$I_b = I_e + \frac{1}{72} I_e$

$\delta_t = \frac{a'(t)}{a(t)}$ simple $\delta_t = \frac{i}{1+it}$ compound $\delta = \ln(1+i)$ $\quad i = \frac{d}{1-d} = d + id \quad d = iv = 1 - v$

level annuity

$a_{\overline{n}|i} = v\frac{1-v^n}{1-v} = \frac{1-v^n}{i} \quad 1 = v^n + i a_{\overline{n}|i}$

$\ddot{a}_{\overline{n}|i} = \frac{1-v^n}{d} = (1+i) a_{\overline{n}|i}$

$s_{\overline{n+k}|} = (1+i)^k s_{\overline{n}|} + \frac{(1+i)^k - 1}{i}$

deferred immediate $PV = a_{\overline{m+n}|} - a_{\overline{m}|} = v^m a_{\overline{n}|}$ due $PV = \ddot{a}_{\overline{m+n}|} - \ddot{a}_{\overline{m}|} = v^m \ddot{a}_{\overline{n}|}$

perpetuity $a_{\overline{\infty}|} = \frac{1}{i} \quad \ddot{a}_{\overline{\infty}|} = \frac{1}{d} \quad \ddot{a}_{\overline{\infty}|} = 1 + a_{\overline{\infty}|} = 1 + \frac{1}{i} = \frac{1}{d} \quad a_{\overline{n}|} = a_{\overline{\infty}|} - v^n a_{\overline{\infty}|}$

k c/p immediate $PV = v^k \frac{1-v^n}{1-v^k} = \frac{1-v^n}{(1+i)^k - 1} = \frac{a_{\overline{n}|}}{s_{\overline{k}|}}$ due $PV = \frac{a_{\overline{n}|}}{a_{\overline{k}|}}$

perpetuity immediate $PV = \frac{1}{i s_{\overline{k}|}}$ due $PV = \frac{1}{i a_{\overline{k}|}}$

m p/c $1/m$ each $a^{(m)}_{\overline{n}|} = \frac{v^{\frac{1}{m}}}{m} \frac{1-v^n}{1-v^{\frac{1}{m}}} = \frac{1-v^n}{i^{(m)}} = \frac{i}{i^{(m)}} a_{\overline{n}|} = s^{(m)}_{\overline{1}|} a_{\overline{n}|} \quad \ddot{a}^{(m)}_{\overline{n}|} = \frac{i}{d^{(m)}} a_{\overline{n}|} = \ddot{s}^{(m)}_{\overline{1}|} a_{\overline{n}|}$

continuous $\lim i^{(m)} = \lim d^{(m)} = \delta \quad \bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta} = \frac{1-e^{-n\delta}}{\delta} = \frac{i}{\delta} a_{\overline{n}|} = \frac{d}{\delta} \ddot{a}_{\overline{n}|}$ perpetuity $\bar{a}_{\overline{\infty}|} = \frac{1}{\delta}$

varying annuity

arithmetic progression immediate $PV = P a_{\overline{n}|} + Q \frac{a_{\overline{n}|} - n v^n}{i}$ due $PV = P \ddot{a}_{\overline{n}|} + Q \frac{a_{\overline{n}|} - n v^n}{d}$

increasing $P = Q = 1$. $(Ia)_{\overline{n}|} = a_{\overline{n}|} + \frac{a_{\overline{n}|} - n v^n}{i} = \frac{\ddot{a}_{\overline{n}|} - n v^n}{i} \quad (I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n v^n}{d}$

$(Is)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i} = \frac{s_{\overline{n+1}|} - (n+1)}{i} \quad (I\ddot{s})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d} = \frac{s_{\overline{n+1}|} - (n+1)}{d}$

decreasing $P = n$, $Q = -1$. $(Da)_{\overline{n}|} = n a_{\overline{n}|} - \frac{a_{\overline{n}|} - n v^n}{i} = \frac{n - a_{\overline{n}|}}{i} \quad (D\ddot{a})_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{d}$

$(Ia)_{\overline{n}|} + (Da)_{\overline{n}|} = (n+1) a_{\overline{n}|}$

perpetuity immediate $PV = P a_{\overline{\infty}|} + Q \frac{a_{\overline{\infty}|}}{i} = \frac{P}{i} + \frac{Q}{i^2}$ due $PV = P \ddot{a}_{\overline{\infty}|} + Q \frac{a_{\overline{\infty}|}}{d} = \frac{P}{d} + \frac{Q}{id}$

