Thomson Kneeland
Mathematical Modelling MA 469
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"The Pricing of Options and Corporate Liabilities" and the Black-Scholes-Merton Model

$$w_2 = rw - rxw_1 - \frac{1}{2}v^2x^2w_{11} \qquad (1)$$

In this paper, we will examine Black-Scholes-Merton Model of options pricing, as derived in "The Pricing of Options and Corporate Liabilities" by Fischer Black and Myron Scholes in 1973. The academic paper was a watershed moment in the world of finance, giving financial institutions a means of hedging portfolio risk through the use of derivatives on an underlying assets. Though flawed in its use for risk mitigation in times of exogenic shocks and market failures, the model did provide a reasonable valuation which was accepted industry wide and led to entire new markets of assets being created over the course of decades. Moreso, from our mathematical standpoint, the model provides a striking methodology to examine from premises to conclusions. The authors clearly state market condition premises, boundary conditions, and more, and combine the heat diffusion equation, Brownian motion, stochastic calculus, and economic premises to reach an option valuation model which is fairly approximate for constant volatility markets. The model is an excellent example of how a PDE model from one area of science can be applied to an entirely different field with excellent results.

Background: What are options?

In order to understand the valuation of options contracts, it is important to get a quick overview of the financial markets to gain a broader understanding of the context within which options operate. Hopefully the reader has some acquaintance with market concepts, but if not, hopefully this guide will provide a brief core understanding needed to understanding the objects being modelled in the Black-Scholes-Merton Equation.

Suppose buyer A agrees to buy a plot of land from seller B at an agreed price X to be paid today; this qualifies as a *spot transaction,* otherwise known as a *cash transaction*.

If on the other hand, buyer A agrees to buy a good from seller B at an agreed price X at a future point in time (perhaps 6 months from now), this is known as a *forward contract*. The agreement or contract for the exchange is entered into at the present time, while the actual cash transaction will take place at the specified price X at a specified future time.

What advantage would parties gain from such a delayed transaction? Should the forward contract price paid be the same as the spot transaction? The answer can be complicated and is situation dependent, but consider a few factors: by agreeing to delay the purchase by 6 months, the buyer has the advantage of retaining their money and earning interest or applying the money to another purpose during that six month period, while seller B will retain usage of the property for the remaining 6 months, perhaps to finish a harvest or earn other income. Regardless, both parties have secured a transaction to some advantage even though the actual exchange of goods occurs at some point in the future.

It should be noted that in locking a forward price, both parties are managing risk against future uncertainty at an agreed upon price. If one locks in a contract for 1,000 walnuts for $1,000 in a year, the price of walnuts in the market may rise or fall in the interim. The buyer will be protected in case the price of walnuts rises to $1,500 due to a walnut tree blight, this saving $500; meanwhile the seller is guaranteed a price of $1,000 even if the price of walnuts falls to $500 due to an oversupply in the market. Both parties are managing risk by securing a transaction today and securing their future interests, while protecting themselves from potential volatility of price movement in the interim.

Accordingly, forwards prices are usually not a present market rates, but reflect an additional price premium due to price volatility, foregone interest, and other factors. Seller B would not sell a forward contract a year from now for $1000 when they could sell it on the market for $1000 in the present day. Such a transaction would forego an entire year of interest. Perhaps, the seller will demand $1,100 for this forward privilege, and the buyer may well agree, for they acquire an extra year to keep their money with the full security of having locked in an agreement for the future. Both counterparties hedge risk and offer each other a mutual benefit in such agreement.

A forward contract offered in a market exchange is called a *futures contract*, and this what we normally see in the news when we see headlines that the price of crude oil has spiked to over $150. Sometimes the news specifies *spot price* (today's price), but typically it refers to December futures, or some other month when the contract expires and the commodity will exchange hands. Futures contracts always involved risk premium, so the question for the buyer and seller is "what model can one use to assess a proper price for a good at a specific point in time?"

An *option contract*, simply referred to as an *option,* takes the forward contract one step further by entering into an agreement whereby the buyer retains the *option* to enter into a specified transaction of price and future date, while being free from the obligation. Buyer A retains the option to decide a later date whether to proceed with the transaction.

