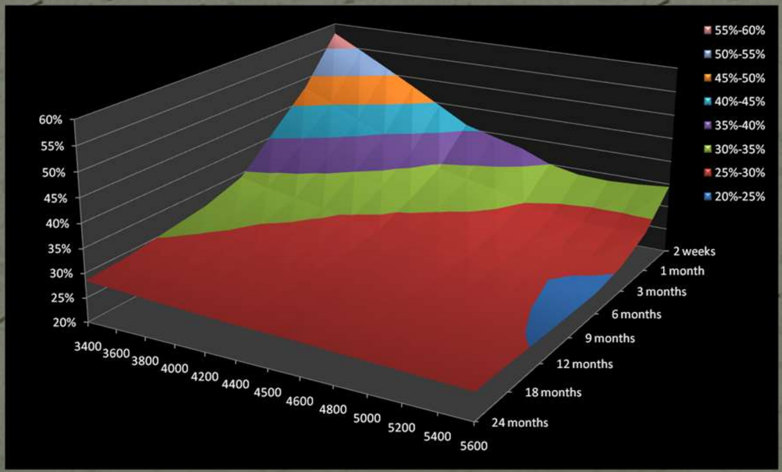
Options Pricing and the Black Scholes Equation

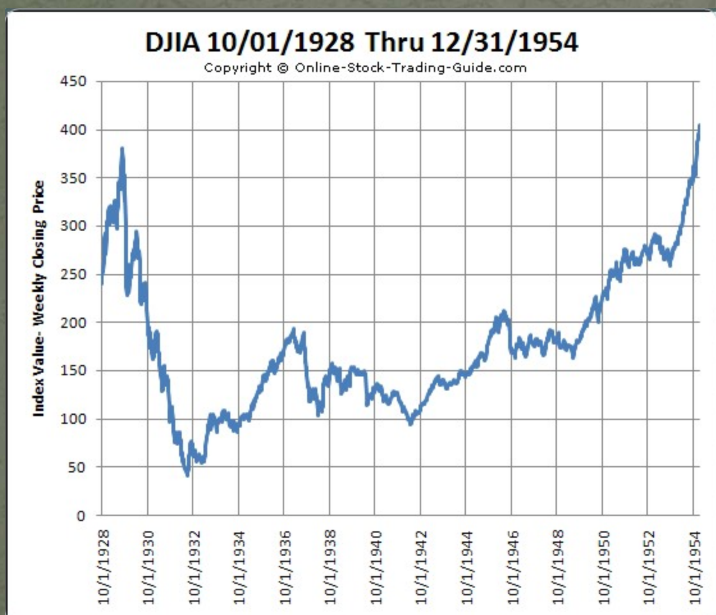
The Revolution of Derivatives in Finance



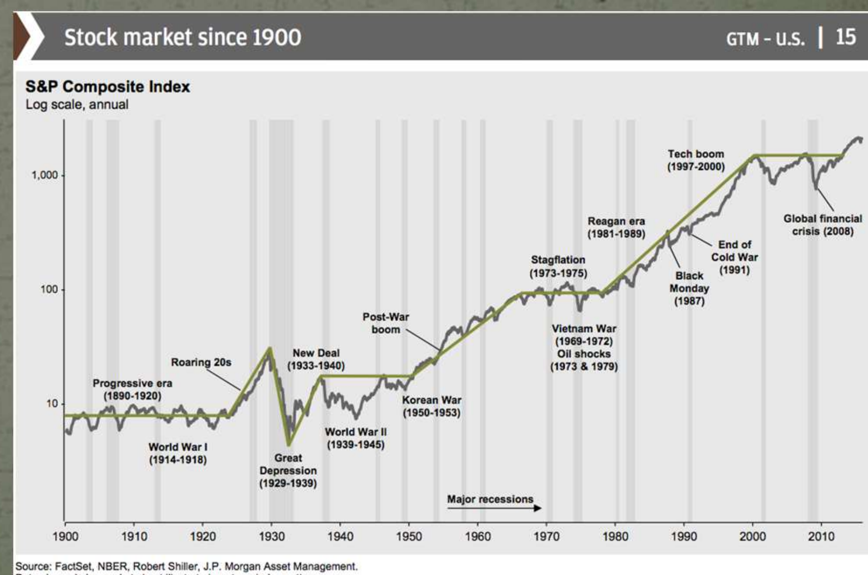
Implied volatility surface, pfadintegral.com

The Financial Markets

- Stocks – Dow Jones, S+P 500, NASDAQ
- Options, warrants
- Treasuries, bonds



Dow Jones Industrial Average, 1928-54



S+P since 1900, FactSet

Stocks

- Security that shows partial ownership in a corporation
- Owner of stock has claim on earnings and assets
- Value? a combination of the corporation's assets as well as "projected earnings" or future worth/growth potential, with a view to debt and obligations

Netflix Daily Price, 12/3/17

Futures

- Forward contract on commodity/currency/market
- Delayed transaction
- Spot price vs futures price
- Value often includes a premium due to the unknown factors in price volatility

CL : Crude Oil Daily, 12/3/17

ZC : Corn, Daily Price 2017

Gold Futures, Dec 17

/GC | Gold Futures, ETH (FEB 18) | 1283.1 | +6.40 +0.50% | B: 1280.8 A: 1284.5

Futures | ALL | Spread: **Single**

Symbol	Days	Exp	Last X	Net Chng
/GCZ7	-4	DEC 17	1279.7 G	+6.50
/GCF8	25	JAN 18	1280.7 G	+5.90
/GCG8 ACTIVE	58	FEB 18	1283.1 G	+6.40
/GCJ8	115	APR 18	1287.1 G	+6.00
/GCM8	178	JUN 18	1292.3 G	+6.70

