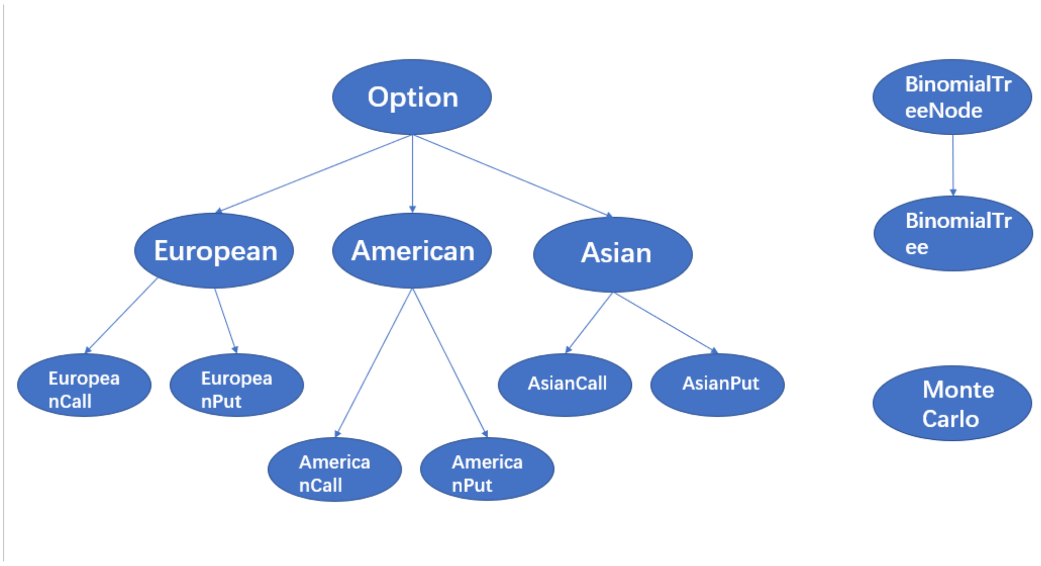
Part 1 Explanation of the code

The classes structure is as follow

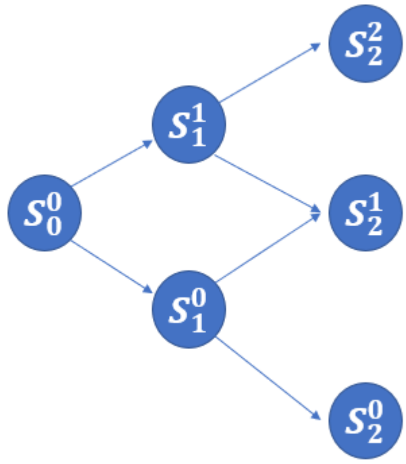


The "Option" class is used to initialize basic data, such as interest rates, volatility, maturity dates, stock prices, etc .; "European", "American" and "Asian" are the core parts, which are priced using different methods; the classes in the bottom layer of the figure are used to store different payoff functions. The classes on the right are used to construct a binary tree. For European and American options, we only need to use the "BinomialTree" class, which will generate the parameters required by the binary tree algorithm. The binary tree algorithm of Asian options needs to track the path, so the "BinomialTreeNode" class is used to record each point, and the points are connected by pointers, which will be explained in detail later.

1. Binomial Tree

1.1 European Option

In the class “European”, first we call the function “GetSharePrice” of the class “BinomialTree” to obtain the last stock price; then use the “EuropeanPayoff” function to obtain the corresponding option value and store into the vector “currentPrice”. The loop is to iterate from the end of the binary tree to the head, so the “currentPrice” vector is continuously updated until the option price at t = 0 is obtained.

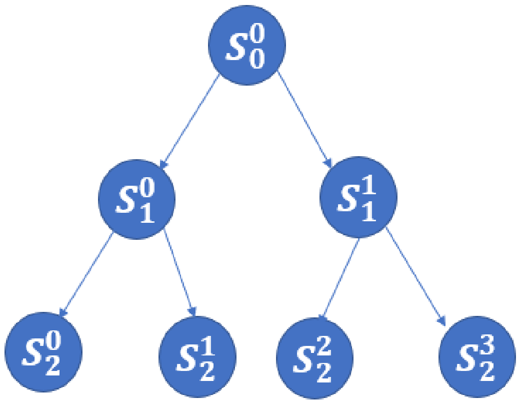


1.2 American Option

The practice of American options is very similar to European options. The only difference is that at each node, it is necessary to determine whether to exercise immediately. This is reflected in “AmericanPayoff”, and when updating the vector “currentPrice”, it is necessary to determine whether to exercise immediately.

