Pinsker (1980)

$$\eta(t) = \theta(t) + \xi(t),$$

where $\theta(t) \in \mathcal{S}(Q)$ and $\xi(t)$ is a Gaussian random process on $E = [0, T]$ with $\mathbb{E}\{\xi(t)\} = 0$

Estimator $\hat{\theta}(t) = \hat{\theta}(t; \eta(s), s \in E) \in L_2(E)$

Ellipsoid $\mathcal{S}(Q) = \{\theta(t) \in L_2(E) : \sum_j a_j \theta_j^2 \leq Q\}$ with nonnegative sequence $a_j \geq 0$, number $Q > 0$ and a complete orthonormal basis $\{\varphi_j\} \subset L_2(E)$, $j = 0, \pm 1, ...$

Equivalently, for $j = 0, \pm 1, \ldots$

$$\eta_j = \theta_j + \xi_j$$
$$\hat{\theta}_j = \hat{\theta}_j(\eta_k, k = 0, \pm 1, \ldots),$$

where $\eta_j = \int_E \eta(t)\varphi_j(t)dt$, $\theta_j = \int_E \theta(t)\varphi_j(t)dt$, $\xi_j = \int_E \xi(t)\varphi_j(t)dt$ with $\sigma_j^2 = \mathbb{E}\{\xi_j^2\}$.

minimax risk

$$R_n = \inf_{\hat{\theta}} \sup_{\theta \in \mathcal{S}(E)} \mathrm{MISE}(\hat{\theta}(t); \theta(t)) = \inf \sup \mathbb{E}_\theta \left\{ \int_E (\hat{\theta}(t) - \theta(t))^2 dt \right\} = \inf \sup \sum_j \mathbb{E}_\theta\{(\theta_j - \tilde{\theta}_j)^2\}$$

Consider linear estimates $\tilde{\theta}_j = \lambda_j \eta_j = \lambda_j(\theta_j + \xi_j)$. Then maximin risk (at $\lambda_j = \theta_j^2/(\theta_j^2 + \sigma_j^2)$)

$$r_n = \sup_{\theta \in \mathcal{S}(E)} \inf_\lambda \sum_j \mathbb{E}_\theta\{(\theta_j - \tilde{\theta}_j)^2\} = \sup_{\theta \in \mathcal{S}(E)} \frac{\theta_j^2 \sigma_j^2}{\theta_j^2 + \sigma_j^2} \leq \inf_{\hat{\theta}} \sup_{\theta \in \mathcal{S}(E)} \mathrm{MISE}(\hat{\theta}(t); \theta(t)) = R_n$$

(1) if for any $d > 0$, $\sum_{j: a_j \leq d} \sigma_j^2 < \infty$, then set $\theta_j^{*2} = \sigma_j^2 \max\{(\mu a_j)^{-1/2} - 1, 0\}$, with $\mu > 0$ s.t. $\sum_j a_j \theta_j^{*2} = Q$.

$$r_n = \sum_j \frac{\theta_j^{*2} \sigma_j^2}{\theta_j^{*2} + \sigma_j^2} = \sum_j \sigma_j^2 \max\{1 - (\mu a_j)^{1/2}, 0\}.$$

$$
\begin{aligned}
\sum_j \mathbb{E}_\theta\{(\theta_j-\tilde\theta_j)^2\} &= \sum_{j:\mu a_j<1} \mathbb{E}_\theta(\theta_j-(1-(\mu a_j)^{1/2})\eta_j)^2 + \sum_{j:\mu a_j\geq 1}\theta_j^2 \\
&= \sum_{j:\mu a_j<1} \mathbb{E}_\theta((1-(\mu a_j)^{1/2})\xi_j-(\mu a_j)^{1/2}\theta_j)^2 + \sum_{j:\mu a_j\geq 1}\theta_j^2 \\
&\leq \sum_{j:\mu a_j<1}(1-(\mu a_j)^{1/2})^2\sigma_j^2 + \sum_j \mu a_j\theta_j^2 \\
&\leq \sum_{j:\mu a_j<1}(1-2(\mu a_j)^{1/2}+\mu a_j)\sigma_j^2 + \mu Q \\
&= \sum_{j:\mu a_j<1}(1-2(\mu a_j)^{1/2}+\mu a_j)\sigma_j^2 + \mu\sum_{j:\mu a_j<1}\sigma_j^2 a_j((\mu a_j)^{-1/2}-1) \\
&= \sum_{j:\mu a_j<1}\sigma_j^2(1-(\mu a_j)^{1/2}) = r_n
\end{aligned}
$$

(2) if there exists some d s.t. $\sum_{j:a_j\leq d}\sigma_j^2=\infty$, set $d_*=\inf d$. If $\mu>0$ exists, r_n follows (1).

