Normal Model

given σ^2

sampling model

$$Y_1, \ldots, Y_n | \theta, \sigma^2 \overset{i.i.d.}{\sim} \text{normal}(\theta, \sigma^2)$$

conditional prior

$$\theta | \sigma^2 \sim \text{normal}\Big(\mu_0, \tau_0^2\Big)$$

posterior

$$\theta | \sigma^2, y_1, \ldots, y_n \sim \text{normal}\Big(\mu_n, \tau_n^2\Big)$$

$\mu_n = \frac{\mu_0/\tau_0^2 + n\bar{y}/\sigma^2}{1/\tau_0^2 + n/\sigma^2}$ and $\tau_n^2 = \frac{1}{1/\tau_0^2 + n/\sigma^2}$

set $\tau_0^2 = \frac{\sigma^2}{\kappa_0}$, $\mu_n = \frac{\kappa_0}{\kappa_0+n}\mu_0 + \frac{n}{\kappa_0+n}\bar{y}$ and $\tau_n^2 = \frac{\sigma^2}{\kappa_0+n} = \frac{\sigma^2}{\kappa_n}$, where $\kappa_n = \kappa_0 + n$

joint prior $p(\theta, \sigma^2) = p(\theta|\sigma^2)p(\sigma^2)$

$$1/\sigma^2 \sim \text{gamma}(\frac{\nu_0^2}{2}, \frac{\nu_0^2}{2}\sigma_0^2)$$
$$\theta|\sigma^2 \sim \text{normal}(\mu_0, \frac{\sigma^2}{\kappa_0})$$
$$Y_1, \ldots, Y_n | \theta, \sigma^2 \overset{i.i.d.}{\sim} \text{normal}(\theta, \sigma^2)$$

posterior

$$1/\sigma^2 | y_1, \ldots, y_n \sim \text{gamma}(\frac{\nu_n^2}{2}, \frac{\nu_n^2}{2}\sigma_n^2)$$
$$\theta|\sigma^2, y_1, \ldots, y_n \sim \text{normal}(\mu_n, \frac{\sigma^2}{\kappa_n})$$

$\nu_n = \nu_0 + n$ and $\sigma_n^2 = \frac{1}{\nu_n}[\nu_0^2\sigma_0^2 + (n-1)s^2 + \frac{\kappa_0 n}{\kappa_n}(\bar{y} - \mu_0)2]$, $s^2 = \frac{1}{n-1}\sum_{l=1}^{n}(y_l - \bar{y})^2$