For example, again suppose buyer A secures an option contract to buy walnuts for $1,000 year from now from seller B. If 1 year from now the price of walnuts is $1,500, that buyer will certainly *exercise* the option to buy, for they can now buy at $1,000 and sell on the open market for $1,500. On the other hand, if the price of walnuts one year from now is $500, the buyer will not exercise their option to buy, for who would pay $1,000 for what is now worth $500? Better to buy $500 on the open market.

But what seller would agree to forego potential profits with a contract like this? Therefore, the seller demands a premium to be paid for the option, dependent upon time, volatility, of price, storage costs of the good, interest and so forth. In this example, perhaps the seller requires $100 for the option to buy 1,000 walnuts for $1,000 in on year's time. This option is called a *call.* In our example, when the price of walnuts has risen from $1,000 to $1,500 a year from now, the seller will exercise their option to buy. The buyer paid $100 premium for this option, buys the walnuts at $1,000 in a years time, and can sell them for $1,500, thus securing $1,500-1,100 = $400 profit. Meanwhile the seller will net $1,000, and an additional $100 for being willing to hold the walnuts for a year for a total sale of $1,100. Both parties benefit from this option agreement and risk has ben hedged.

On the other hand, let us examine the example where the price of walnuts has fallen to $500 a year from now. The buyer will not exercise their call and will let the $100 option contract expire worthless for a loss of $100. But better $100 lost than $500 from their perspective. Meanwhile, the seller has hedged their product in the interim. They have not sold their walnuts for $1,000, and the value of the product is $500 less on the spot market, but in the meantime, they have mitigated risk by accepting the premium from the options contract, thereby securing $100. So the seller, effectively has only lost $500-100 = $400 in this time period rather than the full $500.

Buyers and sellers enter into options contracts not only for the option to buy *(call)*, but also the option to sell *(put).* The roles could have been reversed above, and perhaps the owner of the walnuts would like to sell his stock, but will only sell at $1,200, not at today's spot price. They could buy a *put*, an agreement to sell walnuts at $1200 one year from today, and pay $100 for that privilege. If a counterparty believes the price of walnuts may be $1,500, they certainly would be willing to pay $100 for $1,200 of walnuts a year from now. In such an instance, the seller of the put receives $100, and *must* pay the $1200 for the goods in the event the seller exercises the put.

Looking at these group of financial instruments, it is helpful to classify them further for modelling purposes and understanding. Initial assets eventually purchased (walnuts) are referred to as *underlying assets* which carry an immediate market value. An option contract is a derivation of the initial financial instrument/asset's projected future value, and thus is often referred to as a *derivative*. An option contract specifies a potential future price of exchange called the *exercise price* or *strike price*; it also specifies a future date at which the final transaction will take place, referred to as the *expiration date.* Comparing the two, it should be clear to those with a mathematical understanding that an option contract's present premium is a factor not simply of present market value, but also time to expiration, exercise price, and particularly the risk premium created by the uncertainty of market forces that may occur between the present date and expiration (volatility). Put differently, an asset's value at present is determined by the market auction itself; an option contract derives its value from not only the movement of the asset but the expectation for future movement of the asset. As such, it is clearly a *derivative*.

In fact, for those who are mathematically inclined, but not financially knowledgeable, one could think of the price of a stock or underlying asset as a multivariable function $P = f(x,t,\ldots)$. Accordingly, an option as a derivative could mentally be visualized as dependent on $P'=f'(x,t,\ldots)$. The option price is more dependent on the magnitude and strength of P', rather than ΔP. As such, it is truly a "derivative."

In the Black-Scholes paper we will examine, the authors limit their initial discussion and derivations to option calls, so this paper will restrict examples to calls and "long" transactions for the sake of simplicity for the reader; such will allow a focus on the model itself rather than potential confusion arising from the complexity of options mechanics such as early assignment, combinational option strategies, short sales and more.