(3) if $\mu>0$ does not exists and there exists $d_*>0$, set $\theta_j^{*2}=\sigma_j^2\max\{(d_*^{-1}a_j)^{-1/2}-1,0\}$ and $\theta^{*2}=\left(Q-\sum_{j:a_j<d}a_j\theta_j^{*2}\right)d_*^{-1}$. $r_n=\sum\limits_{j:a_j<d_*}\theta_j^{*2}\sigma_j^2/(\theta_j^{*2}+\sigma_j^2)+\theta^{*2}$

(4) otherwise, $r_n=\infty$

Theorem 1 $R_n = r_n(1+o(1))$

1. For linear estimates

$$
\tilde\theta(t)=\begin{cases}\sum\limits_j \max\{1-(\mu a_j)^{1/2},0\}\eta_j\varphi_j(t), & \exists\mu>0,\\ \sum\limits_j \max\{1-(d_*^{-1}a_j)^{1/2},0\}\eta_j\varphi_j(t), & \nexists\mu>0,\ \exists d_*>0,\\ 0, & d_*=0,\end{cases}
$$

$$
R_n\leq \mathbb{E}_\theta\left\{\int_E(\tilde\theta(t)-\theta(t))^2dt\right\}\leq r_n.
$$

2. If the random variables ξ_j's are independent, then

$$
R_n\geq r_n(1-\psi(\rho)),
$$

where

$$
\rho=\begin{cases}\min\limits_{j:\mu a_j<1} r_n/\sigma_j^2, & \exists\mu>0,\\ \inf\limits_j r_n/\sigma_j^2, & \nexists\mu>0\end{cases}
$$

and the function $\psi(\rho)$ satisfies conditions $0\leq\psi(\rho)<1-c_*$, $c_*>0$ and $\lim\limits_{\rho\to\infty}\psi(\rho)=0$.

Theorem 2 *$E = [0,1]$ and Gaussian $\xi(t) = \epsilon\xi_0(t)$ with independent $\xi_{0j} = \int_0^1 \xi_0(t)\varphi(t)dt$ and $\sigma_j^2 = \mathbb{E}\{\xi_j^2\} = \epsilon^2\mathbb{E}\{\xi_{0j}^2\} = \epsilon^2\sigma_{0j}^2$.*

1. *if for any $d > 0$, $\sum\limits_{j:a_j \le d} \sigma_j^2 < \infty \Longleftrightarrow R_n \to 0$ as $\epsilon^2 \to 0$*
2. *$\sup\limits_{j:a_j<d} \sigma_j^2 = o\Big(\sum\limits_{j:a_j<d} \sigma_j^2\Big)$ as $d \to \infty \Longrightarrow R_n = r_n(1+o(1))$ as $\epsilon^2 \to 0$*

Example Orthonormal basis $\varphi_0(t) = 1$, $\varphi_j(t) = \sqrt{2}\sin(2\pi jt)$, $\varphi_{-j}(t) = \sqrt{2}\cos(2\pi jt)$, $j = 1,2,...$ and white noise $\sigma_j^2 = \epsilon^2\sigma^2$

(1) $a_j = |j|^{2\alpha}$, $\alpha > 0$, i.e. for $\theta(t) \in \mathcal{S}(Q)$, $(2\pi)^{-2\alpha}\int_0^1(\theta^{(\alpha)}(t))^2dt \le Q$. Then

$$R_n = r_n(1+o(1)) = [Q(2\alpha+1)]^{1/(2\alpha+1)}\left(\frac{2\epsilon^2\sigma^2\alpha}{\alpha+1}\right)^{\frac{2\alpha}{2\alpha+1}}(1+o(1)), \qquad \epsilon^2 \to 0.$$