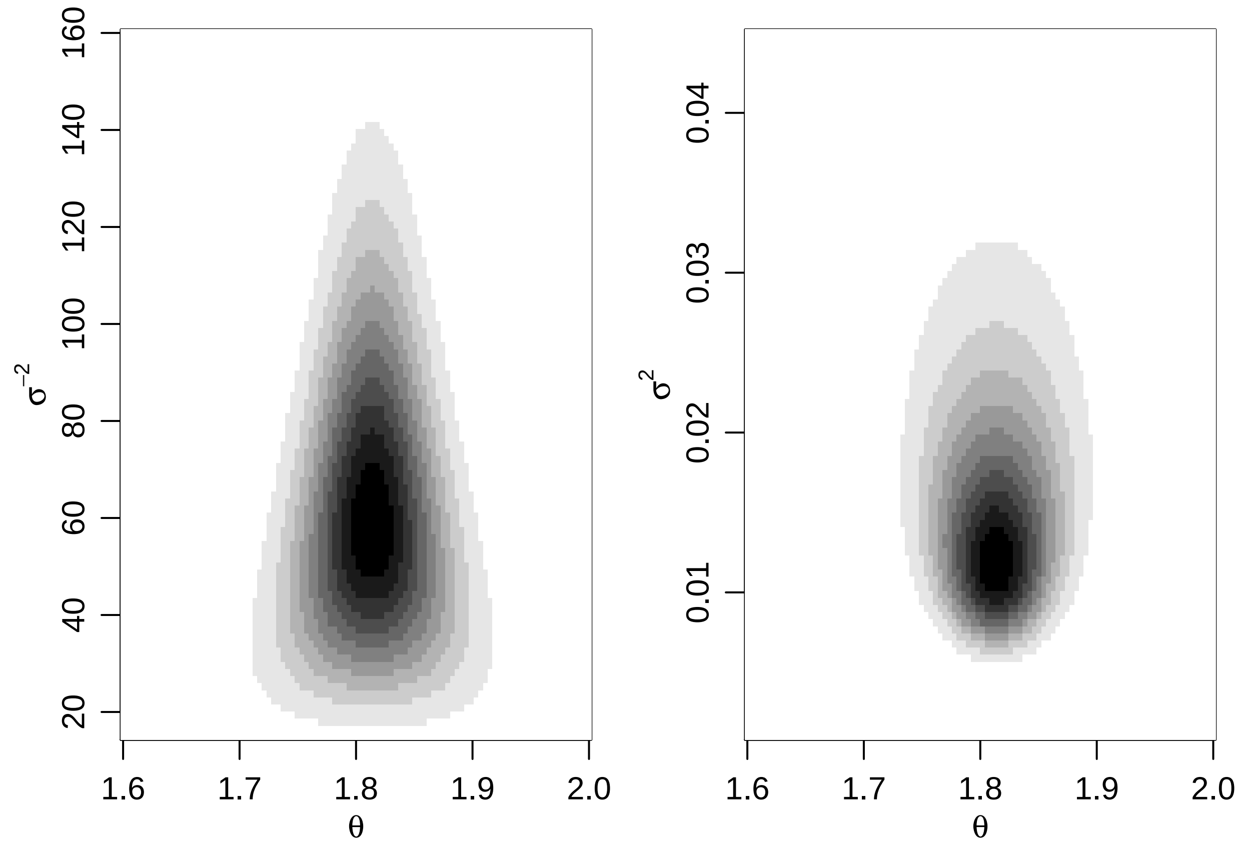
```
> # Fig 5.4 Joint posterior distributions of (theta, sigma^-2) and  (theta, sigma^2)
> # http://www2.stat.duke.edu/~pdh10/FCBS/Replication/chapter5.R
> # prior
> # 1/sigma^2 ~ dgamma(nu_0/2, nu_0 sigma^2_0/2)
> s2_0 <- 0.01 ; nu_0 <- 1
> # theta|sigma^2 ~ dnorm(mu_0, sigma/sqrt(k_0))
> mu_0 <- 1.9 ; k_0 <- 1
>
> ## data
> y <- c(1.64, 1.70, 1.72, 1.74, 1.82, 1.82, 1.82, 1.90, 2.08)
> n <- length(y) ; ybar<-mean(y) ; s2 <- var(y)
>
> ## posterior inference
> k_n <- k_0 + n ; nu_n <- nu_0 + n
> mu_n <- (k_0 * mu_0 + n * ybar) / k_n
> s2_n <- (nu_0 * s2_0 + (n-1) * s2 + k_0 * n * (ybar - mu_0)^2 / k_n) / nu_n
> mu_n
```

```
[1] 1.814
```

```r
> s2_n
```

```
[1] 0.015324
```

```r
> # density of inverse gamma beta^alpha / Gamma(alpha) * x^-(alpha+1) exp(-beta/x)
> dinvgamma<-function(x,a,b) {
+   ld <- a * log(b) - lgamma(a) - (a+1) * log(x) - b / x
+   exp(ld)
+ }
>
> # grid size
> gs <- 100
> theta_g <- seq(1.6, 2.0, length = gs)
> is2_g <- seq(15, 160 , length = gs)
> s2_g <- seq(0.001, 0.045, length = gs)
>
> # log density theta inverse sigma square / sigma square
> ld.th.is2 <- ld.th.s2 <- matrix(0, gs, gs)
> for(i in 1:gs) {
+   for(j in 1:gs) {
+     ld.th.is2[i,j]<- dnorm(theta_g[i], mu_n, 1/sqrt(is2_g[j]*k_n), log = TRUE) +
+                     dgamma(is2_g[j], shape = nu_n/2, rate = nu_n*s2_n/2, log = TRUE)
+     ld.th.s2[i,j]<- dnorm(theta_g[i], mu_n, sqrt(s2_g[j]/k_n), log = TRUE) +
+                     log(dinvgamma(s2_g[j], nu_n/2, nu_n * s2_n/2 ))
+   }
+ }
>
> par(mfrow = c(1,2), mar = c(3,3,1,1), mgp = c(1.75, 0.75, 0))
> grays <- gray((10:0)/10)
> image(theta_g, is2_g, exp(ld.th.is2), col = grays, xlab = expression(theta),
+       ylab = expression(sigma^{-2}) )
> image(theta_g, s2_g, exp(ld.th.s2), col = grays, xlab = expression(theta),
+       ylab = expression(sigma^2) )
```