Glossary

For a quick reference, consult the short glossary here detailing the terms most commonly found in the research discussed:

Break Even Price – Price at which a buyer/seller will net $0 profit based on price at maturity date

Call – an option contract entitling the owner to buy an asset

Derivative – a financial instrument whose price is dependent upon its underlying asset(s); its value is "derived" from various factors besides simply market price: time until expiration, expected change in price/volatility, interest rates, and more

Exercise Price – the agreed price at which an option can be purchased or sold in the option contract

Expiration Date/Maturity Date – the agreed upon date on which an option expires and the buyer and seller conclude transaction of the underlying asset

Forward/Futures Contract – a contract between two parties to buy or sell an asset at a specified price and specified future date

Long Position – ownership of an asset (+100 shares)

Option Contract – a contract between two parties which gives the buyer the option to buy or sell an asset at a specified price and specified future date

Option Premium – current price of an option contract; can also be viewed as the the potential income of the option writer; can also be viewed as "insurance" for the buyer

Option Writer- seller of an option contract

Put – an option contract entitling the owner the option to sell an asset

Short Position – Position in which one "sells" ownership of shares (-100)

Spot transaction – a cash transaction in which two parties agree to buy and sell an asset at market value, settling the transaction "immediately"

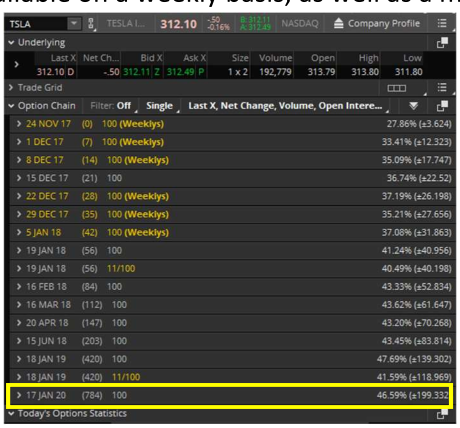
Underlying Asset – financial instrument on which a derivative's price is based

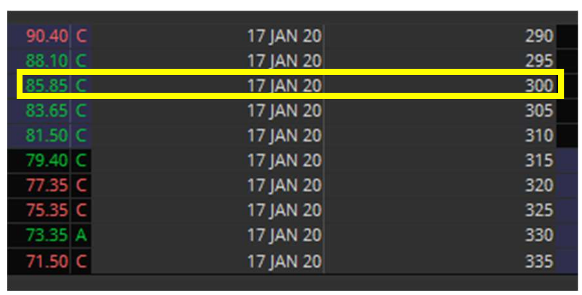
Volatility - a variable measuring the expected fluctuation in price between the present date and expiration date of an option; a quantification of uncertainty. Low volatilty = low expectation of price change, high volatility = high expectation of price change

An Option's Return: Real World Example

Before delving into the factors that affect an option's price, a quick examination of an option's profit or loss may help the reader understand how the financial instrument is used at its simplest.

Take the present day example of the company Tesla which makes electric cars, batteries and more. One can buy Tesla stock on the open market at a price of approximately $312 as of this writing. Options are also available on a weekly basis, as well as a monthly basis as far away as January 2020.





11/24/2017, Tesla options chains, source: TDAmeritrade

In the left frame above, the last line of options shows us from left to right:

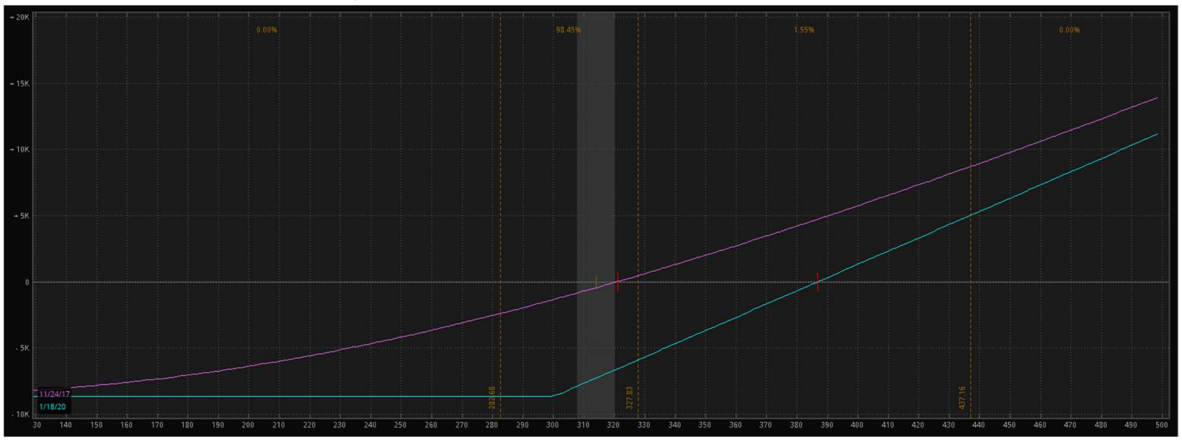
- This series of options expires January 17, 2020 (Maturity/Expiration Date)
- There are 784 days to expiration
- Each option is a contract for 100 shares of Tesla stock
- This Jan 20 option chain has an implied volatility of 46.59% built into its premium

The right frame above shows several January 2020 call options available at different strike prices.