$(Ia)_{\overline{\infty}|} = \frac{1}{i} + \frac{1}{i^2} \quad (I\ddot{a})_{\overline{\infty}|} = \frac{1}{d^2}$

k c/p immediate $PV = \frac{\frac{a_{\overline{n}|}}{a_{\overline{k}|}} - \frac{n}{k} v^n}{i s_{\overline{k}|}}$ due $PV = \frac{\frac{a_{\overline{n}|}}{a_{\overline{k}|}} - \frac{n}{k} v^n}{i a_{\overline{k}|}}$

m p/c $\frac{k}{m} s_{\overline{n}| \frac{i^{(m)}}{m}} = k \frac{i}{i^{(m)}}$ at kth. $(Ia)^{(m)}_{\overline{n}|} = \frac{i}{i^{(m)}} (Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n v^n}{i^{(m)}} \quad (Ia)^{(m)}_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n v^n}{d^{(m)}}$

mthly increasing mthly annuity. $(I^{(m)} a)^{(m)}_{\overline{n}|} = \frac{\ddot{a}^{(m)}_{\overline{n}|} - n v^n}{i^{(m)}} \quad (I^{(m)} \ddot{a})^{(m)}_{\overline{n}|} = \frac{\ddot{a}^{(m)}_{\overline{n}|} - n v^n}{d^{(m)}}$

continuous $PV = \int_0^n f(t) e^{-\int_0^t \delta_r dr} dt$

increasing $f(t) = t \quad (\bar{I}\bar{a})_{\overline{n}|} = \int_0^n t v^t dt = \frac{\bar{a}_{\overline{n}|} - n v^n}{\delta}$

decreasing $f(t) = n - t \quad (\bar{D}\bar{a})_{\overline{n}|} = \int_0^n (n-t) v^t dt = \frac{n - \bar{a}_{\overline{n}|}}{\delta}$

perpetuity $(\bar{I}\bar{a})_{\overline{\infty}|} = \frac{1}{\delta^2}$

geometric progression

increasing $1+k$ immediate $PV = \begin{cases} nv & k = i \\ \frac{1 - \left(\frac{1+k}{1+i}\right)^n}{i-k} & k \neq i \end{cases}$ due $PV = \begin{cases} n & k = i \\ (1+i)\frac{1 - \left(\frac{1+k}{1+i}\right)^n}{i-k} & k \neq i \end{cases}$

decreasing $1-k$ immediate $PV = \frac{1 - \left(\frac{1-k}{1+i}\right)^n}{i+k}$ due $PV = (1+i)\frac{1 - \left(\frac{1-k}{1+i}\right)^n}{i+k}$

perpetuity $k < i$ immediate $PV = \frac{1}{i-k}$ due $PV = \frac{1+i}{i-k}$

$NPV(i) = \sum_{k=0}^n v^{t_k} (R_{t_k} - C_{t_k}) = \sum_{k=0}^n v^{t_k} c_{t_k}$

reinvest at j, $FV = 1 + is_{\overline{n}|j}$

dollar weighted rate of return $B = A + C + I = A + [\sum_{0\le t\le 1} c_t] + [iA + \sum_{0\le t\le 1} c_t[(1+i)^{1-t} - 1]]$

$(1+i)^{1-t} \approx (1-t)i \quad i \approx \frac{I}{A+\sum_{0\le t\le 1} c_t(1-t)} = \frac{I}{\text{exposure associated with } i}$

assumed lump sum at $k \approx \frac{1}{C}\sum_{0\le t\le 1} tc_t$, $i \approx \frac{I}{A+(1-k)C}$. $\quad i \approx \frac{2I}{A+B-I}$ assume $k = 0.5$

time weighted rate of return $(1+i) = (1+j_1)(1+j_2)\dots(1+j_n)$

loan L with PMT P. outstanding loan balance $B_t^p = Pa_{\overline{n-t}|} = L(1+i)^t - Ps_{\overline{t}|} = B_t^r$

amortization.