Note different expiration dates for contracts:
Further date = higher premium

Modelling Analogy

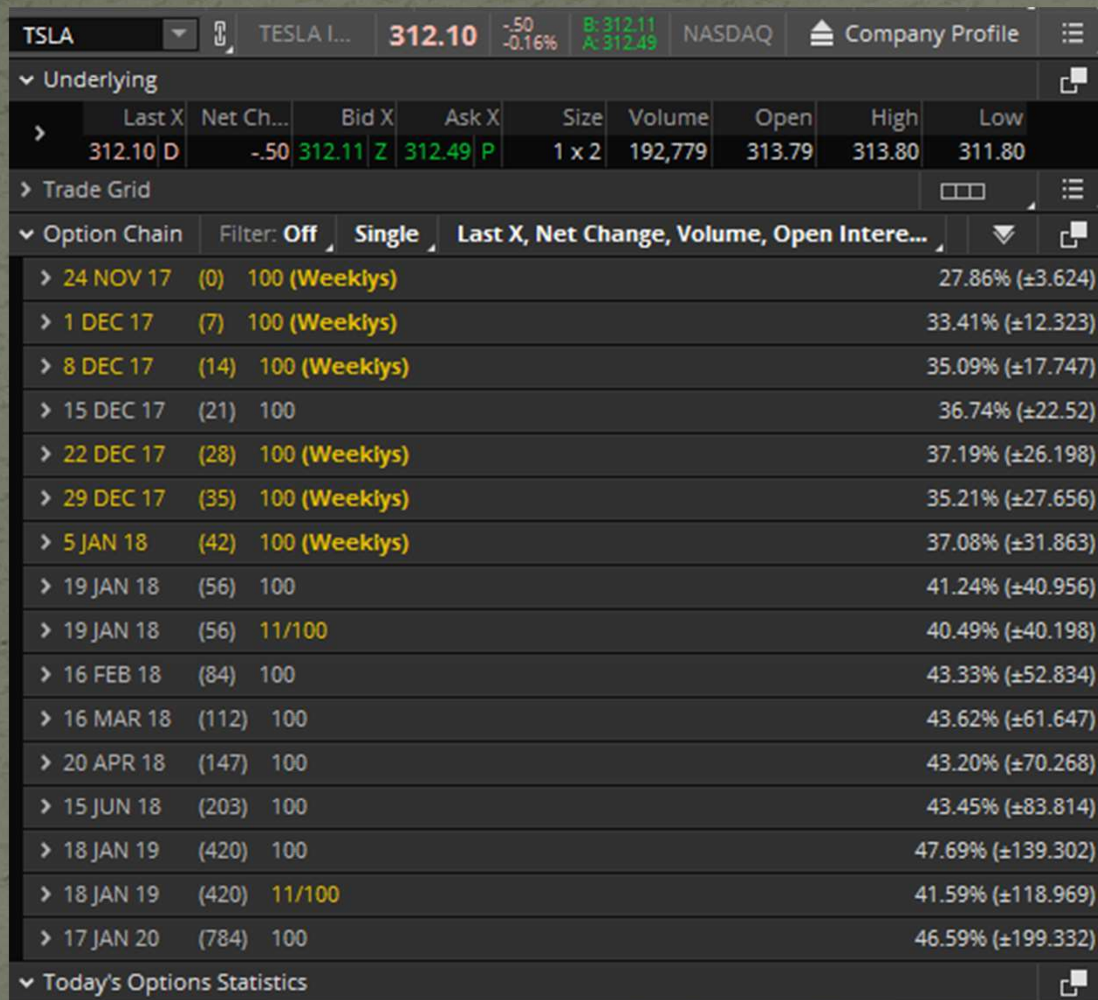
- Stock/Commodity = particle with path in price/time space
- Future = forward contract to buy particle at time T (or take delivery of physical product)

  - If a commodity or stock is an "underlying instrument" with a path, why not create a "derivative" based on its projected future path?
  - Derivative's value would depend upon speed of motion, volatility of particle path, time, and more

Options

- One step beyond Forward Contract
- "option" to enter transaction at future date and price, without the obligation to do so
    - Future unknowns make a valuation particularly difficult
    - Must account for "volatility"; potential for future information as well as market crashes

Tesla Options



TSLA | TESLA I... | 312.10 | -.50 -0.16% | B: 312.11 A: 312.49 | NASDAQ | Company Profile

Underlying

Last X	Net Ch...	Bid X	Ask X	Size	Volume	Open	High	Low
312.10 D	-.50	312.11 Z	312.49 P	1 x 2	192,779	313.79	313.80	311.80

Trade Grid

Option Chain | Filter: Off | Single | Last X, Net Change, Volume, Open Intere...

24 NOV 17	(0)	100 (Weeklys)	27.86% (±3.624)
1 DEC 17	(7)	100 (Weeklys)	33.41% (±12.323)
8 DEC 17	(14)	100 (Weeklys)	35.09% (±17.747)
15 DEC 17	(21)	100	36.74% (±22.52)
22 DEC 17	(28)	100 (Weeklys)	37.19% (±26.198)
29 DEC 17	(35)	100 (Weeklys)	35.21% (±27.656)
5 JAN 18	(42)	100 (Weeklys)	37.08% (±31.863)
19 JAN 18	(56)	100	41.24% (±40.956)
19 JAN 18	(56)	11/100	40.49% (±40.198)
16 FEB 18	(84)	100	43.33% (±52.834)
16 MAR 18	(112)	100	43.62% (±61.647)
20 APR 18	(147)	100	43.20% (±70.268)
15 JUN 18	(203)	100	43.45% (±83.814)
18 JAN 19	(420)	100	47.69% (±139.302)
18 JAN 19	(420)	11/100	41.59% (±118.969)
17 JAN 20	(784)	100	46.59% (±199.332)

Today's Options Statistics

Note volatility on right

OPTIONS CHAIN
Tesla Jan 20 Call

Price		Expiration	Strike/Exercise Price
90.40	C	17 JAN 20	290
88.10	C	17 JAN 20	295
85.85	C	17 JAN 20	300
83.65	C	17 JAN 20	305
81.50	C	17 JAN 20	310
79.40	C	17 JAN 20	315
77.35	C	17 JAN 20	320
75.35	C	17 JAN 20	325
73.35	A	17 JAN 20	330
71.50	C	17 JAN 20	335

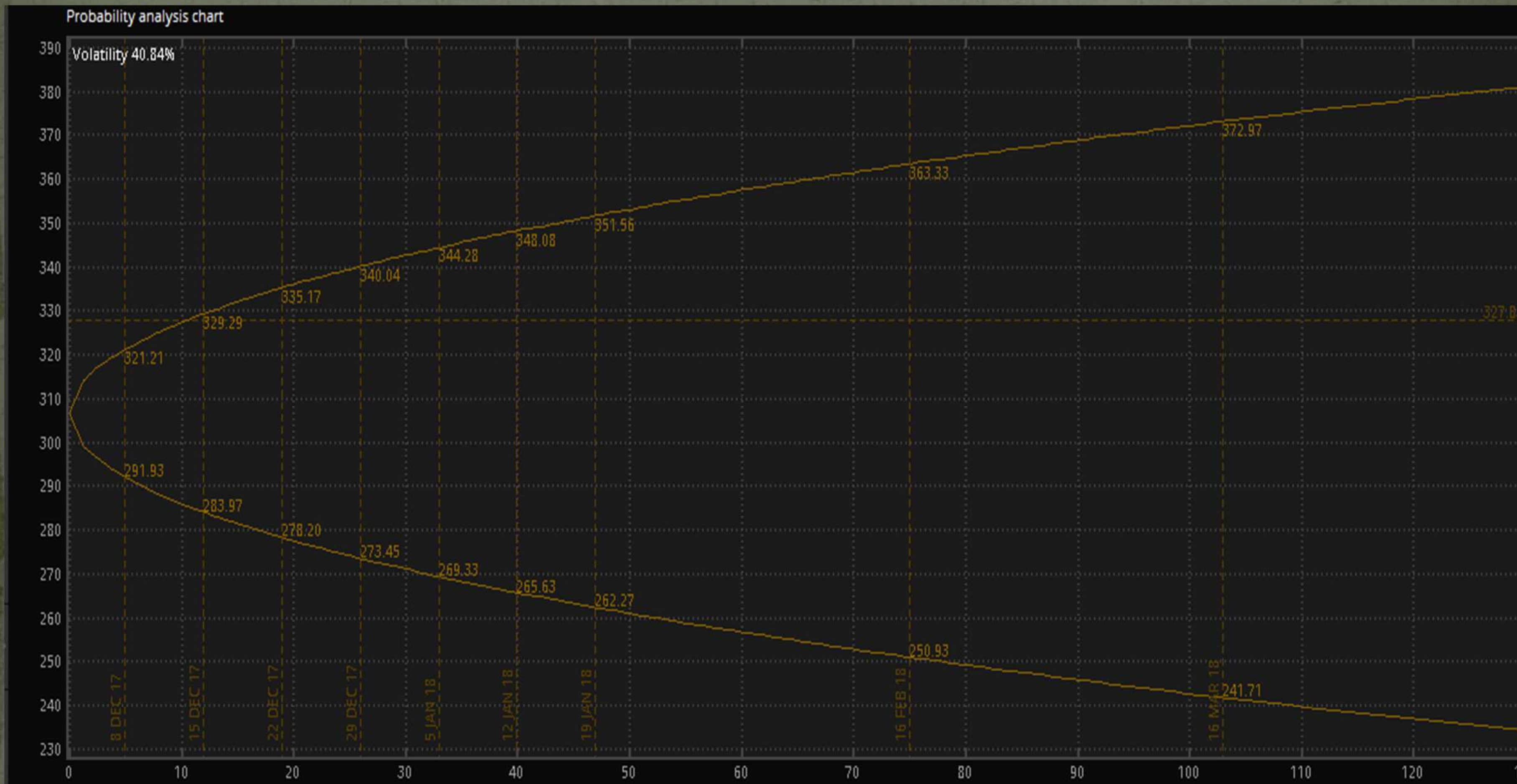
11/24/2017, Tesla options chains, source: TDAmeritrade