- The first column shows the price of the call option (need a multiplier of x100), so this option costs $8,585.
- The second column shows the Expiration Date
- The last column shows the strike price.

Put into market terms, one may buy the Jan 20 call @ 300 for a premium of 85.85 . The market is valuing this option to purchase Tesla shares in January 2020 for $8,585. If one were to buy the shares today and hold, the purchase price would be $312.10 x 100 = $31,210. The option call offers leverage of one's capital while limiting downside risk by approximately $31,000-8,500 = $22,500. But what is the potential return from this option at expiration? A *parity graph* represents the value of an option at expiration. This is not a model, but an actual mathematical inevitability based upon purchase price.

Parity Graph
Expected Return - Tesla Jan 2020 Call @ 300

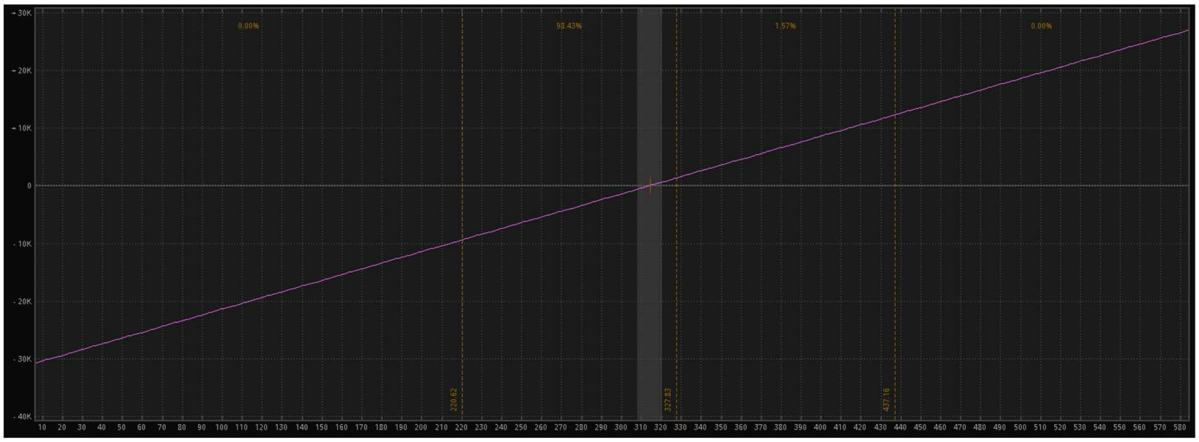


Purple: projected change in price of option at present time for different prices using Black-Scholes model; Turquoise: profit at expiration; X-axis: strike price, Y-axis: profit

Source: TDAmeritrade 11/24/17

Note the differences in the two "curves"; the purple model shows a continuous modelling of change in option price using the Black-Scholes model for different strike prices. The Black-Scholes Model is valuing this option is if the underlying asset (Tesla) were at a price of $320 today (red line). Expected profit would change with the curve as the price of Tesla stock goes up or down. Note that today's price is $312, so the model is not valuing the option price perfectly. Of note in visualizing the call's profit at expiration (turquoise), note the flat lefthand side of the curve <300. If Tesla's stock price is anywhere below 300 at option expiration, the buyer of the call will lose their entire $8,585. If Tesla's stock is at 450 at Jan 20 expiration, the buyer of the call is guaranted approximately $6,000 profit. To break even, the stock must end at 385 or higher (red line on right) . The shaded area in the center of the chart represents expected price tomorrow based on present volatity. This chart shows the overall risk and return for the call option at expiration. But what is the risk of a buyer simply buying 100 shares of the underlying asset at expiration rather than the derivative?