$L = a_{\overline{n}|}$ with $P = 1$.

outstanding loan balance $B_k^p = a_{\overline{n-k+1}|} - v^{n-k+1} = a_{\overline{n-k}|}$.

interest paid $iB_{k-1}^p = ia_{\overline{n-k+1}|} = 1 - v^{n-k+1}$. principal paid v^{n-k+1}.

k c/p.

outstanding loan balance $B_{mk}^p = \frac{a_{\overline{n-(m-1)k}|}}{s_{\overline{k}|}} - v^{n-(m-1)k} = \frac{a_{\overline{n-mk}|}}{s_{\overline{k}|}}$.

interest paid $[(1+i)^k - 1]B_{(m-1)k}^p = [(1+i)^k - 1]\frac{a_{\overline{n-(m-1)k}|}}{s_{\overline{k}|}} = 1 - v^{n-(m-1)k}$.

principal paid $v^{n-(m-1)k}$.

m p/c.

outstanding loan balance $B_{\frac{t}{m}}^p = a^{(m)}_{\overline{n-\frac{t-1}{m}}|} - \frac{1}{m}v^{n-\frac{t-1}{m}} = a^{(m)}_{\overline{n-\frac{t}{m}}|}$.

interest paid $\frac{i^{(m)}}{m}B_{\frac{t-1}{m}}^p = \frac{i^{(m)}}{m}a^{(m)}_{\overline{n-\frac{t-1}{m}}|} = \frac{1}{m}(1 - v^{n-\frac{t-1}{m}})$.

principal paid $\frac{1}{m}v^{n-\frac{t-1}{m}}$.

sinking fund.

$L = 1$. interest paid i. interest rate on SF j. SF deposit $1/s_{\overline{n}|j}$.

at bgn of t period, SF BAL $s_{\overline{t-1}|j}/s_{\overline{n}|j}$. during t period, interest earned on SF $js_{\overline{t-1}|j}/s_{\overline{n}|j}$.

at end of t period, SF BAL $s_{\overline{t}|j}/s_{\overline{n}|j}$ and net amt of loan $1 - s_{\overline{t}|j}/s_{\overline{n}|j}$.

in t period, net interest paid $i - (js_{\overline{t-1}|j}/s_{\overline{n}|j})$

change in amt of loan $(1 - s_{\overline{t-1}|j}/s_{\overline{n}|j}) - (1 - s_{\overline{t}|j}/s_{\overline{n}|j}) = (1+j)^{t-1}/s_{\overline{n}|j}$.

amortization $\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$ sinking fund.

level PMT=1 reinvest at j. $PV = a_{\overline{n}|i\&j}$. interest paid $ia_{\overline{n}|i\&j}$. SF deposit $1 - ia_{\overline{n}|i\&j}$.

$(1 - ia_{\overline{n}|i\&j})s_{\overline{n}|j} = a_{\overline{n}|i\&j}$. $a_{\overline{n}|i\&j} = \frac{s_{\overline{n}|j}}{1+is_{\overline{n}|j}}$.

$\frac{1}{a_{\overline{n}|i\&j}} = \frac{1}{s_{\overline{n}|j}} + i = \frac{1}{a_{\overline{n}|j}} + (i-j)$. $a_{\overline{n}|i\&j} = \frac{a_{\overline{n}|j}}{1+(i-j)a_{\overline{n}|j}}$.

bond (price P, YTM i)

(face val F, redemption val C, coupon rate r, modified coupon rate $g = \frac{Fr}{C}$, period n)

PV of C at YTM, $K = Cv^n$. base amt $G = \frac{Fr}{i}$

basic formula $P = Fra_{\overline{n}|} + Cv^n = Fra_{\overline{n}|} + K$

premium/discount formula $P = Fra_{\overline{n}|} + C(1 - ia_{\overline{n}|}) = C + (Fr - Ci)a_{\overline{n}|}$

base amt formula $P = Gia_{\overline{n}|} + Cv^n = G + (C - G)v^n$

Makeham formula $P = Cga_{\overline{n}|} + K = K + \frac{g}{i}(C - K)$

k c/p, $P = Fr\frac{a_{\overline{n}|}}{s_{\overline{k}|}} + Cv^n$

m p/c, $P = mFra^{(m)}_{\overline{n}|} + Cv^n$

premium/discount $P - C = (Fr - Ci)a_{\overline{n}|} = C(g - i)a_{\overline{n}|}$