1.3 Asian Option

Because it is related to the path, we use the pointer to track the path generated by the stock price. First, we construct a node class "BinomialTreeNode" used to generate each node, containing two elements (the value of the node and the pointer to the previous node parent). The class "BinomialTree" inherits from "BinomialTreeNode", where the function "MakeTreeNode" is used to generate a new node.



In the class "Asian", function “PriceAsianOptionWithBinomialTree ()” is divided into three steps to solve. After processing all the nodes of one layer and then processing the next layer. The first step is to generate a binary tree with pointers linking nodes; the second step is to find the average corresponding to each node at the bottom of the binary tree; the third step is similar to European options, to find the corresponding option value when t = 0. See the code comments for specific details, I wrote in great detail.

2. PDE

2.1 European Option

The PDE formula for European options is

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 = rV$$

Where V is the option value, S is the share price, r is the interest rate, and σ^2 is the volatility. When the range of S is [S1, S2], the boundary conditions for call options are

$$V(S,T) = \max(S-K,0)$$

$$V(S1,t) = V(S2,t) = K$$

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Then we make some transformations on this PDE

$$\frac{\partial V}{\partial t} = rV - rS\frac{\partial V}{\partial S} - \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2$$

$$\Leftrightarrow \frac{\partial V}{\partial t} = rV - rS\frac{V(S+\Delta S)-V(S)}{\Delta S} - \frac{1}{2}\sigma^2 S^2\frac{V(S+\Delta S)+V(S-\Delta S)-2V(S)}{\Delta S^2}$$

Assuming $S = \Delta S * n$, $V_n(t) = V(\Delta S * n, t)$, then the above formula is equivalent to

$$\frac{V(t)-V(t-\Delta t)}{\Delta t} = rV_n - rn(V_{n+1}-V_n) - \frac{1}{2}\sigma^2 n^2(V_{n+1}+V_{n-1}-2V_n)$$

Equal to

$$V(t-\Delta t) = V(t) - \Delta t * (rV_n - rn(V_{n+1}-V_n) - \frac{1}{2}\sigma^2 n^2(V_{n+1}+V_{n-1}-2V_n))$$

See the code comments for the next programming process.

2.2 American Option

The PDE of American options is exactly the same as European options.

The only difference is that every time a new option price is obtained, it is necessary to judge whether to exercise immediately.

3. Monte Carlo

3.1 European Option

The basic idea of Monte Carlo is:

- Sample the random path of S
- Calculate the return of options
- Repeat the first two steps to obtain many samples, and calculate the average of these samples
- Discount the average value, the result is the option price

The formula for calculating the S value is

$$S(T) = S(0)\exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma\varepsilon\sqrt{T}\right]$$

Where $\varepsilon \sim N(0,1)$, created by Monte Carlo simulate.

3.2 American Option

Because every day we need to judge whether we should exercise right immediately, so we use the “least square Monte Carlo method”, the principle is complicated. For call options, this method is only applicable to shareprice> strike, and for put options, this method is only applicable to shareprice <strike.

The method is divided into three steps. The first step is to perform N Monte Carlo simulations to obtain a (N + 1) points stock price sample path. Repeat M times to obtain M stock price sample paths. For example, a American put options, take M = 10 and N = 3, the randomly generated path is as follows

	t=0	t=1	t=2	t=3
1	1.00	0.98	0.93	0.95
2	1.00	1.09	1.18	1.11
3	1.00	0.96	1.02	0.98
4	1.00	0.98	0.97	0.94
5	1.00	0.97	0.9	0.88
6	1.00	0.94	0.93	0.91
7	1.00	0.99	0.99	0.99
8	1.00	0.99	1.01	0.86
9	1.00	1.08	1.07	1.05
10	1.00	0.99	1.02	1.08

Take the strike = 1.05, then the value of each path at t = 3 is max (strike-S, 0), discount it to t = 2, and record it as Y; consider "stock price at t = 2" as X.