Parity Graph – Expected Return of Underlying Asset: TSLA



Profit/Return of 100 shares Tesla stock, time independent *Source: TDAmeritrade 11/24/17*

As one can see, buying the underlying asset risks far more capital, as Tesla stock could conceivably go bankrupt and the stock price to 0; the curve shows us that such would represent a loss of approximately $31,000. On the other hand, if Tesla proceeds to 450, there would be a profit of apprximately $14,000. This is not a model, but a time independent linear function that shows actual return based on price of the underlying asset bought at today's price. The profit/loss will be the same whether the price change occurs today or a year from now. As such, we see that option price is time dependent, whereas the underlying asset's price is time independent.

The Problem: How to Model an Option's Price?

Before entering into a discussion of the Black-Scholes Model, lastly let us take a look at some of the factors that can affect an option's price. An option's price at expiration at its base level must be a function of time t days until expiration. Thus any model requires an examination of what are the critical variables that affect the price $w(t)$. Will these be constants, independent variables, or dependent variables? By no means an exhaustive list, here are many factors that could affect option price:

- Change in underlying asset price
- Time (t): days to expiration
- Interest rate: present interest rate one benefits from holding cash
- Dividends: dividends that a stock or asset pays
- Volatility: statistical measure of an asset's fluctuation in price
  - Historical Volatility: actual volatility calculated based on historical data of price movement
  - Implied Volatility: an option's valuation of expected future volatility

To understand volatility better, a visual representation serves well. (Display graph of Tesla stock with implied volatility and future projection)

Historical Perspective: Options

Options and forward contracts have been in use throughout history, as long as the civilized world has had barter systems exchanging physical commodities. The first known options existed in ancient Greece for speculation on olive harvests. In the United States, the early 20th century featured "bucket shops" in which speculators could bet on the rise or fall of a given stock/security through options without actually buying or selling the underlying asset. The first exchange traded options were established with the opening of the Chicago Board Options Exchange (CBOE) in 1973, which still exists today. Black and Scholes published their paper "The Pricing of Options and Corporate Liabilities" in the same year.

Historical Perspective: Options Models

As there was no regulated and centralized marketplace for options prior to 1973, there was less of an academic/marketplace demand for a generally accepted model of options valuation. One would assume options trade was more the domain of speculators betting on the movement of underlying assets rather than an options pricing model.

In 1877, Charles Castelli published "The Theory of Options in Stocks and Shares" in London, an examination of some options trading strategies. In 1900, perhaps the first options pricing model was published by French mathematician Louis Bachelier as "The Theory of Speculation" as his doctoral thesis. Bachelier introduced many concepts of stochastic analysis into financial options valuation, albeit stochastic analysis did not exist yet. In his thesis, Bachelier modeled "Brownian motion" as a stochastic process and applied it to stock movements. Brownian motion refers to the random movement of particles suspended in a gas, liquid or other medium. In the finance world, the analogy certainly makes sense, as we need to model stocks (particles) given some overall market environment constantly subject to change (change of temperature due to heat which increases motion, for example). Bachelier's model generally was left to the academic world and did not enter the world of financial application until revived by Black, Scholes, and Merton.

Black and Scholes' paper cites other modelling work from the 60's that had been done on "warrants". Such is beyond the scope of this paper, but in brief, warrants represent a contractual option to buy stock XYZ at a future expiration date just like call options. However, warrants are only issued by the company itself (Tesla) and not private parties, speculators or exchanges. Warrants typically have expiration dates in years rather than months, and are "dilutive" in that they create new shares when exercised, rather than applying a sale to existing shares.

Black, Scholes, and Merton: The Authors

The model being examined in this paper was published as "The Pricing of Options and Corporate Liabilities" by Fischer Black and Myron Scholes by the University of Chicago Press in 1973. Though the paper was published by Black and Scholes, Robert Merton from MIT was also instrumental in the model's development. Earlier in 1973, Merton published the "Theory of Rational Option Pricing" which was instrumental to and cited in in Black and Scholes' work along with "A Complete Model of Warrant Pricing that Maximizes Utility". All three of these authors released several papers in the early 1970's examining facets of the topic at hand, but "The Pricing of Options and Corporate Liabilities" is the publication which produced the Black-Scholes Equation itself, detailing the valuation of options pricing.

$$w_2 = rw - rxw_1 - \frac{1}{2}v^2x^2w_{11}$$

Accordingly, the model is typically called "The Black-Scholes Options Pricing Model", but given Merton's contribution, it is often referred to as the "Black-Scholes-Merton Model." We shall call it "BSM" for the sake of brevity hereafter.

