Relational Colour Refinement for Non-Relational Signatures

Theodor Jurij Teslia

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RWTH Aachen University

Classical Colour Refinement

Colour Refinement

- Also called CR or 1-dimensional Weisfeiler-Leman algorithm
- Iterative graph algorithm
- Constructs colour for every vertex, based on colours of neighbours

Definition (Colour Refinement)
For graph $G = (V, E)$, for every $v \in V$ and $i \in \mathbb{N}$:

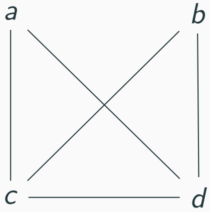
$$C_0(v) := 0$$

and

$$C_{i+1}(v) := (C_i(v), \{\!\{ C_i(u) : \{v, u\} \in E \}\!\}).$$

Example for CR

G:



- $C_0(a) = C_0(b) = C_0(c) = C_0(d) = 0$

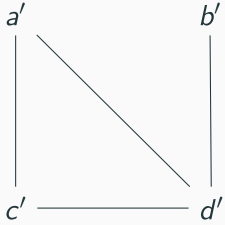
- $C_1(a) = (C_0(a), \{\!\{0, 0\}\!\}) = C_1(b)$
- $C_1(c) = (C_0(c), \{\!\{0, 0, 0\}\!\}) = C_1(d)$

- $C_2(a) = (C_1(a), \{\!\{C_1(c), C_1(c)\}\!\}) = C_2(b)$
- $C_2(c) = (C_1(c), \{\!\{C_1(a), C_1(a), C_1(c)\}\!\}) = C_2(d)$

Distinguished graphs

- CR distinguishes two graphs G and H, if
- there exists $C_i(v)$ in colouring of G or H, such that number of vertices with colour $C_i(v)$ is different in G than in H

H:



- Colours in first round equal
- $C_1(b') = (C_0(b'), \{\!\{ C_0(d') \}\!\}) = (0, \{\!\{0\}\!\})$ does not appear in G

$\Rightarrow$ Colour Refinement distinguishes G and H.

Characterisations of CR

- There are equivalent characterisations for CR
- Due to bibliography:
  CR distinguishes G and H if, and only if, there exists $\varphi \in \mathsf{C}_2$, such that $G \models \varphi$ and $H \not\models \varphi$
- Due to bibliography:
  CR distinguishes G and H if, and only if, there exists tree T, such that $\hom(T, G) \neq \hom(T, H)$

Examples for G and H:

- $\varphi \coloneqq \exists^{\geq 1} x . \neg \exists^{\geq 2} y . E(x, y)$
- $T \coloneqq (\{v, u\}, \{\{v, u\}\})$

Relational Colour Refinement

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- Introduced by bibliography