	Y	X
1	0.10×0.99584	0.93
2		
3	0.07×0.99584	1.02
4	0.11×0.99584	0.97
5	0.17×0.99584	0.90
6	0.14×0.99584	0.93
7	0.06×0.99584	0.99
8	0.19×0.99584	1.01
9		
10	0.00×0.99584	1.02

Fitted by least square method

$$E\left[Y \mid X\right] = -0.093 + 1.083X - 0.908X^2$$

Substitute X and get the following table, the second column is the payoff if immediately exercise options, and the third column is the payoff from continuing to hold options, iterating until t = 0.

1	0.12	0.13
2		
3	0.03	0.07
4	0.08	0.10
5	0.15	0.15
6	0.12	0.13
7	0.06	0.09
8	0.04	0.08
9		
10	0.03	0.07

Finally, we can get the optimal exercise time of each path and discount the option price of each path. Then the average of all paths is the price of American option.

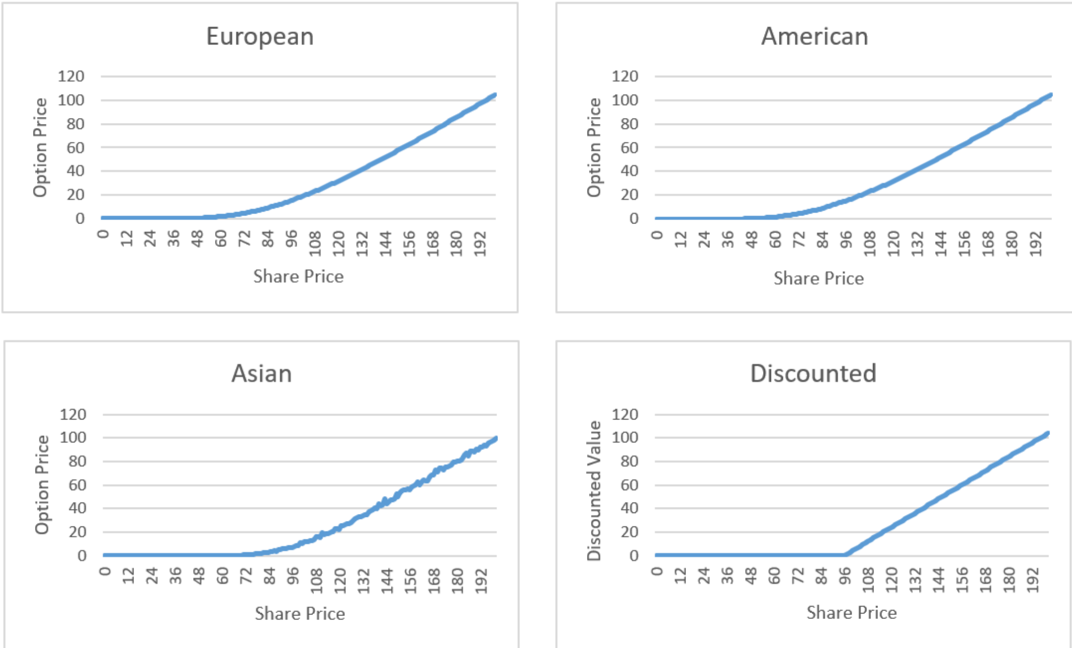
3.3 Asian Option

The idea of Asian options is the same as that of European options, but the path needs to be simulated and averaged, so the amount of calculation is relatively large. In my case, the Asian option was sampled once a day, 365 days a year.