Types of Options and Terminology

- Call – the option to buy at a specified time/price
- Put – the option to sell at a specified time/price

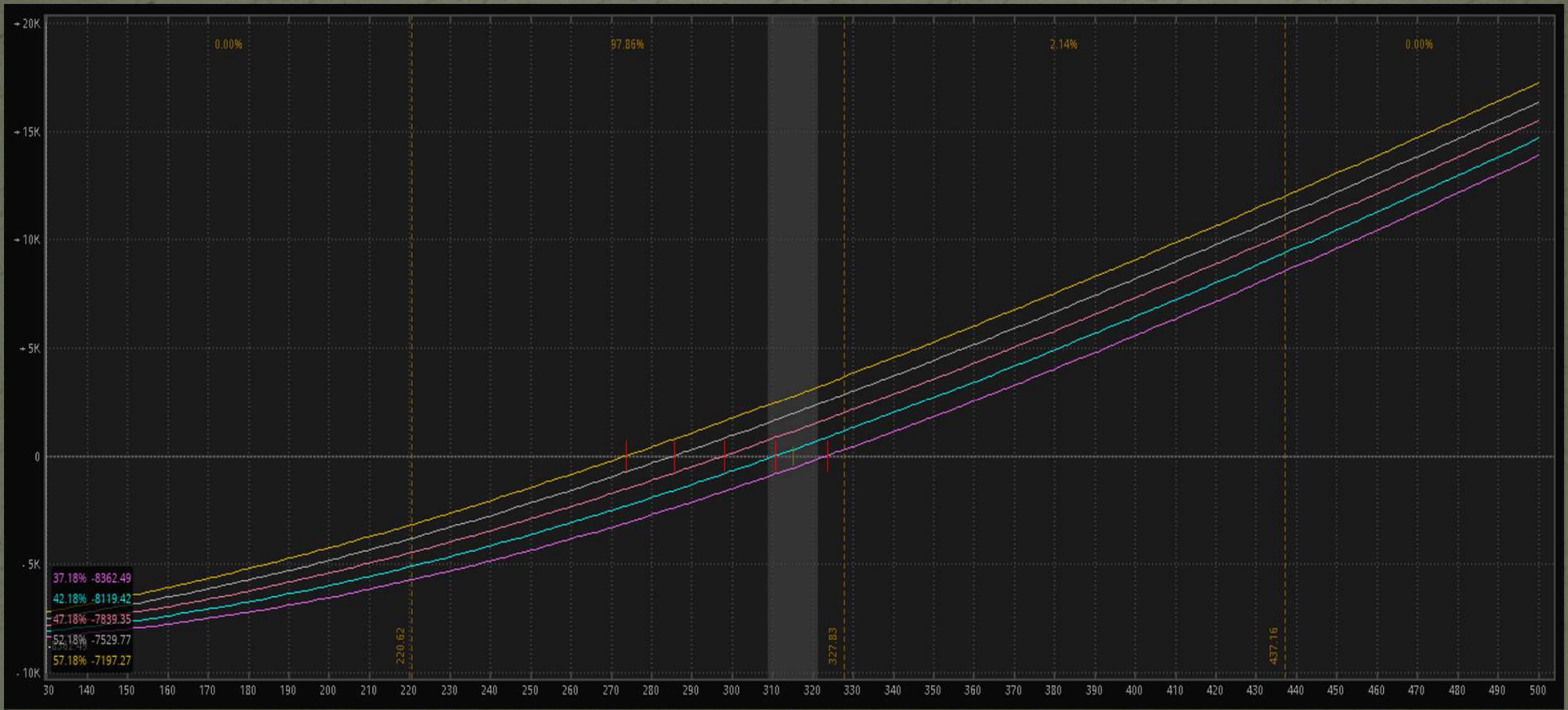
- Exercise Price –agreed price at which option can be bought or sold
- Expiration Date – date transaction will conclude
- Volatility – quantification of future uncertainty
  - Historical
  - Implied (future)

Tesla Jan 20 300 options – Forecast Range based on implied volatility 40.84%
Boundary (curve) = 1 standard deviation



	8 DEC 17 (5)	15 DEC 17 (12)	22 DEC 17 (19)	29 DEC 17 (26)	5 JAN 18 (33)	12 JAN 18 (40)	19 JAN 18 (47)	16 FEB 18 (75)	16 MAR 18 (103)	20 APR 18 (138)	15 JUN 18 (194)	18 JAN 19 (411)
above 437.16	0.00%	0.00%	0.01%	0.05%	0.16%	0.36%	0.64%	2.28%	4.16%	6.37%	9.28%	15.62%
327.83-437.16	7.68%	17.36%	22.29%	25.27%	27.22%	28.53%	29.40%	30.56%	30.11%	28.90%	26.79%	20.85%
220.62-327.83	92.32%	82.64%	77.68%	74.53%	72.18%	70.23%	68.49%	62.67%	57.97%	53.22%	47.43%	35.04%
below 220.62	0.00%	0.00%	0.02%	0.15%	0.43%	0.88%	1.46%	4.50%	7.77%	11.51%	16.50%	28.49%