semiconjugate prior $p(\theta, \sigma^2) = p(\theta)p(\sigma^2)$

$$\theta|\sigma^2 \sim \text{normal}(\mu_0, \tau_0^2)$$
$$1/\sigma^2 \sim \text{gamma}(\frac{\nu_0^2}{2}, \frac{\nu_0^2}{2}\sigma_0^2)$$
$$Y_1, \ldots, Y_n|\theta, \sigma^2 \overset{\text{i.}i.d.}{\sim} \text{normal}(\theta, \sigma^2)$$

posterior

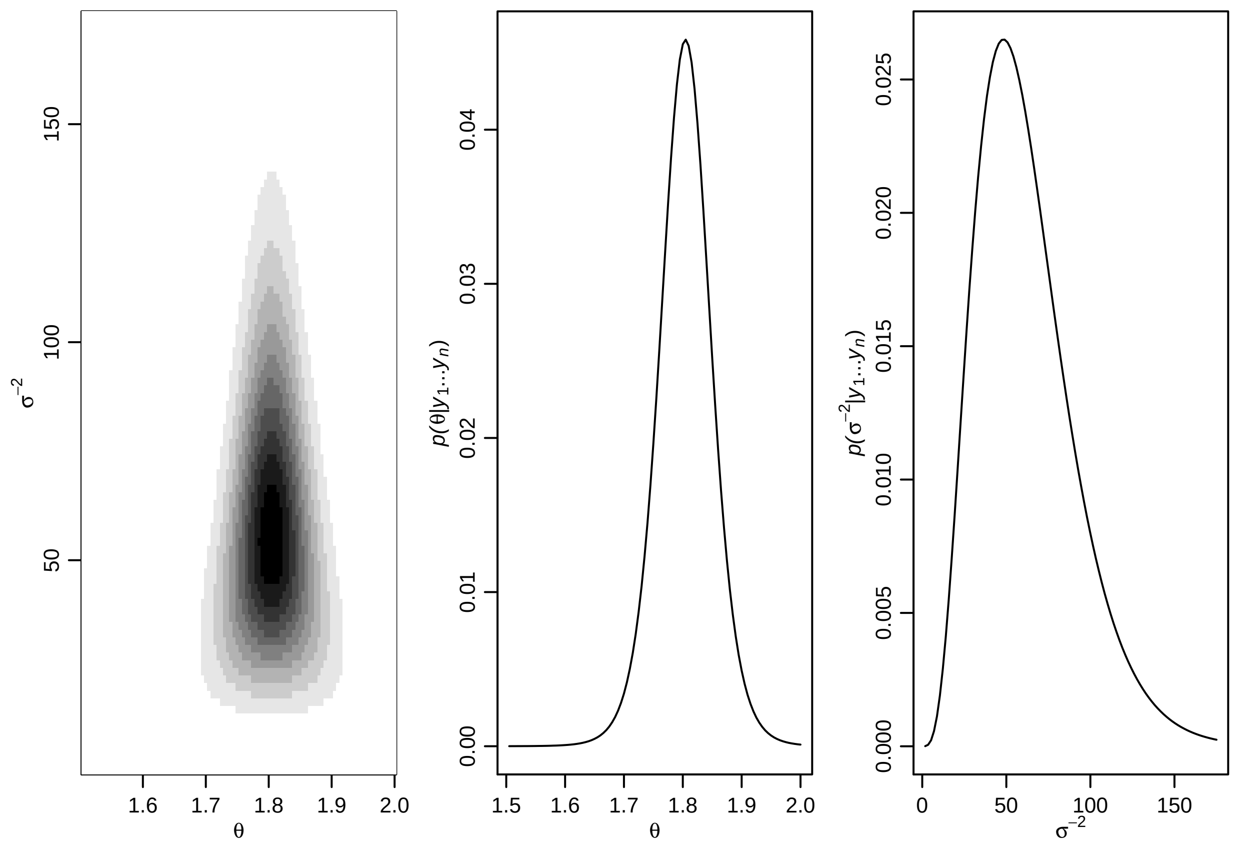
$$\theta|y_1, \ldots, y_n \sim \text{normal}(\mu_n, \tau_n^2)$$
$$1/\sigma^2|y_1, \ldots, y_n \sim \text{some distribution not standard}$$
$$1/\sigma^2|\theta, y_1, \ldots, y_n \sim \text{gamma}(\frac{\nu_n^2}{2}, \frac{\nu_n^2}{2}\sigma_n^2(\theta))$$