(2) $a_j = e^{2\alpha|j|}, \alpha > 0$. Then

$$R_n = r_n(1+o(1)) = \frac{\epsilon^2\sigma^2}{\alpha}\ln\Big(\frac{\alpha Q}{\epsilon^2\sigma^2}\Big)(1+o(1)), \qquad \epsilon^2 \to 0.$$

Theorem 3 *$E = [0,T]$ and Gaussian $\xi_T(t)$ with independent $\xi_{T,j}$'s. Orthogonal basis $\varphi_{T,j}(t)$'s, nonnegative sequence $\{a_{T,j}, j = 0, \pm 1, \dots\}$ and $\sum_j a_{T,j}\theta_j^2 \le Q_T$*

1. *$\lim\limits_{T\to\infty} T^{-1}R_n \Longleftrightarrow \lim\limits_{T\to\infty} T^{-1}r_n = 0$.*
2. *If $\rho_T \to \infty$ as $T \to \infty$, then $R_n = r_n(1+o(1))$ as $T \to \infty$.*

Example Orthonormal basis $\varphi_0(t) = T^{-1/2}$, $\varphi_{T,j}(t) = \sqrt{2/T}\sin(2\pi jt)$, $\varphi_{T,-j}(t) = \sqrt{2/T}\cos(2\pi jt)$, $j = 1,2,...$, number $Q_T = QT$ and white noise $\sigma_{T,j}^2 = \sigma^2$

(1) $a_{T,j} = |j|^{2\alpha}/T^{2\alpha}$. Then

$$R_n = r_n(1+o(1)) = T[Q(2\alpha+1)]^{1/(2\alpha+1)}\left(\frac{2\alpha\sigma^2}{\alpha+1}\right)^{2\alpha/(2\alpha+1)}(1+o(1)), \qquad T \to \infty.$$

(2) $a_{T,j} = e^{2\alpha|j|/T}, \alpha > 0$. Then

$$R_n = r_n(1+o(1)) = \frac{T\sigma^2}{\alpha}\ln\Big(\frac{\alpha Q}{\sigma^2}\Big)(1+o(1)), \qquad T \to \infty.$$

Proof sketches

Define a sequence of random variable ζ_j's independent of ξ_j's s.t. $\sum_j \mathbb{E}\{(\zeta_j - \hat{\zeta}_j)^2\} \ge r_n(1+\psi(\rho))$ with $\eta_k = \zeta_k + \xi_k$. Note Bayes estimator $\hat{\zeta}_j = \mathbb{E}\{\zeta_j|\eta_k, k = 0, \pm 1, \dots\}$ is admissible

When $\exists\mu > 0$, define partition $\mathcal{N} = \{a_j : \mu a_j < 1\} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$

$\eta = \zeta + \xi$ with independent ζ and $\xi \sim N(0, \sigma^2)$

(1) $\mathcal{N}_1 = \{a_j : q^{-1} < \theta_j^{*2}/\sigma_j^2 < q\}$

Since $\theta_j^{*2} = \max\{(\mu a_j)^{-1/2} - 1, 0\}$, $a_i/a_j < q^2$ for $a_i, a_j \in \mathcal{N}_1$.

$\zeta \sim N(0, \theta^2)$. Then $\mathbb{E}\{(\zeta - \mathbb{E}\{\zeta|\eta\})^2\} = \frac{\theta^2\sigma^2}{\theta^2+\sigma^2} = \theta^2(1 + \theta^2/\sigma^2)^{-1} = \sigma^2(1 + \sigma^2/\theta^2)^{-1}$

(2) $\mathcal{N}_2 = \{a_j : \theta_j^{*2}/\sigma_j^2 \leq q^{-1}\}$

$\zeta \sim \mathbb{P}(\zeta = \theta) = \mathbb{P}(\zeta = -\theta) = 0.5$. Then $\mathbb{E}\{(\zeta - \mathbb{E}\{\zeta|\eta\})^2\} = \theta^2(1 + o(1))$ as $\theta^2/\sigma^2 \to 0$

(3) $\mathcal{N}_3 = \{a_j : \theta_j^{*2}/\sigma_j^2 \geq q\}$

$\zeta \sim U(-\theta, \theta)$. Then $\mathbb{E}\{(\zeta - \mathbb{E}\{\zeta|\eta\})^2\} = \sigma^2(1 + o(1))$ as $\theta^2/\sigma^2 \to \infty$