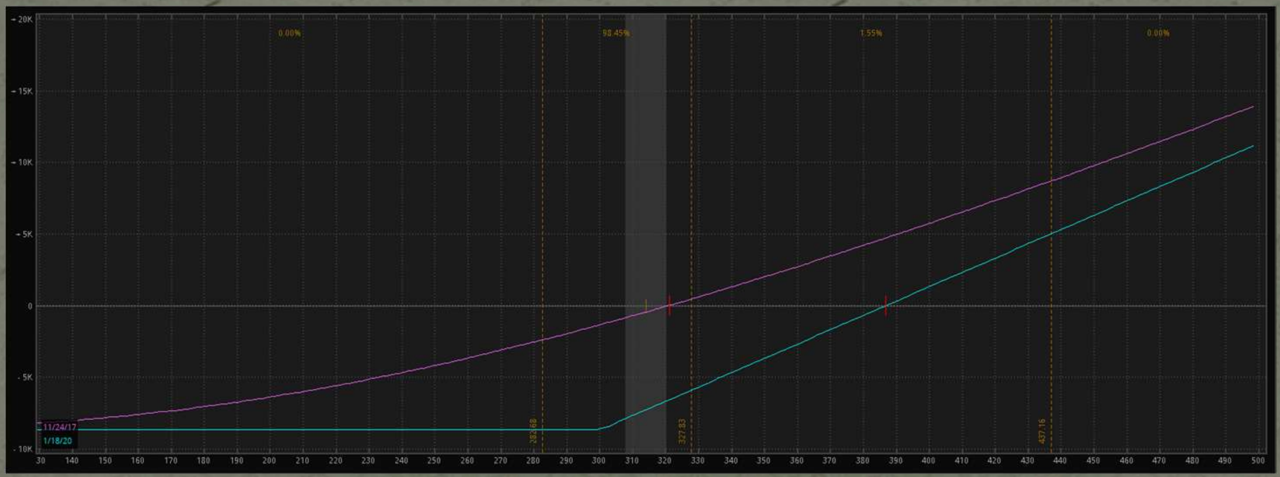
Tesla 300 Jan 20 Call



Expected change in P/L curve based on changes in volatility

Why Options?

- Many speculative uses for advanced trading strategies
- Hedging and “Insurance” on a portfolio position in case of market crashes/market volatility
- Insurance; the buyer pays a “premium”



Parity Graph, Tesla 300 Jan 20 Call

Variables for price determination?

- Exercise/Strike Price (intrinsic value)
- Change in underlying asset price (delta)
- Time to expiration (t) (extrinsic value)
- Interest rate (r)
- Dividends of underlying security
- Volatility

Examples

- Let's say Apple (AAPL) stock is $100/share today
- Buy Jan 18 call @ 100 for $3
  - If AAPL is $110 in January, the change in value of the position is:

  final stock price at – option strike price – option price

  $110-100-3 = $7

- Buy Jan 18 call @ 105 for $1
  - If AAPL is $105 in January, the change in value of the position is:

    $105-105-1 = -$1

History of Options

- Ancient Greece: olive harvests
- Early 20th century: US bucket shops

CBOE – Chicago Board Options Exchange 1973

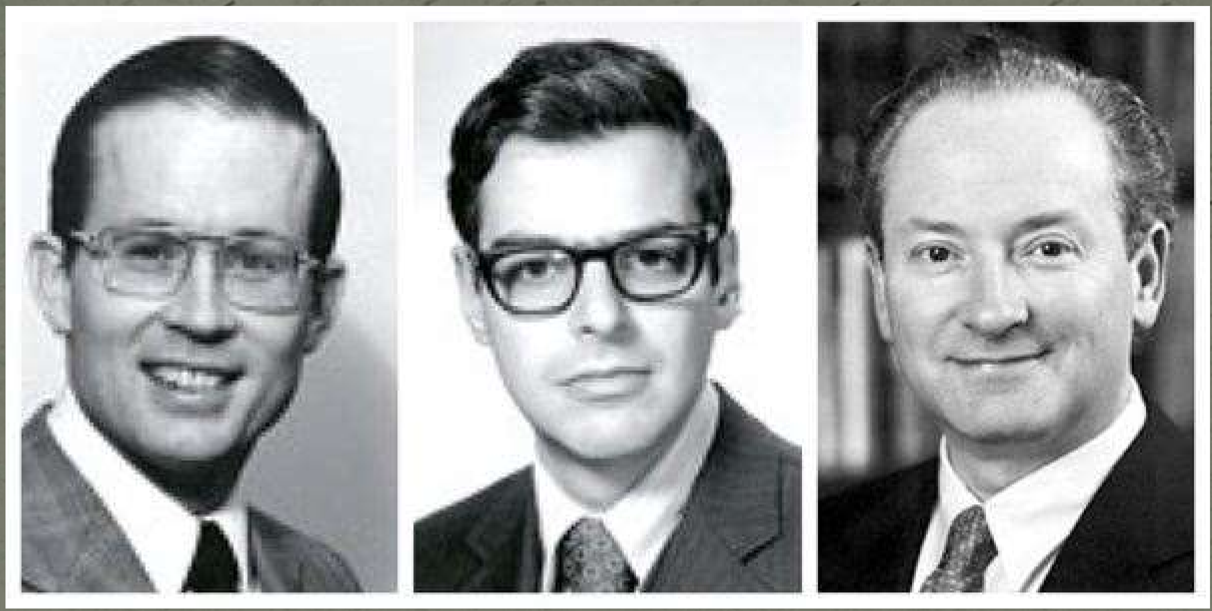
History of Options Models

- No regulated marketplace = no demand for model
- 1877 – Charles Castelli:

  The Theory of Options in Stocks and Shares
- 1900 – Louis Bachelier's doctoral thesis:

  The Theory of Speculation
  - formulated stochastic analysis
  - Modelled Brownian motion and applied it to stock movements
  - Neglected by academic/financial community

The Black, Scholes, Merton Model



Fischer Black Myron Scholes Robert Merton

Background

- Students/Academic researchers in 60's
- Merton – MIT
- Scholes- University of Chicago, MIT
- Black – Harvard University

- Black and Scholes published *The Pricing of Options and Corporate Liabilities* in 1973
  - Relied upon Merton's help and his previous work on CAPM (Capital Asset Pricing Model)

The Model: Assumptions

- 1 call option
- If current price of stock is much lower than option's exercise price, option price should be zero
- If expiration date near, price of an option should approximate:
  - Option price = asset price – option exercise price
- Option never worth more than underlying asset
- Markets are "efficient" (CAPM)
  - Formulated by Sharpe/Pintner – 1964-65
  - Asset prices reflect all publicly available information
  - Inefficiencies exploited and reduced to 0
- Market entails "ideal conditions"

Ideal Market Conditions?

- Short term interest rate constant and known
- No dividends
- Option only exercised at maturity
- No transaction fees
- Fractional shares allowed
- “random walk” of stock price

“The Random Walk”

