

Pinsker (1980)

$$\eta(t) = \theta(t) + \xi(t),$$

where $\theta(t) \in \mathcal{S}(Q)$ and $\xi(t)$ is a Gaussian random process on $E = [0, T]$ with $\mathbb{E}\{\xi(t)\} = 0$

Estimator $\hat{\theta}(t) = \hat{\theta}(t; \eta(s), s \in E) \in L_2(E)$

Ellipsoid $\mathcal{S}(Q) = \{\theta(t) \in L_2(E) : \sum_j a_j \theta_j^2 \leq Q\}$ with nonnegative sequence $a_j \geq 0$, number $Q > 0$ and a complete orthonormal basis $\{\varphi_j\} \subset L_2(E)$, $j = 0, \pm 1, \dots$

Equivalently, for $j = 0, \pm 1, \dots$

$$\eta_j = \theta_j + \xi_j$$

$$\hat{\theta}_j = \hat{\theta}_j(\eta_k, k = 0, \pm 1, \dots),$$

where $\eta_j = \int_E \eta(t) \varphi_j(t) dt$, $\theta_j = \int_E \theta(t) \varphi_j(t) dt$, $\xi_j = \int_E \xi(t) \varphi_j(t) dt$ with $\sigma_j^2 = \mathbb{E}\{\xi_j^2\}$.

minimax risk

$$R_n = \inf_{\hat{\theta}} \sup_{\theta \in \mathcal{S}(E)} \text{MISE}(\hat{\theta}(t); \theta(t)) = \inf \sup_{\theta} \mathbb{E}_{\theta} \left\{ \int_E (\hat{\theta}(t) - \theta(t))^2 dt \right\} = \inf \sup_{\theta} \sum_j \mathbb{E}_{\theta} \{(\theta_j - \tilde{\theta}_j)^2\}$$

Consider linear estimates $\tilde{\theta}_j = \lambda_j \eta_j = \lambda_j (\theta_j + \xi_j)$. Then maximin risk (at $\lambda_j = \theta_j^2 / (\theta_j^2 + \sigma_j^2)$)

$$r_n = \sup_{\theta \in \mathcal{S}(E)} \inf_{\lambda} \sum_j \mathbb{E}_{\theta} \{(\theta_j - \tilde{\theta}_j)^2\} = \sup_{\theta \in \mathcal{S}(E)} \frac{\theta_j^2 \sigma_j^2}{\theta_j^2 + \sigma_j^2} \leq \inf_{\hat{\theta}} \sup_{\theta \in \mathcal{S}(E)} \text{MISE}(\hat{\theta}(t); \theta(t)) = R_n$$

(1) if for any $d > 0$, $\sum_{j: a_j \leq d} \sigma_j^2 < \infty$, then set $\theta_j^{*2} = \sigma_j^2 \max\{(\mu a_j)^{-1/2} - 1, 0\}$, with $\mu > 0$ s.t.

$$\sum_j a_j \theta_j^{*2} = Q.$$

$$r_n = \sum_j \frac{\theta_j^{*2} \sigma_j^2}{\theta_j^{*2} + \sigma_j^2} = \sum_j \sigma_j^2 \max\{1 - (\mu a_j)^{1/2}, 0\}.$$

$$\begin{aligned}
\sum_j \mathbb{E}_\theta \{(\theta_j - \tilde{\theta}_j)^2\} &= \sum_{j:\mu a_j < 1} \mathbb{E}_\theta (\theta_j - (1 - (\mu a_j)^{1/2})\eta_j)^2 + \sum_{j:\mu a_j \geq 1} \theta_j^2 \\
&= \sum_{j:\mu a_j < 1} \mathbb{E}_\theta ((1 - (\mu a_j)^{1/2})\xi_j - (\mu a_j)^{1/2}\theta_j)^2 + \sum_{j:\mu a_j \geq 1} \theta_j^2 \\
&\leq \sum_{j:\mu a_j < 1} (1 - (\mu a_j)^{1/2})^2 \sigma_j^2 + \sum_j \mu a_j \theta_j^2 \\
&\leq \sum_{j:\mu a_j < 1} (1 - 2(\mu a_j)^{1/2} + \mu a_j) \sigma_j^2 + \mu Q \\
&= \sum_{j:\mu a_j < 1} (1 - 2(\mu a_j)^{1/2} + \mu a_j) \sigma_j^2 + \mu \sum_{j:\mu a_j < 1} \sigma_j^2 a_j ((\mu a_j)^{-1/2} - 1) \\
&= \sum_{j:\mu a_j < 1} \sigma_j^2 (1 - (\mu a_j)^{1/2}) = r_n
\end{aligned}$$

(2) if there exists some d s.t. $\sum_{j:a_j \leq d} \sigma_j^2 = \infty$, set $d_* = \inf d$. If $\mu > 0$ exists, r_n follows (1).