Assumptions in the BSM Model

Any model must examine its underlying assumptions and simplifications. The BSM model is no different, and after an overview of options, various assumptions used to build the model are laid out.

1. The model prices a call option
2. One call option is a contract with the option to buy 1 share of stock XYZ at the expiration date. In today's markets, these are typically done in increments of 100 shares, while the futures market options is a contract for 1 future
3. If the current price of a stock is much less than the option's exercise price, the option's value should be close to 0, as there is less probability of it having value at expiration
4. The current value of an option will be approximately the value of the underlying asset minus the price of a pure discount bond at strike price with the same maturity date. If the expiration date is far in the future, the value of the bond will be low, and one can expect the value of the option to equal the price of the stock.
5. If the expiration date of the option is near, the value of the option will be approximately the price of the underlying asset minus the exercise price.
6. One would never value an option more than the underlying asset, for in such an instance, one should simply buy the underlying asset.
7. The market entails "ideal conditions". (see below)

With 3, 4, and 5 above, we see initial conditions that the model should meet and can be used as the model is derived. These assumptions also lend themselves to criticisms of the model, to be discussed later.

Point 7 is expounded upon prior to the BSM derivation, which I merely enumerate here:

a) "The short term interest rate is known and is constant through time"
b) "Stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price"
c) The stock pays no dividends
d) The option can only be exercised at maturity
e) No transaction fees for buying/selling the stock or option
f) Fractional shares allowed
g) No penalties for short selling

With these simplifications, the authors are very clear that the value of the option can be determined with known quantities. Thus:

h) one can create a "riskless hedge" based upon an underlying asset and its concomitant option

Assumption 7b) The Random Walk of the Markets and Brownian Motion/Wiener Process

The random walk theory of markets was essentially utilized by Bachelier in "The Theory of Speculation" published in 1900, viewing the movement of stock prices as Brownian motion above. As a physical process, Brownian Motion consists of the random motion of particles suspended in some medium; mathematically, it is an application of the Wiener Process, a continuous stochastic process.

Black and Scholes specifically state a key idea that "our assumption that the stock price follows a continuous random walk and that the return has a constant variance rate". It is worth examining this key tenet in greater depth outside of their paper to understand the author's premises.

Viewing the markets as a random walk makes the assumption that the markets are *efficient,* that is, stock prices reflect all available information and expectations at any point in time. As such, movements in price are a reflection of random movement, or essentially a coin toss. With this premise, we can make the journey into probability theory and statistics.

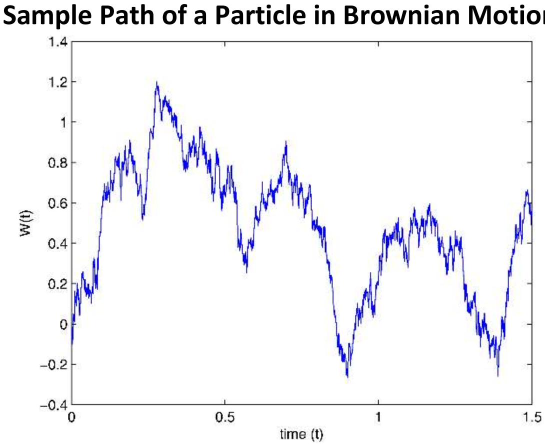
Given the probability P(x) =.5 that a stock will go up or down in price, one could simulate the motion of that market over time increments Δt and statistically expect outcomes to distribute as a Bernoulli distribution with results (1,-1) for each Δt.

$$f(\Delta t)=(1,-1)$$

We know under the Central Limit Theorem, that given enough independent trials, the binomial/Bernoulli distribution will approach a normal distribution. As such, when viewing these independent trials of stock price movements, we are dealing with a discrete-time process based on each movement ("up" or "down") which results in a normally distributed set of outcomes.

It is difficult to fully grasp the mathematics of Brownian motion, which requires stochastic calculus and Ito calculus, but here is my best attempt. Brownian motion represents a continuous time process, rather than a discrete time process. Nonetheless, it still consists of random variables normally distributed and a mean of 0, and as such, can be obtained from the symmetric random walk, ie Bernoulli variable with a binomial distribution.