BV $B_t = Fra_{\overline{n-t}|} + Cv^{n-t}$, $B_0 = P$, $B_n = C$

bond amortization schedule. $B_t = B_0 - \sum_{k=1}^{t} P_k$. coupon Fr

interest earned $I_t = iB_{t-1} = i[Fra_{\overline{n-t+1}|} + Cv^{n-t+1}] = Cg + C(i - g)v^{n-t+1}$

principal adjustment $P_t = Fr - I_t = C(i - g)v^{n-t+1}$

straight line method $P_t = \frac{P-C}{n}$, $I_t = Fr - P_t$

accrued coupon Fr_k. flat price $B^f_{t+k} = (B_{t+1} + Fr)v^{1-k} = Fr_k + B^m_{t+k}$ market price

theoretical method $B^f_{t+k} = (1 + i)^k B_t$, $Fr_k = Fr\frac{(1+i)^k-1}{i}$, $B^m_{t+k} = (1 + i)^k B_t - Fr\frac{(1+i)^k-1}{i}$

practical method $B^f_{t+k} = (1+ki)B_t$, $Fr_k = kFr$, $B^m_{t+k} = (1+ki)B_t - kFr = (1-k)B_t + kB_{t+1}$

semi-theoretical method $B^f_{t+k} = (1 + i)^k B_t$, $Fr_k = kFr$, $B^m_{t+k} = (1 + i)^k B_t - kFr$

premium/discount $B^m_{t+k} - C$

preferred stock. dividend growth k. $PV = \frac{D}{i-k}$

Fisher equation $i' = \frac{i-r}{1+r} \approx i - r$

forward rate $(1 + i_n)^n(1 + f_n^{n+k})^k = (1 + i_{n+k})^{n+k}$

method of equated time. average term to maturity $\bar{t} = \frac{\sum tR_t}{\sum R_t}$

Macaulay duration $\bar{d} = \frac{\sum tv^t R_t}{\sum v^t R_t}$, $\frac{d}{di}\bar{d} < 0$

$P = Fr(v + v^2 + \cdots + v^n) + Fv^n = F(1 + (r - i)a_{\overline{n}|})$, $-P' = F(r(Ia)_{\overline{n}|} + nv^n)$

$\bar{d} = \frac{1+i}{i} - \frac{1+i+(r-i)n}{r[(1+i)^n-1]+i}$. $r = i$, $\bar{d} = \ddot{a}_{\overline{n}|}$

first-order Macaulay approximation of the present-value $P(i) \approx P(i_0)\left(\frac{1+i_0}{1+i}\right)^{\bar{d}(i_0)}$

volatility of PV. modified duration $\bar{\nu} = -\frac{P'(i)}{P(i)} = -\frac{d}{di}\ln(P(i)) = \frac{\bar{d}}{1+i} = v\bar{d}$

first-order modified approximation of the present-value $P(i) \approx P(i_0)[1 - (i - i_0)\bar{v}(i_0)]$

second-order modified approximation of the present value $P(i) \approx P(i_0)\left(1 - (i - i_0)\bar{v}(i_0) + \frac{(i-i_0)^2}{2}\bar{c}(i_0)\right)$

portfolio modified duration $\bar{\nu} = \sum \frac{Pk(i)}{P(i)}\bar{\nu}_k$ MV weighted

Redington immunization (1) $P(i) = 0$ (2) $P'(i) = 0$ (3) $P''(i) > 0$

convexity $\bar{c} = \frac{P''(i)}{P(i)}$. $\frac{P(i+\epsilon)-P(i)}{P(i)} \approx -\epsilon\bar{\nu} + \frac{\epsilon^2}{2}\bar{c}$

Macaulay duration $c_{Mac} = \frac{\sum t^2 v^{t+2} R_t}{\sum v^t R_t}$. Modified duration $c_{mod} = \frac{\sum t(t+1)v^{t+2}R_t}{\sum v^t R_t} = \frac{c_{Mac}+d_{Mac}}{(1+i)^2}$

portfolio convexity $\bar{c} = \sum \frac{Pk(i)}{P(i)}\bar{c}_k$ MV weighted

full immunization (1) $P(i) = 0$ (2) $P'(i) = 0$ (3) A at $t = k - a$, $-L$ at $t = k$, B at $t = k + a$

$L = Ae^{a\delta} + Be^{-a\delta}$, $Aae^{a\delta} = Bbe^{-b\delta}$, $B = A\frac{a}{b}e^{(a+b)\delta}$

$S(\delta') = PV_{AB}(\delta') + PV_L(\delta') > 0$ for $\delta' \neq \delta$