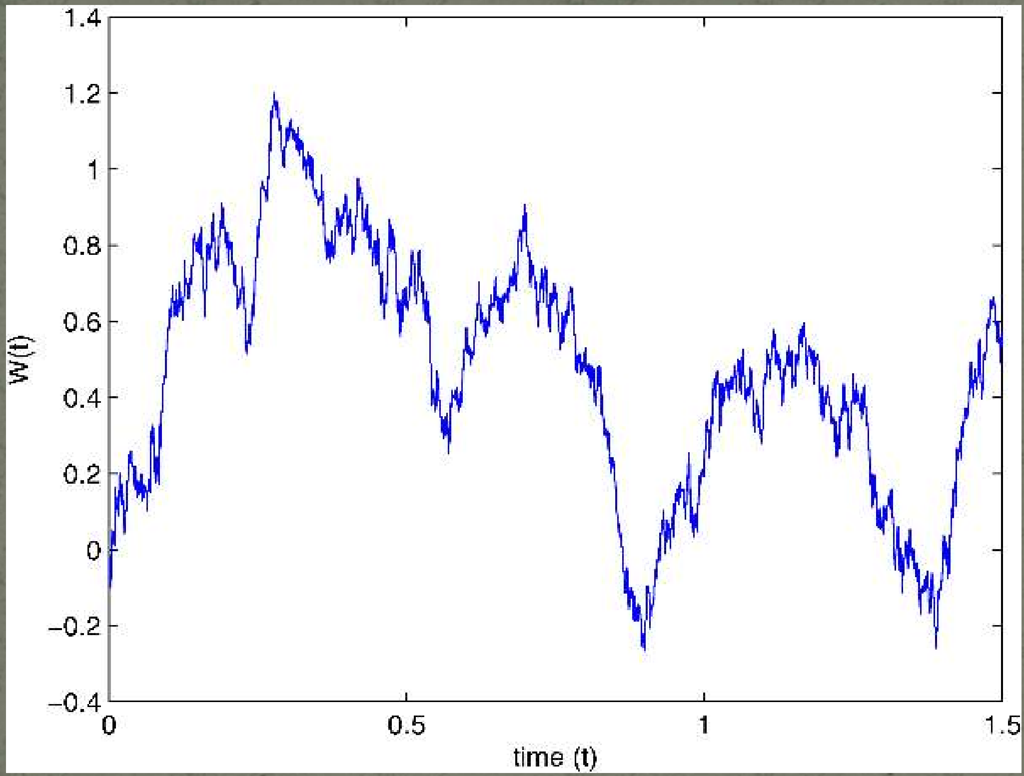
- Stock market prices evolve according to a “random walk”
- Jules Regnault – 1863
- Louis Bachelier – 1900
- Paul Cootner – MIT – *The Random Character of Stock Market Prices* - 1964
- Eugene Fama – *Random Walks in Stock Market Prices* - 1965
- Popularized in 1973 – Burton Malkiel: *A Random Walk Down Wall Street*

Brownian Motion

- Random movement of particles suspended in a medium (gas/liquid, etc)
- How to model particle trajectory with collisions and random displacements?
- Use *Wiener/Diffusion process* – solution to stochastic differential equation (PDE)
- Gaussian distribution has probability density u(x,t) which satisfies heat/diffusion equation:

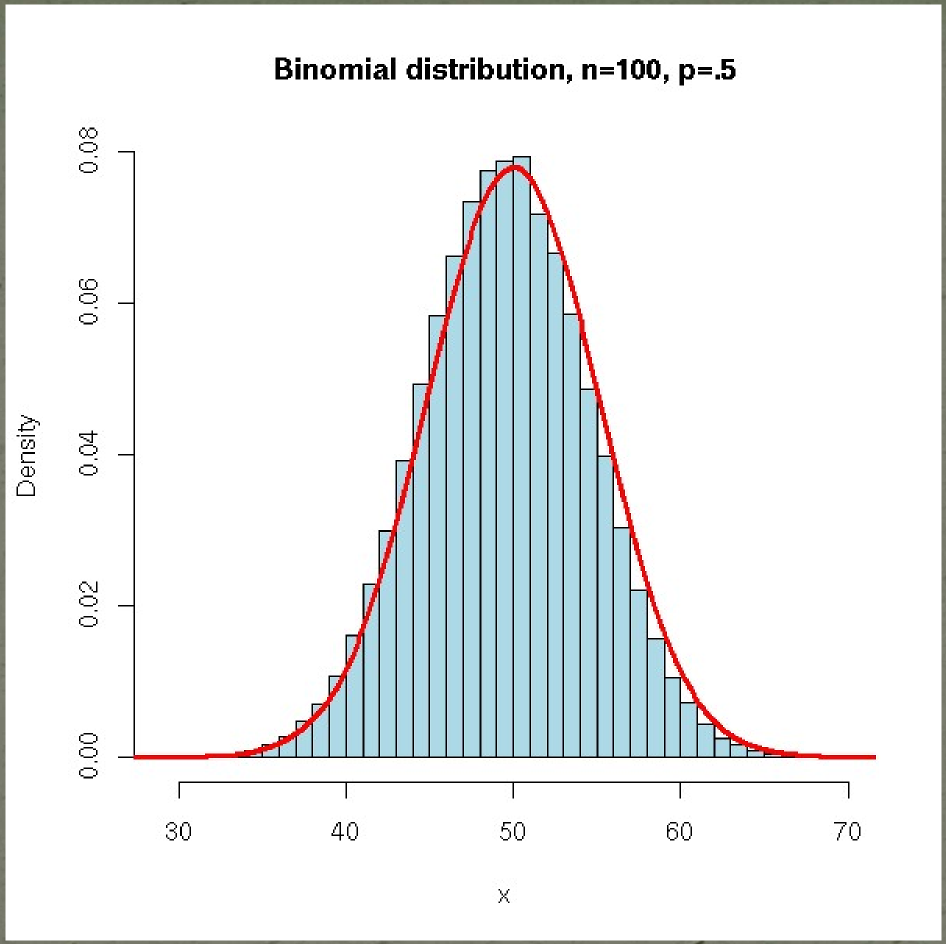
$$\partial_t u = \frac{1}{2}\partial_x^2 u \, .$$

Sample Path of a Particle in Brownian Motion



Bernoulli Random Variable
"Up" = 1 or "Down" = -1
P(x)=.5

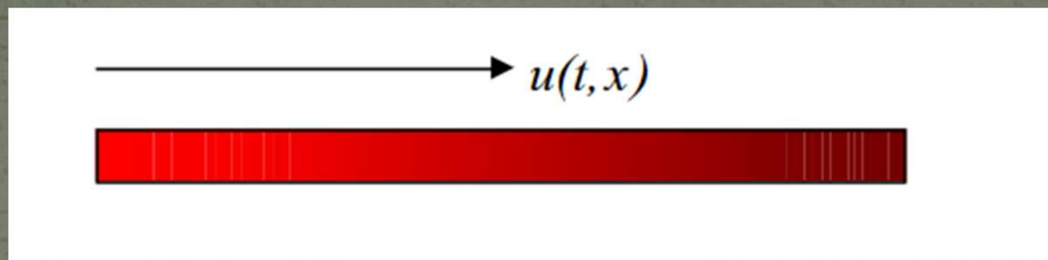
B(100,.5)



Central Limit Theorem -> set of outcomes will approach normal distribution

A Brief Mathematical Look

u(t,x) = temperature of rod



Satisfied by:

$$\partial_t u = \frac{1}{2}\partial_x^2 u \,.$$

Recall the probability density of the normal distribution:

$$N(\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\partial_t u = \frac{1}{2}\partial_x^2 u \,.$$

Given $\mu = 0$ and let $t = \sigma^2$, the normal distribution is a solution

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x)^2}{2t}}$$

And finally after other steps:

$u(t,x) = E[\varphi(X_t)]$,where $X_t = N(x,t)$.