$\mu_n = \frac{\mu_0 \tau_0^2 + n\bar{y}/\sigma^2}{1/\tau_0^2 + n/\sigma^2}$ and $\tau_n^2 = \frac{1}{1/\tau_0^2 + n/\sigma^2}$

$\nu_n = \nu_0 + n$ and $\sigma_n^2(\theta) = \frac{1}{\nu_n}[\nu_0^2\sigma_0^2 + ns_n^2(\theta)]$, where $s_n^2(\theta) = \frac{1}{n}\sum_{l=1}^{n}(y_l - \theta)^2 = \frac{1}{n}[(n-1)s^2 + n(\bar{y} - \theta)^2]$

```
> # Fig 6.1 Joint and marginal posterior distributions based on a discrete approximation (theta, sigma^
> # http://www2.stat.duke.edu/~pdh10/FCBS/Replication/chapter6.R
> # prior
> # 1/sigma^2 ~ dgamma(nu_0/2, nu_0 sigma^2_0/2)
> s2_0 <- 0.01 ; nu_0 <- 1
> # theta ~ dnorm(mu_0, tau_0)
> mu_0 <- 1.9  ; t2_0 <- 0.95^2
```

```r
>
> # data
> y <- c(1.64,1.70,1.72,1.74,1.82,1.82,1.82,1.90,2.08)
> n <- length(y) ; ybar <- mean(y) ; s2 <- var(y)
>
>
> # grid size
> gs <- 100
> theta_g <- seq(1.505, 2.00, length = gs)
> is2_g <- seq(1.75, 175, length = gs)
>
> posterior_g <- matrix(0, nrow = gs, ncol = gs)
>
> for(i in 1:gs) {
+   for(j in 1:gs) {
+     posterior_g[i,j]<- dnorm(theta_g[i], mu_0, sqrt(t2_0)) *
+                      dgamma(is2_g[j], nu_0/2, s2_0*nu_0/2 ) *
+                      prod(dnorm(y, theta_g[i], 1/sqrt(is2_g[j])))
+   }
+ }
>
> posterior_g <- posterior_g/sum(posterior_g)
>
> par(mfrow = c(1,3), mar = c(3,3,1,1), mgp = c(1.70,0.70,0))
> image( theta_g, is2_g, posterior_g, col=grays, xlab=expression(theta),
+        ylab=expression(sigma^{-2}))
>
> theta_p<- apply(posterior_g, 1, sum)
> plot(theta_g, theta_p, type="l", xlab=expression(theta),
+      ylab=expression(paste(italic("p("), theta, "|",
+                            italic(y[1]), "...", italic(y[n]), ")", sep="")))
>
> is2_p <- apply(posterior_g, 2, sum)
> plot(is2_g,is2_p,type="l",xlab=expression(sigma^{-2}),
+      ylab=expression(paste(italic("p("), sigma^{-2}, "|",
+                            italic(y[1]), "...", italic(y[n]), ")", sep="")))
```



