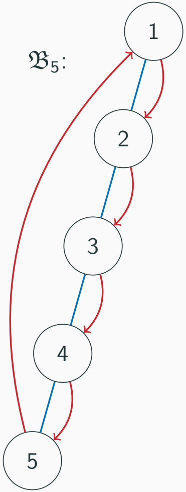
$\mathfrak{B}_4$:



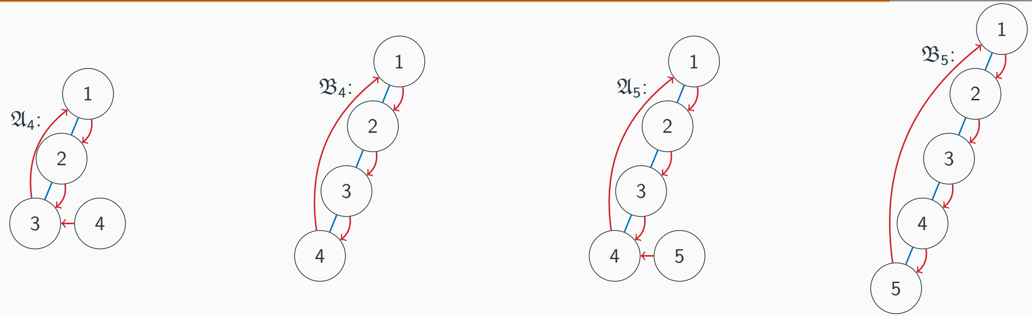
$\mathfrak{A}_5$:



$\mathfrak{B}_5$:



Structures that are distinguished by nRCR but not total structures ii



- Obviously distinguished by naive RCR
- If structure has R_f-loops or R_f-2-cycles, then no homomorphisms to either structure
- Because total, it has to contain larger R_f-cycles, but then cannot be acyclic

Results of combinatorial characterisation of naive RCR

We have the following results:

$$\text{Naive RCR distinguishes } \mathfrak{A} \text{ and } \mathfrak{B}$$
$$\Updownarrow$$
$$\text{There exists acyclic structure that dist. encodings } \mathfrak{A}' \text{ and } \mathfrak{B}' \text{ by hom. count}$$
$$\Updownarrow$$
$$\text{There exists acyclic and functional structure that dist. encodings by hom. count}$$
$$\Uparrow, \text{ but } \not\Downarrow$$
$$\text{There exists acyclic, total and functional structure that dist. encodings by hom. count}$$
$$\Updownarrow$$
$$\text{There exists acyclic, non-relational structure that dist. } \mathfrak{A} \text{ and } \mathfrak{B} \text{ by hom. count}$$

Restricting RCR to Symmetric Structures

Restricting the Class of Structures

- For what subclass $\mathcal{S}$ of relational structures do we have the following equivalence:

$$\text{Two structures from } \mathcal{S} \text{ get distinguished by RCR}$$
$$\text{iff.}$$
$$\text{There exists an acyclic structure from } \mathcal{S} \text{ that dist. the structures by hom. count}$$

- Does not hold for class of total structures
    - Encodings of classes of structures from before are total, but not total and acyclic structure dist. them by hom. count
- Another class to investigate: Class of symmetric structures

Restriction to Symmetric Structures

- Relational Structure is symmetric, if for every k-ary relation R and for every k-tuple $\mathbf{x} \in R$, every permutation of the elements in $\mathbf{x}$ is also in R
- For two symmetric structures we can show

  Acyclic structure dist. the structures by hom. count

  iff.