Bachelier incorporated these results from the Brownian Motion and the Heat equation into the options pricing in *The Theory of Speculation*

BSM utilizes geometric Brownian motion

Wiener Process = Brownian Motion

- Continuous stochastic process
  - Probability paths are continuous and independent of past outcome
- $W_0 = 0$
- Normally distributed
- Results independent of past results; completely random
- Requires an SDE (Stochastic Differential Equation) incorporating a stochastic term

Stochastic Calculus

- Assume X_t is an diffusion process (Ito Process)

W_t: Weiner Process, a_t: stochastic term (white noise)

b_t: σ

Ito Process:

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t = a_t dt + b_t dW_t \qquad (6)$$

Given that X_t is an Ito Process, suppose g(x) is a twice continuously differentiable function. Then:

$Y_t = g(X_t)$ is also an Ito Process

Ito's Formula:

$$dY_t = \frac{\partial g}{\partial x}(X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t)(dX_t)^2 \qquad (7)$$

Substitute (6):

$$dY_t = \left(\frac{\partial g}{\partial x}(X_t)a_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t)(b_t)^2\right)dt + \frac{\partial g}{\partial x}(X_t)dW_t \qquad (8)$$

Back to the BSM Model: Base Assumption = Riskless hedge

- The authors assume that one can create a riskless position in the market which is a balance of stock value and short options

  # of options X option price = 1 share of stock

$$\text{Let } \frac{\partial w(x,t)}{\partial x} = w_1(x,t)$$

$$O w_1(x,t) = 1 \; share$$

$$O = \frac{1}{w_1(x,t)} \qquad (9)$$

w(x,t) = option price, O = # options, x = stock price

Riskless hedge

- Buy stock and "sell short" options
- This riskless hedge is assumed to be maintained until an option's expiration
- In other words, no risk through arbitrage
- Would require continuous position adjustment

The Derivation Begins:

- Assume a stock/option position as a riskless hedge requiring O options at price w(x,t)

$$O = \frac{1}{w_1(x,t)} \quad (10)$$

For a stock price x, the total equity invested in a position:

$$x - Ow(x,t)$$

And in a riskless hedge we expect the total equity to be 0

(1 share of X)(Price of X) – (number of options)(Price of option)=0

$$x - Ow(x,t) = 0$$

$$x = Ow(x,t)$$

Substitute for O (10):

$$x = \frac{w(x,t)}{w_1(x,t)} \qquad (11)$$

For small values of t (Δt), Equation (11) becomes:

$$\Delta x = \frac{\Delta w(x,t)}{w_1(x,t)} = \frac{\Delta w}{w_1}$$

$$\Delta w = w_1 \Delta x \qquad (12)$$

Fundamental Theorem of Calculus for small t:

$$\Delta w = w(x + \Delta x, t + \Delta t) - w(x,t) \qquad (13)$$

Recall Ito's Formula which we have transformed with substitutions:

$$dY_t = \left(\frac{\partial g}{\partial x}(X_t)a_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t)(b_t)^2\right)dt + \frac{\partial g}{\partial x}(X_t)dW_t \qquad (8)$$

Given that we have a riskless hedge, the authors combine (8) Brownian motion and (13) FTOC:

$$\Delta w = w_1 \Delta x + \frac{1}{2} w_{11} v^2 x^2 \Delta t + w_2 t \qquad (14)$$

(where v^2= variance)

Recalling (12), which is the change in the value of the equity in a short time period :

$$\Delta x = \frac{\Delta w}{w_1}, \text{ given small } \Delta t \qquad (12)$$

Substitute (16) Δw into (12):

$$\Delta x = \Delta x + (\frac{1}{2} w_{11} v^2 x^2 + w_2) \frac{\Delta t}{w_1} \qquad (17)$$

Where does price change come from if markets are efficient (CAPM)?

$$Interest = equity\ X\ interest\ rate\ X\ time$$

$$\Delta\ equity = \left(x - \frac{w}{w_1}\right) r \Delta t \qquad (18)$$

The last terms of (16) must equal (17), but negative since it is a hedged position and this balances inversely. Combining (16) and (17):

$$-\left(\frac{1}{2} w_{11} v^2 x^2 + w_2\right) \frac{\Delta t}{w_1} = \left(x - \frac{w}{w_1}\right) r \Delta t \qquad (19)$$

Divide by Δt:

$$-\left(\frac{1}{2} w_{11} v^2 x^2 + w_2\right) \frac{1}{w_1} = \left(x - \frac{w}{w_1}\right) r$$

Solve for differential equation of the value of the option's second derivative, w_2:

$$-\left(\frac{1}{2} w_{11} v^2 x^2 + w_2\right) = (x w_1 - w) r$$

The model in Diff EQ Form

$$\frac{1}{2}w_{11}v^2x^2 + w_2 = wr - xw_1r$$

And we have our final modelled equation, the Black-Scholes-Merton Model

$$w_2 = rw - rxw_1 - \frac{1}{2}w_{11}v^2x^2 \qquad (20)$$

- Given we have modelled the second derivative of the option price, how to solve for the option price?

Boundary conditions

- Price of the call option today (t)
- Price of the call option at expiration (t*)
  - If option strike price above exercise price, option expires worthless
  - If option strike price below exercise price, value = difference

$$
\begin{aligned}
w(x,t^*) &= x - c, \qquad x \geq c \\
w(x,t^*) &= 0, \qquad x < c
\end{aligned}
$$

More substitutions and transformations:

$$y_2 = y_{11}$$

$$y(u,s) = 1/\sqrt{2\pi} \int_{-u/\sqrt{2s}}^{\infty} c\left[e^{(u+q\sqrt{2s})\left(\frac{1}{2}v^2\right)/\left(r-\frac{1}{2}v^2\right)} - 1\right] e^{-q^2/2}\, dq. \tag{12}$$

Finally, with more substitutions we arrive at the final derivations:

$$\boldsymbol{w(x,t) = xN(d_1) - ce^{r(t-t*)}N(d_2)} \qquad \textbf{(21)}$$

$$d_1 = \frac{\ln\left(\frac{x}{c}\right) + (r + \frac{1}{2}v^2)(t^* - t)}{v(\sqrt{t^* - t}}$$

$$d_2 = \frac{\ln\left(\frac{x}{c}\right) + (r - \frac{1}{2}v^2)(t^* - t)}{v(\sqrt{t^* - t}}$$

Where $N(d_1)$ is the normal density function. Now, one can solve for the value of an option given stock price x, interest rate r, time remaining to expiration, variance v^2, and exercise price c.

All steps of the process make sense, even if the substitutions do not!

Solving for the first derivative:

$$\boldsymbol{w_1(x,t) = N(d_1)} \qquad \textbf{(22)}$$

The Full BSM Model

$$w(x,t) = xN(d_1) - ce^{r(t-t*)}N(d_2) \qquad (21)$$

$$w_1(x,t) = N(d_1) \qquad (22)$$

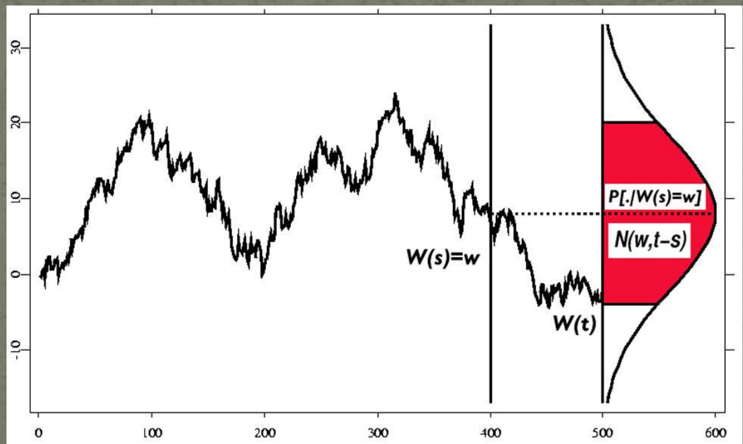
$$w_2(x,t) = rw - rxw_1 - \frac{1}{2}w_{11}v^2x^2 \qquad (20)$$

$$d_1 = \frac{\ln\left(\frac{x}{c}\right) + (r + \frac{1}{2}v^2)(t^* - t)}{v(\sqrt{t^* - t}}$$

$$d_2 = \frac{\ln\left(\frac{x}{c}\right) + (r - \frac{1}{2}v^2)(t^* - t)}{v(\sqrt{t^* - t}}$$