Gibbs sampler

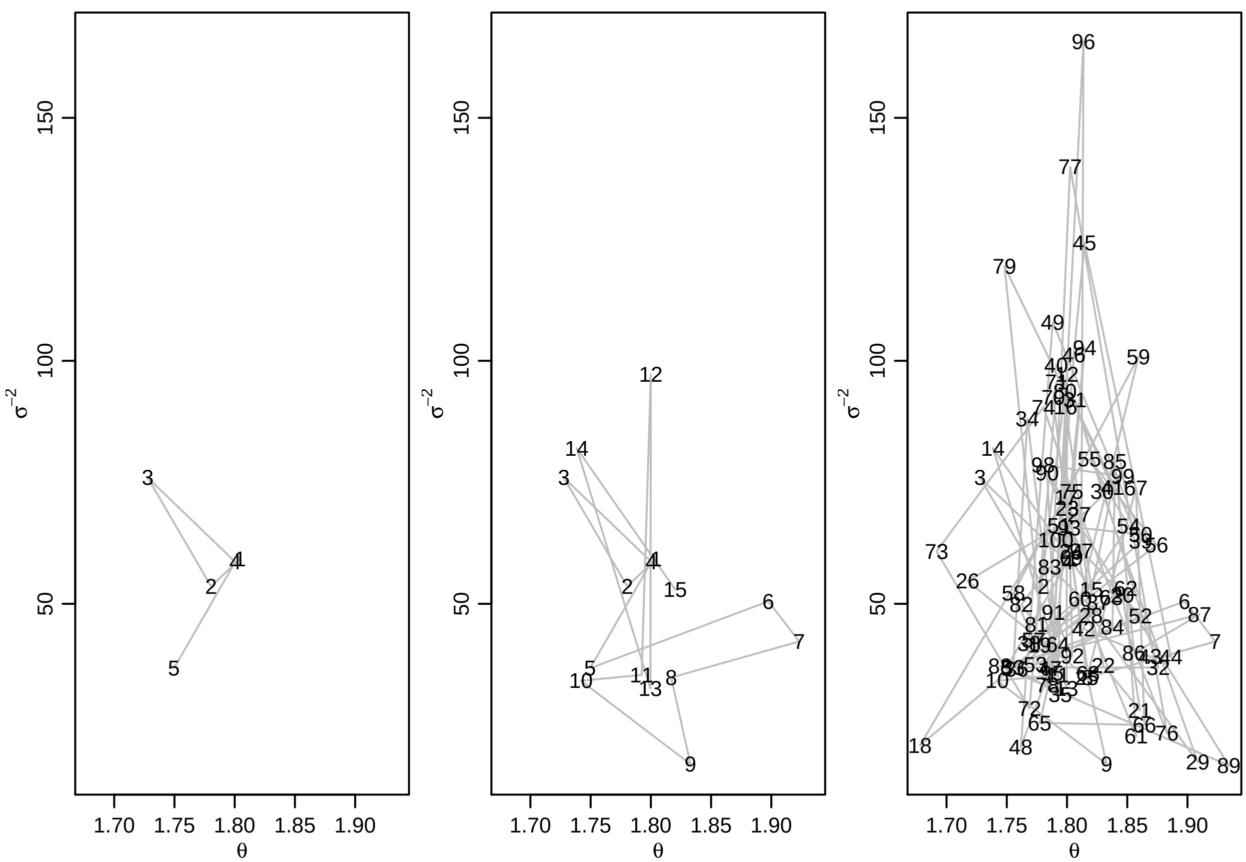
full conditional distribution $p(\theta|\sigma^2, y_1, \ldots, y_n)$ and $p(\sigma^2|\theta, y_1, \ldots, y_n)$