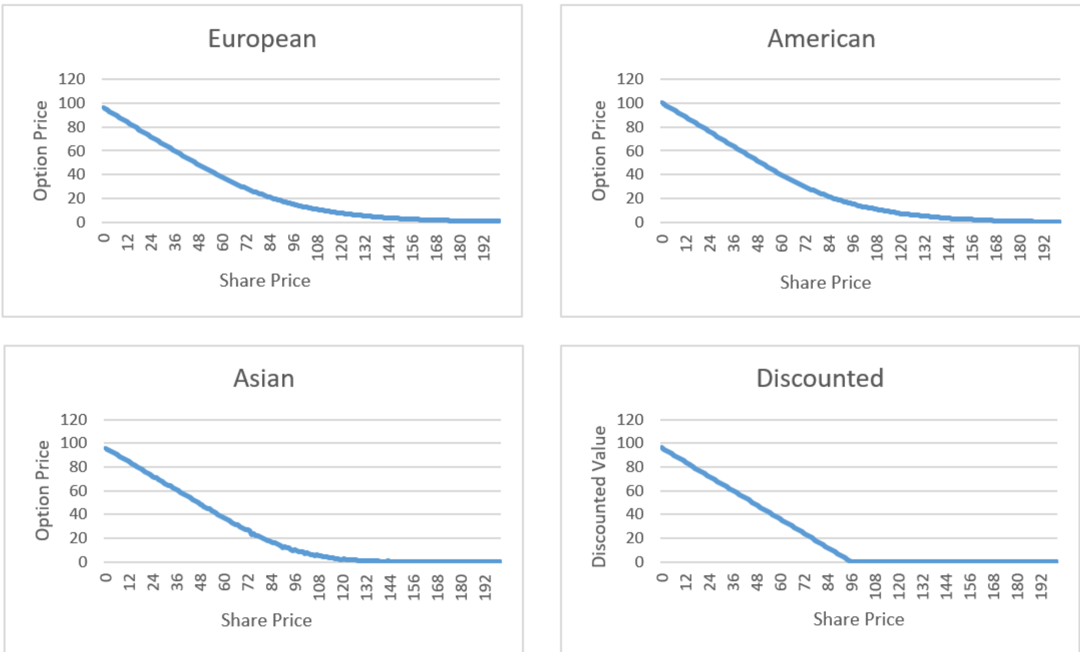
Part 2 SPECIFIC TESTS

Interest rate = 0.04, Strike = 100, volatility = 0.4, I use “Binomial Tree” to calculate European option and American option and use Monte Carlo for Asian option.