Sample Path of a Particle in Brownian Motion



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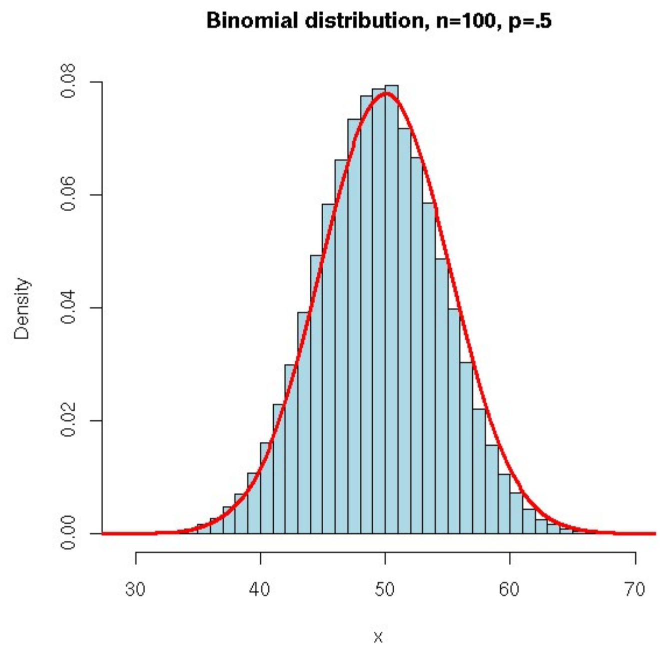
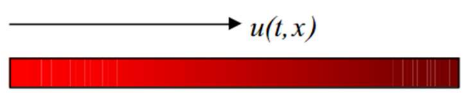


Given enough iterations, the Central Limit Theorem tells us that the Binomial Distribution will approach a normal distribution, as demonstrated above through software with $\mu = 50$ (our starting position).

We proceed back to physics, Brownian motion and the heat equation.

Let u(t,x) be the temperature in a rod (spatially a one dimension model?) that satisfies the heat equation:



$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2},\ t > 0 \qquad (2)$$

Recalling the probability density of the normal distribution:

$$N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (3)$$

Knowing $\mu = 0$ and letting $\sigma^2 = t$, we find that this function fulfills the heat equation:

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x)^2}{2t}} \qquad (4)$$

However I could not prove this mathematically. Skipping some math and substitutions, we eventually arrive at:

$$u(t,x) = E[\varphi(X_t)] \text{ ,where } X_t = N(x,t). \qquad (5)$$

A stochastic differential equation (SDE) is a type of partial differential equation (PDE) which incorporates a term which is a stochastic/random process (collection of random variables which changes over time), and thus whose solution also incorporates a stochastic term.

Black and Scholes utilize "Geometric" Brownian Motion and incorporate a stochastic term into their model, specifically an Ito Process which is essentially Brownian motion using functions a_t for "drift" and b_t for "diffusion". The addition of this stochastic term adds "white noise", which allows for Brownian motion with a non-zero mean. A non-zero mean seems necessary to me considering the upward drift of overall stock prices historically (regardless of inflation, interest rates and other potential factors). Put another way, price movements in the market are *not* normally distributed.

Ito Process:

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t = a_t dt + b_t dW_t \qquad (6)$$

Given that X_t is an Ito Process, suppose g(x) is a twice continuously differentiable function. Then:

$Y_t = g(X_t)$ is also an Ito Process

Ito's Formula:

$$dY_t = \frac{\partial g}{\partial x}(X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t)(dX_t)^2 \qquad (7)$$

Substitute (6):

$$dY_t = \left(\frac{\partial g}{\partial x}(X_t)a_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t)(b_t)^2\right)dt + \frac{\partial g}{\partial x}(X_t)dW_t \qquad (8)$$

I do not understand the mathematics involved here, but I thank MIT Open Courseware and their course on Advanced Stochastic Processes for giving some equations to which I could transform Ito Calculus Equations into the forms of the paper being studied. These equations are utilized in the BSM, but as substitutions without additional explanation or arithmetic. The BSM Model also specifically employs Geometric Brownian Motion, and thus incorporates a logarithmic function.