$$\theta|\sigma^2, y_1, \ldots, y_n \stackrel{\theta \perp \sigma}{=} \theta|y_1, \ldots, y_n \sim \text{normal}(\mu_n, \tau_n^2)$$

$$1/\sigma^2|\theta, y_1, \ldots, y_n \sim \text{gamma}(\frac{\nu_n^2}{2}, \frac{\nu_n^2}{2}\sigma_n^2(\theta))$$

```
> # http://www2.stat.duke.edu/~pdh10/FCBS/Replication/chapter6.R
> ## initial values
> set.seed(123)
> nsamp <- 1000
> mphi <- matrix(nrow = nsamp, ncol = 2)
> mphi[1,] <- phi <- c(ybar, 1/s2)
>
> ## Gibbs sampling algorithm
> for(s in 2:nsamp) {
+
+   # generate a new theta value from its full conditional
+   # mu_n = [mu_0/\tau_0^2 + n*ybar/sigma^2]/[1/tau_0^2 + n/sigma^2]
+   mu_n <-  (mu_0/t2_0 + n*ybar*phi[2]) / ( 1/t2_0 + n*phi[2])
+   # tau_n^2 = 1/[1/tau_0^2 + n/sigma^2]
+   t2_n <- 1 / (1/t2_0 + n*phi[2])
+   phi[1] <- rnorm(1, mu_n, sqrt(t2_n))
+
+   # generate a new sigma^2 value from its full conditional
+   nu_n <- nu_0 + n
```

```
+   # sigma_n^2(\theta) = 1 / nu_n * [nu_0^2*sigma_0^2 + (n-1)*s^2 + n(ybar-theta)^2]
+   s2_n <- (nu_0*s2_0 + (n-1)*s2 + n*(ybar-phi[1])^2) /nu_n
+   phi[2] <- rgamma(1, nu_n/2, nu_n*s2_n/2)
+
+   mphi[s,]<-phi
+ }
>
> # Fig 6.2 The first 5, 15 and 100 iterations of a Gibbs sampler
> par(mfrow = c(1,3), mar = c(2.75,2.75,0.5,0.5), mgp = c(1.70,0.70,0))
> m1<-5
> plot(mphi[1:m1,], type = "l", xlim = range(mphi[1:100,1]), ylim = range(mphi[1:100,2]),
+      lty = 1, col = "gray", xlab = expression(theta), ylab = expression(sigma^{-2}))
> text(mphi[1:m1,1], mphi[1:m1,2], c(1:m1) )
>
> m1<-15
> plot(mphi[1:m1,], type = "l", xlim = range(mphi[1:100,1]), ylim = range(mphi[1:100,2]),
+      lty = 1, col = "gray", xlab = expression(theta), ylab = expression(sigma^{-2}))
> text(mphi[1:m1,1], mphi[1:m1,2], c(1:m1) )
>
> m1<-100
> plot(mphi[1:m1,], type = "l", xlim = range(mphi[1:100,1]), ylim = range(mphi[1:100,2]),
+      lty = 1, col = "gray", xlab = expression(theta), ylab = expression(sigma^{-2}))
> text(mphi[1:m1,1], mphi[1:m1,2], c(1:m1) )
```