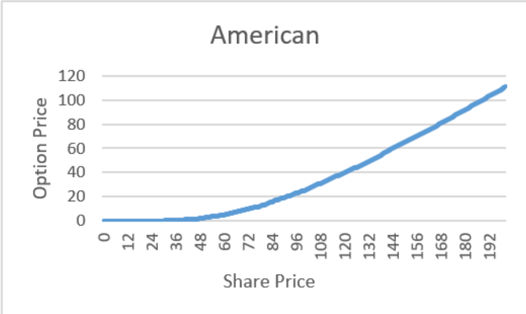
Expiry = 1, Call option

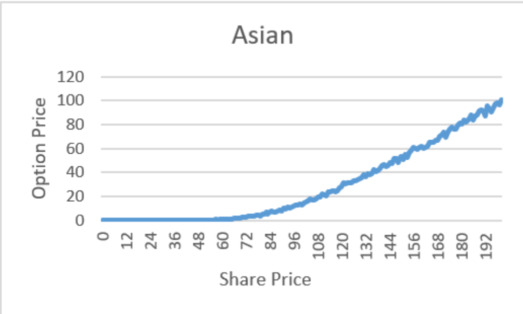


Expiry = 1, Put option

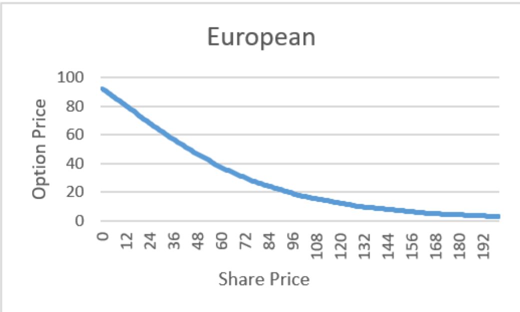


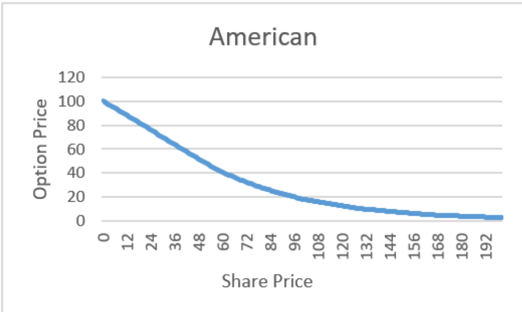
Expiry = 2, Call option

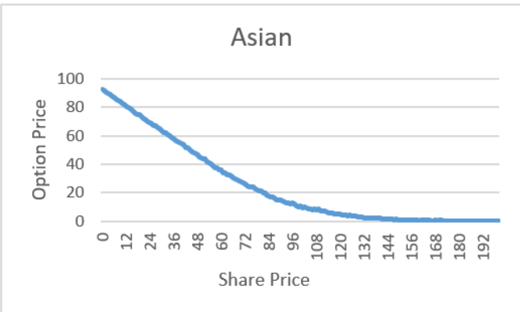