(3) if $\mu > 0$ does not exist and there exists $d_* > 0$, set $\theta_j^{*2} = \sigma_j^2 \max\{(d_*^{-1} a_j)^{-1/2} - 1, 0\}$ and $\theta^{*2} = (Q - \sum_{j:a_j < d_*} a_j \theta_j^{*2}) d_*^{-1}$. $r_n = \sum_{j:a_j < d_*} \theta_j^{*2} \sigma_j^2 / (\theta_j^{*2} + \sigma_j^2) + \theta^{*2}$

(4) otherwise, $r_n = \infty$

Theorem 1 $R_n = r_n(1 + o(1))$

1. For linear estimates

$$\tilde{\theta}(t) = \begin{cases} \sum_j \max\{1 - (\mu a_j)^{1/2}, 0\} \eta_j \varphi_j(t), & \exists \mu > 0, \\ \sum_j \max\{1 - (d_*^{-1} a_j)^{1/2}, 0\} \eta_j \varphi_j(t), & \nexists \mu > 0, \exists d_* > 0, \\ 0, & d_* = 0, \end{cases}$$

$$R_n \leq \mathbb{E}_\theta \left\{ \int_E (\tilde{\theta}(t) - \theta(t))^2 dt \right\} \leq r_n.$$

2. If the random variables ξ_j 's are independent, then

$$R_n \geq r_n(1 - \psi(\rho)),$$

where

$$\rho = \begin{cases} \min_{j:\mu a_j < 1} r_n / \sigma_j^2, & \exists \mu > 0, \\ \inf_j r_n / \sigma_j^2, & \nexists \mu > 0 \end{cases}$$

and the function $\psi(\rho)$ satisfies conditions $0 \leq \psi(\rho) < 1 - c_*$, $c_* > 0$ and $\lim_{\rho \rightarrow \infty} \psi(\rho) = 0$.

Theorem 2 $E = [0, 1]$ and Gaussian $\xi(t) = \epsilon \xi_0(t)$ with independent $\xi_{0j} = \int_0^1 \xi_0(t) \varphi(t) dt$ and $\sigma_j^2 = \mathbb{E}\{\xi_j^2\} = \epsilon^2 \mathbb{E}\{\xi_{0j}^2\} = \epsilon^2 \sigma_{0j}^2$.

1. if for any $d > 0$, $\sum_{j: a_j \leq d} \sigma_j^2 < \infty \iff R_n \rightarrow 0$ as $\epsilon^2 \rightarrow 0$
2. $\sup_{j: a_j < d} \sigma_j^2 = o\left(\sum_{j: a_j < d} \sigma_j^2\right)$ as $d \rightarrow \infty \implies R_n = r_n(1 + o(1))$ as $\epsilon^2 \rightarrow 0$

Example Orthonormal basis $\varphi_0(t) = 1$, $\varphi_j(t) = \sqrt{2} \sin(2\pi jt)$, $\varphi_{-j}(t) = \sqrt{2} \cos(2\pi jt)$, $j = 1, 2, \dots$ and white noise $\sigma_j^2 = \epsilon^2 \sigma^2$

- (1) $a_j = |j|^{2\alpha}$, $\alpha > 0$, i.e. for $\theta(t) \in \mathcal{S}(Q)$, $(2\pi)^{-2\alpha} \int_0^1 (\theta^{(\alpha)}(t))^2 dt \leq Q$. Then

$$R_n = r_n(1 + o(1)) = [Q(2\alpha + 1)]^{1/(2\alpha+1)} \left(\frac{2\epsilon^2 \sigma^2 \alpha}{\alpha + 1} \right)^{\frac{2\alpha}{2\alpha+1}} (1 + o(1)), \quad \epsilon^2 \rightarrow 0.$$