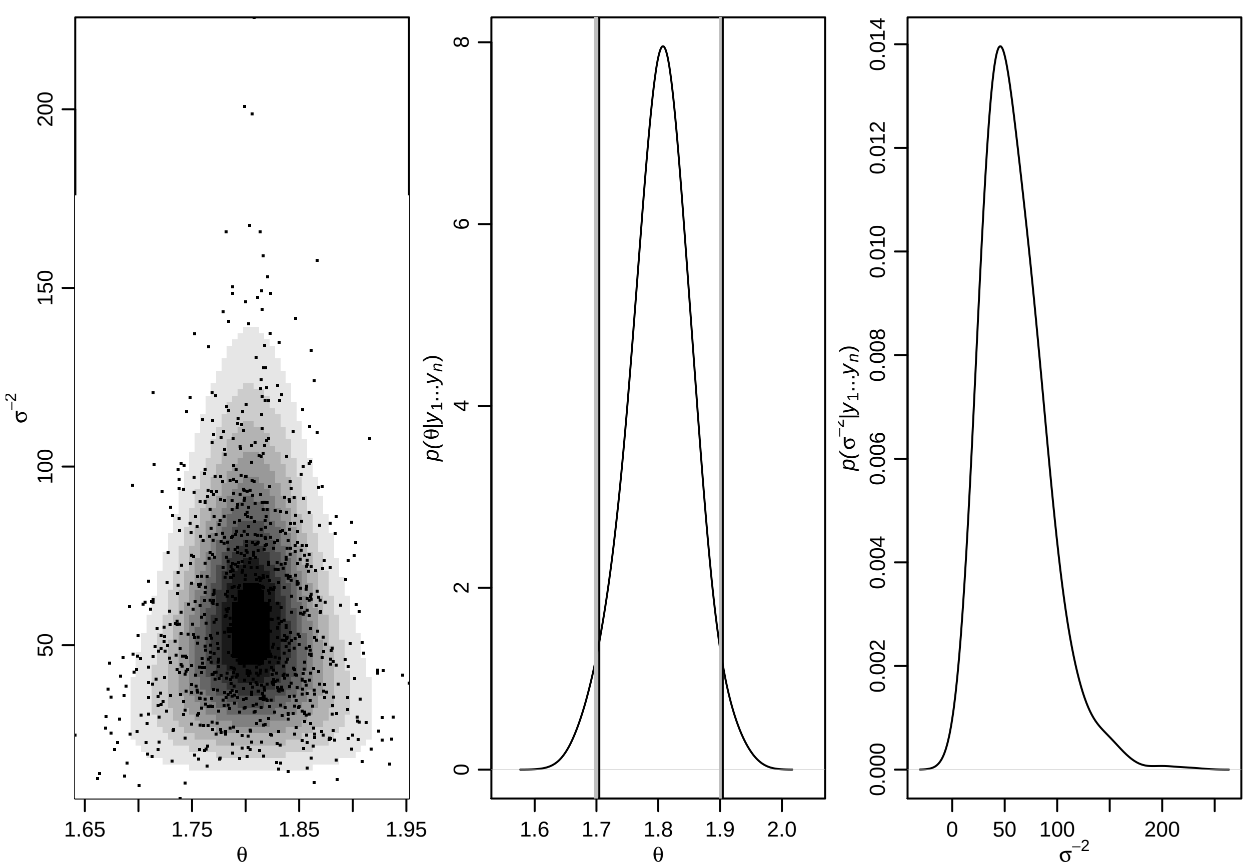



```
> # Fig 6.2 distribution of Gibbs sample
> par(mfrow = c(1,3), mar = c(2.75,2.75,0.5,0.5), mgp = c(1.70,0.70,0))
```