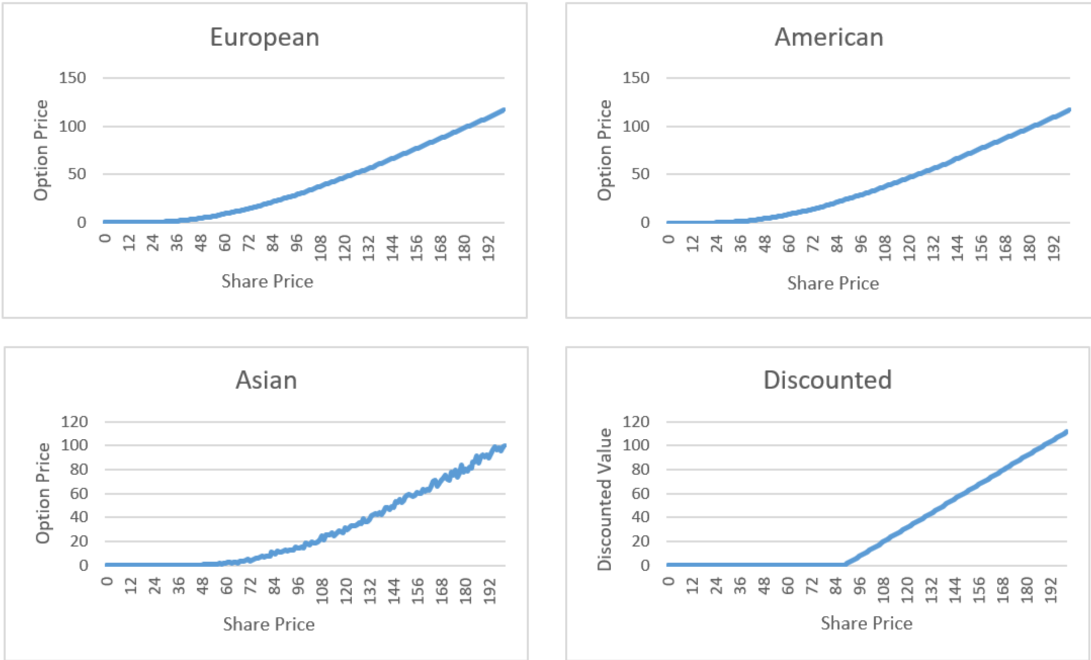
Expiry = 2, Put option



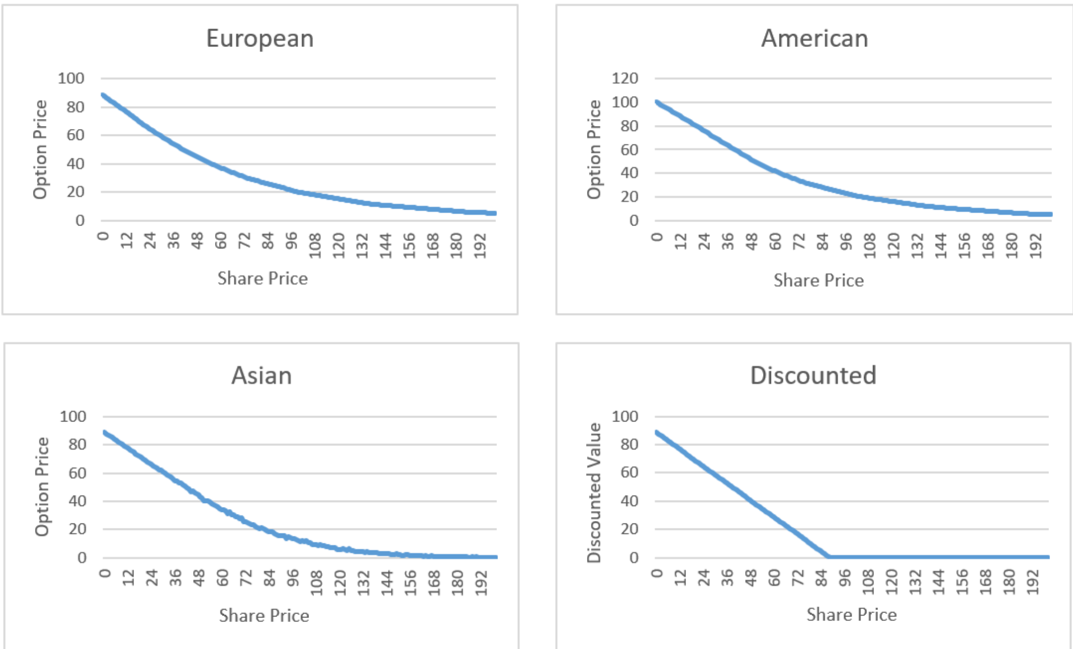




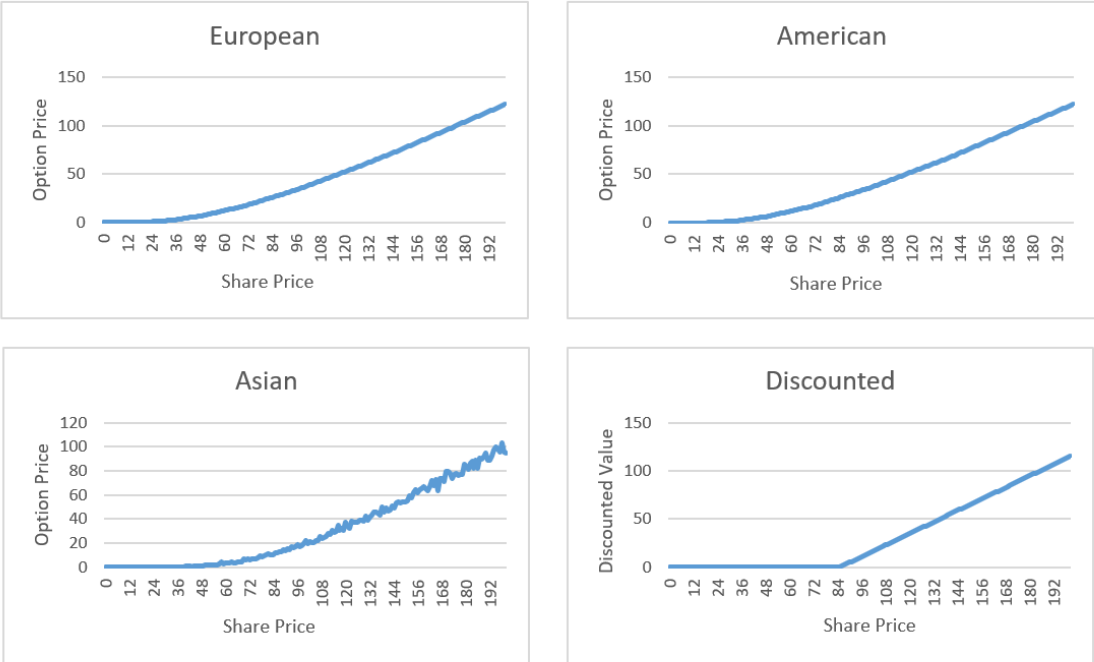
Expiry = 3, Call option



Expiry = 3, Put option



Expiry = 4, Call option



Expiry = 4, Put option