Model Success

- Transformed financial industry as financial markets could finally value derivatives, hedge portfolios, and mitigate risk
- Accepted through financial world, despite simplifications
- Earned Merton and Scholes the 1997 Nobel Prize in Economic Sciences

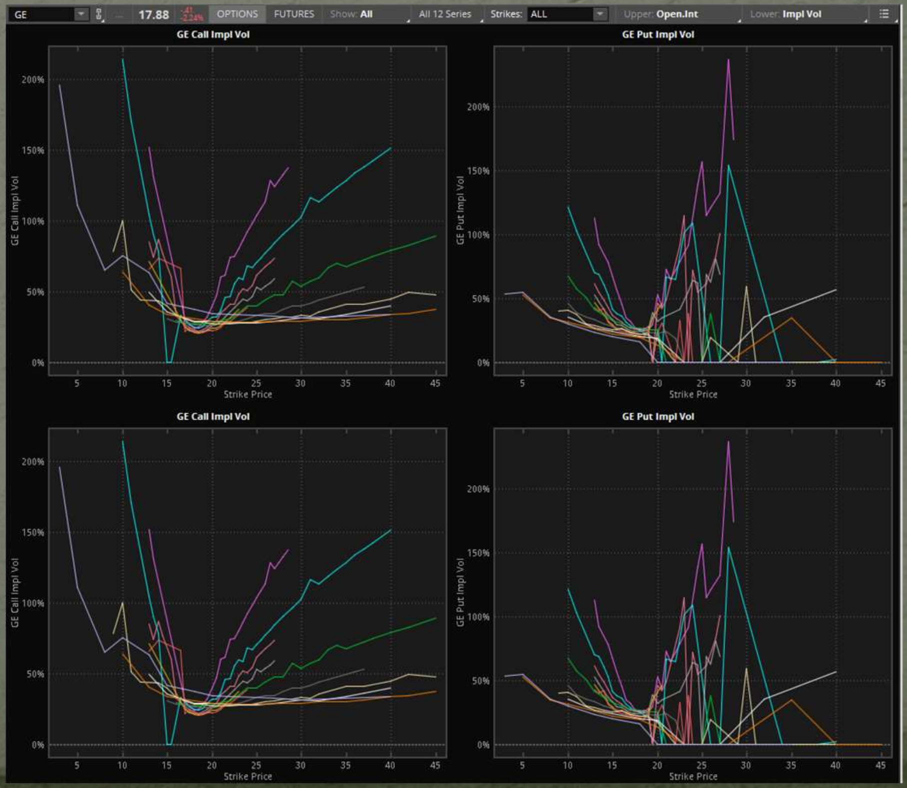


Basiswert: DAX 4.130,81 +41,68 +1,02% 11.11.2004 Java-Applet: aktiv Neu Starten

WKN	Typ	Bid	Zeit	Ask	Zeit	Strike	StopLoss	Währung	BV	Fälligkeit
SAL60F	Long	**2.470**	7:05:14 PM	**2.490**	7:05:14 PM	3.900,00	3.900,00	XXP	0,01	23.12.2004
SAL6UC	Long	**2.650**	7:05:24 PM	**2.670**	7:05:24 PM	3.900,00	3.900,00	XXP	0,01	24.03.2005
SAL60G	Long	**2.240**	7:05:14 PM	**2.260**	7:05:14 PM	3.925,00	3.925,00	XXP	0,01	23.12.2004
SAL6U9	Long	**1.970**	7:05:14 PM	**1.990**	7:05:14 PM	3.950,00	3.950,00	XXP	0,01	23.12.2004
SAL6UA	Long	**1.730**	7:05:14 PM	**1.750**	7:05:14 PM	3.975,00	3.975,00	XXP	0,01	23.12.2004
SAL6UB	Long	**1.470**	7:05:14 PM	**1.490**	7:05:14 PM	4.000,00	4.000,00	XXP	0,01	23.12.2004
SAL4VM	Short	**0.160**	7:05:14 PM	**0.180**	7:05:14 PM	4.150,00	4.150,00	XXP	0,01	23.12.2004
SAL4VN	Short	**0.410**	7:05:14 PM	**0.430**	7:05:14 PM	4.175,00	4.175,00	XXP	0,01	23.12.2004
SAL1S6	Short	**0.660**	7:05:24 PM	**0.670**	7:05:24 PM	4.200,00	4.200,00	XXP	0,01	23.12.2004

Shortcomings

- Trading costs not zero
- Interest Rate not same for all investors
- Options often exercised early
- If markets are efficient, there would be no profits on hedged positions



Implied Volatility Charts
For different strike prices

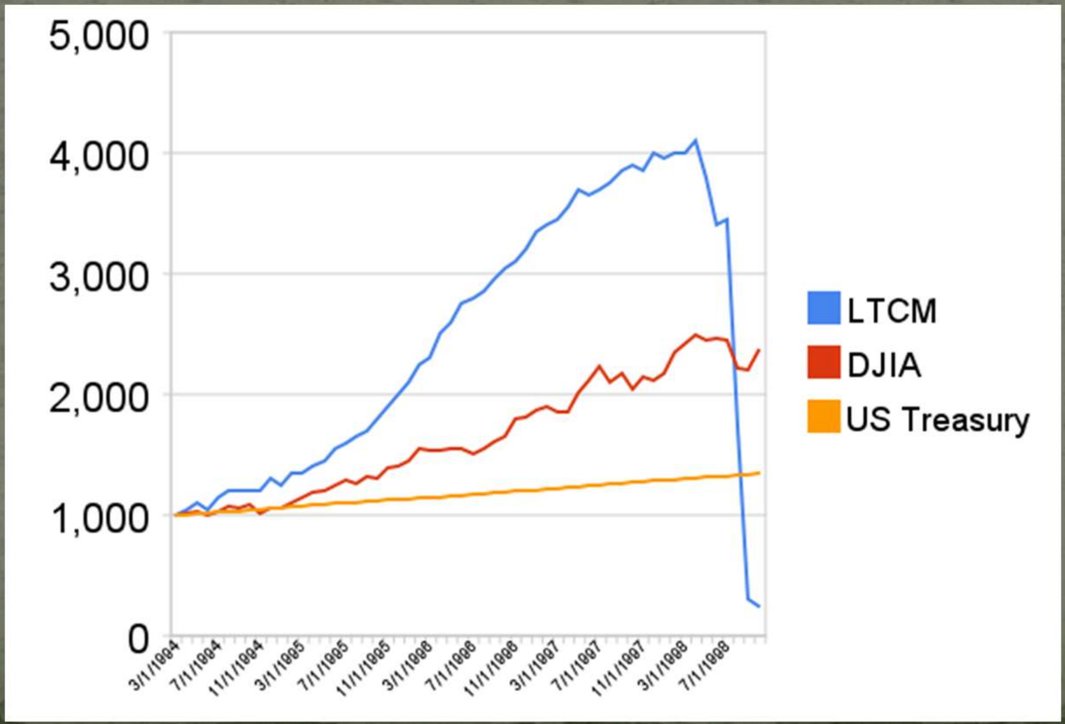
Are Stock Prices Continuous?