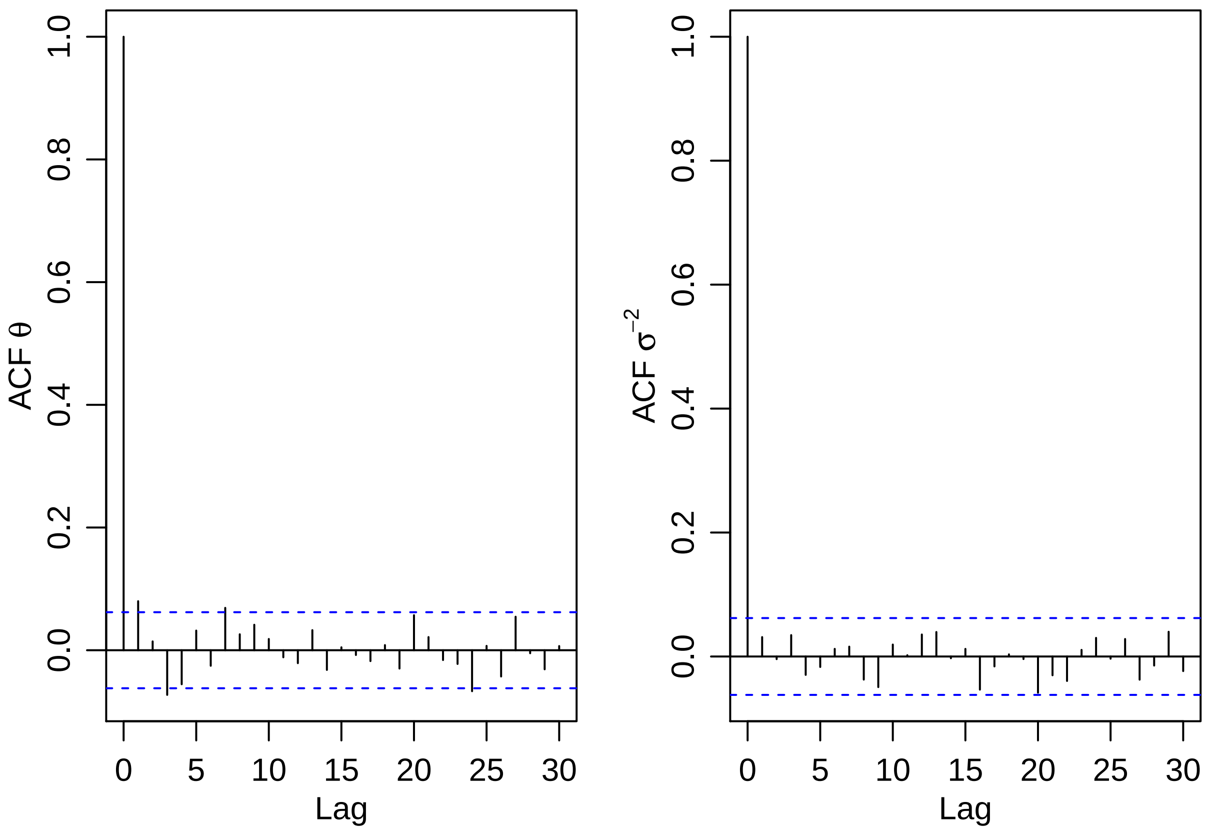
```r
>
> image(theta_g, is2_g, posterior_g, col = grays,
+        xlab = expression(theta), ylab = expression(sigma^{-2}),
+        xlim = range(mphi[,1]), ylim = range(mphi[,2]) )
> points(mphi[,1], mphi[,2], pch = ".", cex = 1.25 )
>
> plot(density(mphi[,1], adj = 2),  xlab = expression(theta), main = "", xlim = c(1.55,2.05),
+      ylab = expression(paste(italic("p("), theta, "|", italic(y[1]),
+                               "...", italic(y[n]), ")", sep = "")))
> abline(v = quantile(mphi[,1], prob = c(0.025,0.975)), lwd = 2, col = "gray")
>
> ## t-test based confidence interval
> ybar + qt(c(.025,.975), n-1) *sqrt(s2/n)
```

```
[1] 1.704583 1.904306
```

```r
> abline(v =  ybar + qt(c(0.025,0.975), n-1) * sqrt(s2/n), col = "black", lwd = 1)
>
> plot(density(mphi[,2], adj = 2), xlab = expression(sigma^{-2}), main = "",
+      ylab = expression(paste(italic("p("), sigma^{-2}, "|", italic(y[1]),
+                               "...", italic(y[n]), ")",sep = "")))
```




```r
> par(mfrow = c(1,2), mar = c(3,3,1,1), mgp = c(1.75, 0.75, 0))
> acf(mphi[,1], ylab = expression(paste("ACF ",theta)))
> acf(mphi[,2], ylab = expression(paste("ACF ",sigma^{-2})))
```


ACF θ
1.0
0.8
0.6
0.4
0.2
0.0
0
5
10
15
20
25
30
Lag
ACF σ⁻²
Lag