- (2) $a_j = e^{2\alpha|j|}$, $\alpha > 0$. Then

$$R_n = r_n(1 + o(1)) = \frac{\epsilon^2 \sigma^2}{\alpha} \ln \left(\frac{\alpha Q}{\epsilon^2 \sigma^2} \right) (1 + o(1)), \quad \epsilon^2 \rightarrow 0.$$

Theorem 3 $E = [0, T]$ and Gaussian $\xi_T(t)$ with independent $\xi_{T,j}$'s. Orthogonal basis $\varphi_{T,j}(t)$'s, nonnegative sequence $\{a_{T,j}, j = 0, \pm 1, \dots\}$ and $\sum_j a_{T,j} \theta_j^2 \leq Q_T$

1. $\lim_{T \rightarrow \infty} T^{-1} R_n \iff \lim_{T \rightarrow \infty} T^{-1} r_n = 0$.
2. If $\rho_T \rightarrow \infty$ as $T \rightarrow \infty$, then $R_n = r_n(1 + o(1))$ as $T \rightarrow \infty$.

Example Orthonormal basis $\varphi_0(t) = T^{-1/2}$, $\varphi_{T,j}(t) = \sqrt{2/T} \sin(2\pi jt)$, $\varphi_{T,-j}(t) = \sqrt{2/T} \cos(2\pi jt)$, $j = 1, 2, \dots$, number $Q_T = QT$ and white noise $\sigma_{T,j}^2 = \sigma^2$

- (1) $a_{T,j} = |j|^{2\alpha}/T^{2\alpha}$. Then

$$R_n = r_n(1 + o(1)) = T[Q(2\alpha + 1)]^{1/(2\alpha+1)} \left(\frac{2\alpha \sigma^2}{\alpha + 1} \right)^{2\alpha/(2\alpha+1)} (1 + o(1)), \quad T \rightarrow \infty.$$

- (2) $a_{T,j} = e^{2\alpha|j|/T}$, $\alpha > 0$. Then

$$R_n = r_n(1 + o(1)) = \frac{T \sigma^2}{\alpha} \ln \left(\frac{\alpha Q}{\sigma^2} \right) (1 + o(1)), \quad T \rightarrow \infty.$$

Proof sketches

Define a sequence of random variable ζ_j 's independent of ξ_j 's s.t. $\sum_j \mathbb{E}\{(\zeta_j - \hat{\zeta}_j)^2\} \geq r_n(1 + \psi(\rho))$ with $\eta_k = \zeta_k + \xi_k$. Note Bayes estimator $\hat{\zeta}_j = \mathbb{E}\{\zeta_j | \eta_k, k = 0, \pm 1, \dots\}$ is admissible

When $\exists \mu > 0$, define partition $\mathcal{N} = \{a_j : \mu a_j < 1\} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$

$\eta = \zeta + \xi$ with independent ζ and $\xi \sim N(0, \sigma^2)$

$$(1) \mathcal{N}_1 = \{a_j : q^{-1} < \theta_j^{*2}/\sigma_j^2 < q\}$$

Since $\theta_j^{*2} = \max\{(\mu a_j)^{-1/2} - 1, 0\}$, $a_i/a_j < q^2$ for $a_i, a_j \in \mathcal{N}_1$.

$$\zeta \sim N(0, \theta^2). \text{ Then } \mathbb{E}\{(\zeta - \mathbb{E}\{\zeta|\eta\})^2\} = \frac{\theta^2\sigma^2}{\theta^2 + \sigma^2} = \theta^2(1 + \theta^2/\sigma^2)^{-1} = \sigma^2(1 + \sigma^2/\theta^2)^{-1}$$

$$(2) \mathcal{N}_2 = \{a_j : \theta_j^{*2}/\sigma_j^2 \leq q^{-1}\}$$

$\zeta \sim \mathbb{P}(\zeta = \theta) = \mathbb{P}(\zeta = -\theta) = 0.5$. Then $\mathbb{E}\{(\zeta - \mathbb{E}\{\zeta|\eta\})^2\} = \theta^2(1 + o(1))$ as $\theta^2/\sigma^2 \rightarrow 0$

$$(3) \mathcal{N}_3 = \{a_j : \theta_j^{*2}/\sigma_j^2 \geq q\}$$

$\zeta \sim U(-\theta, \theta)$. Then $\mathbb{E}\{(\zeta - \mathbb{E}\{\zeta|\eta\})^2\} = \sigma^2(1 + o(1))$ as $\theta^2/\sigma^2 \rightarrow \infty$