Are Price Movements Distributed Normally?

AMZN
AMAZON.COM INC COM
1162.00
Share
Wk
Style
Drawings
Studies
Patterns
AMZN Max W [NASDAQ]
D: 3/13/00
O: 64
H: 66.94
L: 60
C: 64.81
R: 6.94
Y: 328.16
VolumeProfile (AUTOMATIC, 1.0, CHART, 1, yes, 1000, yes, yes, 70, 50)
N/A
N/A
N/A
2017 © TD Ameritrade IP Company, Inc.
amazon.com
Hi: 1213.41
Lo: 1.31
1162
328.16
108.22
21.64
1300
1200
1100
1000
900
800
700
600
500
400
300
200
0
-100
98
99
00
3/13
01
02
03
04
05
06
07
08
09
10
11
12
13
14
15
16
17
18
19
Drawing set: Default

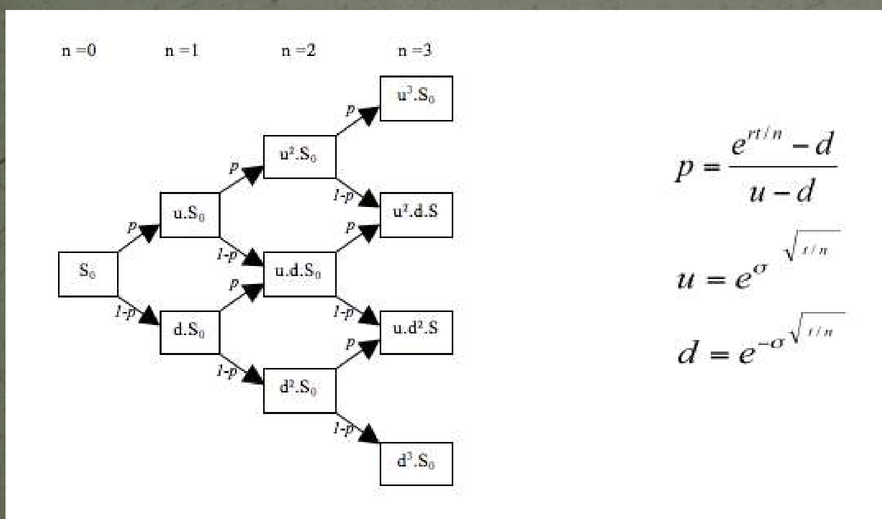
Option Pricing Failures: Long Term Capital Management

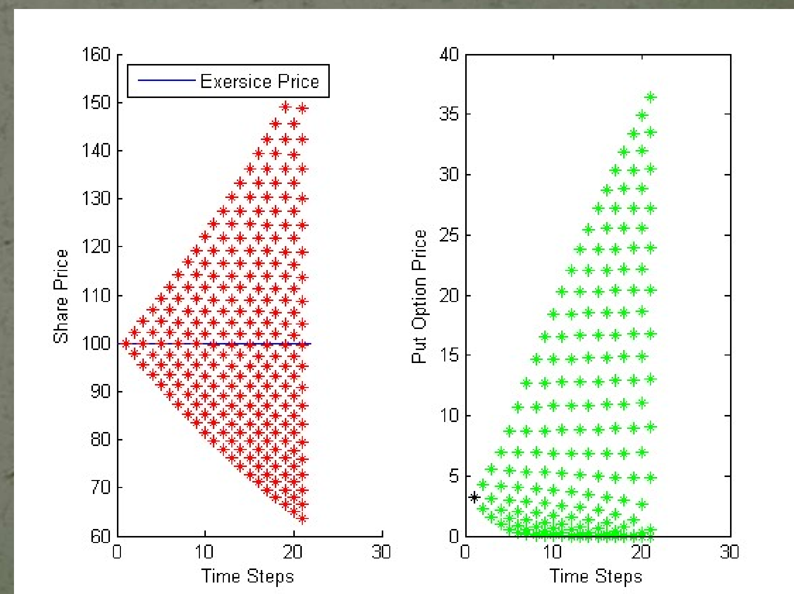
- A misassumption of potential risk
- An over leveraged derivatives portfolio took LTCM to bankruptcy in months
- In 1998, portfolio value dropped from $7.5 billion to $250 million
- Portfolio value was dropping $550 million/day in August 1998



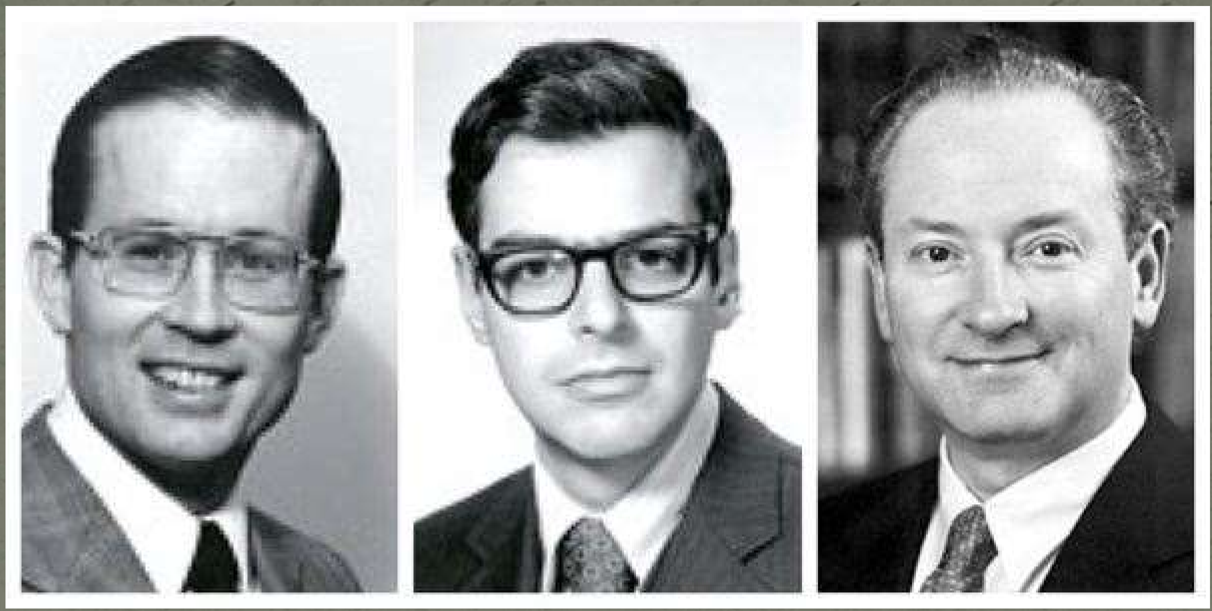
Later Option Pricing Models

- The authors continued to make improvements to the BSM model as they worked in the finance industry for decades to come
- Binomial Options Pricing Model – lattice based numerical method
- Black-Derman-Toy Model built further on this





The Black, Scholes, Merton Model



Fischer Black Myron Scholes Robert Merton