  Acycic, symmetric structure dist. the structure by hom. count
- From this, restriction to symmetric structures is possible

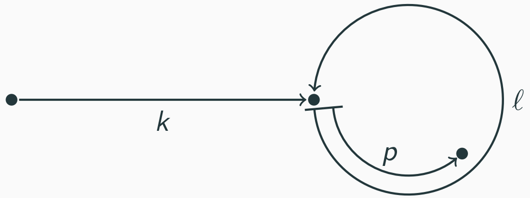
Sketch of a Proof

Statement of the Lemma

- First lemma for logical characterisation of transitive expansion
- A formula ψ of the form $f^m(x_1) = x_2 \in \mathrm{GF(C)}_1$ can be translated to a formula $\vartheta(x_1, x_2) \in \mathrm{GF(C)}_1$, such that:
    1. They are equivalent for structures with n elements
    2. There does not appear a term f^i with $i > n$ in ϑ
    3. ϑ is of the form $\bigvee \Phi$ and if ϑ is fulfilled, then there exists exactly one $\varphi \in \Phi$ which is satisfied

Proof Idea

- For $f^0(x), f^1(x), \ldots, f^m(x)$, if $m > n$, there have to be $i, j \leq n$ such that $f^i(x) = f^j(x)$
- We get path to a cycle, a cycle and a last part of it
- Define set $\mathcal{I}(n, m)$ as set of all such decomposition (k, ℓ, p)



Sketch of the Proof i

- Define $\vartheta(x_1, x_2) := \bigvee_{(k,\ell,p)\in\mathcal{I}(n,m)} \zeta_{(k,\ell,p)}(x_1, x_2)$ where

$$\begin{aligned}\zeta_{(k,\ell,p)}(x_1, x_2) :=& f^{k+p}(x_1) = x_2 \wedge f^k(x_1) = f^{k+\ell}(x_1)\\ &\wedge \mathrm{E}_f^{k,\ell}(x_1)\\ &\wedge \bigwedge_{0<\ell'<\ell} f^k(x_1) \neq f^{k+\ell'}(x_1)\end{aligned}$$

and

$$\mathrm{E}_f^{k,\ell}(t(x_1)) := \begin{cases} \top & \text{if } k = 0 \\ f^{k-1}(t(x_1)) \neq f^{k-1+\ell}(t(x_1)) & \text{otherwise.} \end{cases}$$

Sketch of the Proof ii

- $f^{k+p}(x_1) = x_2 \wedge f^k(x_1) = f^{k+\ell}(x_1)$ ensures that (k, ℓ, p) decomposes the path into a path to a cycle and the cycle itself
- $\mathsf{E}_f^{k,\ell}(x_1) \wedge \bigwedge_{0<\ell'<\ell} f^k(x_1) \neq f^{k+\ell'}(x_1)$ ensures that only the lexicographically smallest decomposition is satisfied

- If ψ is satisfied, a smallest decomposition (k, ℓ, p) exists that describes the path of f
- Then it can be shown that $\zeta_{(k,\ell,p)}$ is satisfied, and because only the lexicographically smallest (k, ℓ, p) is satisfied, it is the only one

Sketch of the Proof iii

- If ϑ is satisfied, some $\zeta_{(k,\ell,p)}$ is satisfied
- This means that (k, ℓ, p) describes the path of f, therefore ψ is also satisfied

Conclusion

Conclusion

- We presented classical CR and Scheidt's and Scheikardt's RCR algorithm
- We defined two possible ways to apply their algorithm to non-relational signatures
  - Naive RCR
  - RCR_k
- We showed our results for the logical characterisations
  - Naive RCR gets characterised by the nesting free fragment of counting logic
  - RCR_k gets characterised by the natural extension of GF(C) to non-relational signatures where terms have a maximal alternation depth of k
- We disproved the characterisation by homomorphism counting
  - Functionality can be enforced
  - Totality cannot
- We showed results for the restriction to two subclasses of the relational structures
  - The restriction to total structures does not preserve the characterisation by hom. counting
  - The restriction to symmetric structures does preserve it

Equality between terms t and alternations α

- For a term t and a $\alpha \in \mathsf{Alters}_n^k(\sigma)$ we say $t = \alpha$, if:
- If $t = f^i(x)$, the i-times application of one function symbol f, and $\alpha = f^i$
- If $t = f^i(g^j(s(x)))$, where f and g are function symbols and s is a term, and $\alpha = f^i\alpha'$ and $s = \alpha'$
- Informally, if t is written using $\circ$, i.e. $f^i \circ g^j(x)$ instead of $f^i(g^j(x))$, the $\circ$ are omitted and then this equals α