<u>Assumption 7h) The Riskless Hedge:</u>

Let O = # of options sold; $w(x, t)$=option price (PDE) | x=stock price, t=time

For each share of stock we want to hedge,

$$\text{Let } \frac{\partial w(x,t)}{\partial x} = w_1(x, t)$$

$$Ow_1(x, t) = 1\ share$$

$$O = \frac{1}{w_1(x,t)} \qquad (9)$$

In the riskless hedge, one would buy the underlying asset (long position) and sell $\frac{1}{w_1(x,t)}$ call options on it (short position). The authors are claiming that it is possible to create an aggregate position of stocks and options on a single asset, such that there is no risk and overall parity of capital can be maintained. Keep in mind that $w_1(x, t)$ is a function of time, so as the option price changes, the number of options needed in the hedge would have to change accordingly. Such a "riskless hedge" must be constantly maintained as market price changes over time; the authors assume that such is a given for this model (which is physically impossible, given fluctuations, never mind trading costs).

The Black-Scholes-Merton Model

Given the many above conditions and presuppositions, we are finally ready to begin deriving the BSM Model. The riskless hedge of 7b is a key supposition in the initial derivation. Given the fact that one can always execute a continuous riskless hedge, recall the riskless hedge:

$$O = \frac{1}{w_1(x,t)} \tag{10}$$

For a stock price x, the total equity invested in a position:

$$x - Ow(x,t)$$

And in a riskless hedge we expect the total equity to be 0

(1 share of X)(Price of X) – (number of options)(Price of option)=0

$$x - Ow(x,t) = 0$$
$$x = Ow(x,t)$$

Substitute for O (10):

$$x = \frac{w(x,t)}{w_1(x,t)} \tag{11}$$

For small values of t (Δt), Equation (11) becomes:

$$\Delta x = \frac{\Delta w(x,t)}{w_1(x,t)} = \frac{\Delta w}{w_1}$$
$$\Delta w = w_1 \Delta x \tag{12}$$

From hereon $w = w(x,t)$, $w_1 = w_1(x,t)$, and recall that $w_1$is the partial derivative in relation to stock price x. From the Fundamental Theorem of Calculus, for small intervals of t we know:

$$\Delta w = w(x + \Delta x, t + \Delta t) - w(x,t) \tag{13}$$

Recall Ito's Formula which we have transformed with substitutions:

$$dY_t = \left(\frac{\partial g}{\partial x}(X_t)a_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t)(b_t)^2\right)dt + \frac{\partial g}{\partial x}(X_t)dW_t \tag{8}$$

Given that we have a riskless hedge, the authors combine (8) Brownian motion and (13) FTOC:

$$\Delta w = w_1 \Delta x + \frac{1}{2} w_{11} v^2 x^2 \Delta t + w_2 \Delta t \tag{14}$$

Where v^2 is the variance rate of the return on the stock. Frankly, I cannot make full sense of this step. The first two terms makes sense, as:

$$\Delta w = dY_t \qquad and \;\; w_1 \Delta x = \frac{\partial g}{\partial x}(X_t)dX_t$$

However, the 3rd and 4th terms only seem possible if the authors used a slightly modified Ito's Formula, which I shall hereby attempt.

Ito's Formula, with a Kneeland Transformation:

$$dY_t = \frac{\partial g}{\partial x}(X_t)a_t dt + \frac{1}{2}\frac{\partial^2 g}{\partial x \partial t}(X_t)(b_t)^2 dt + \frac{\partial g}{\partial x}(X_t)dW_t \tag{15}$$

The authors' resultant derivation:

$$\Delta w = w_1 \Delta x + \frac{1}{2} w_{11} v^2 x^2 \Delta t + w_2 t \tag{16}$$

At least here in (16), the middle term now involves partial differentiation with respect to the x and t variables, rather than the second partial derivative with respect to x. I still do not understand how the author's derived $v^2 x^2$ for the middle term. I will simply have to take the author's mathematics at face value.

Recalling (12), which is the change in the value of the equity in a short time period :

$$\Delta x = \frac{\Delta w}{w_1}, \text{ given small } \Delta t \qquad (12)$$

Substitute (16) Δw into (12):

$$\Delta x = \Delta x + (\tfrac{1}{2} w_{11} v^2 x^2 + w_2) \frac{\Delta t}{w_1} \qquad (17)$$

These additional terms point towards a further change in equity (profit). However, by definition, the hedged position has no risk. A price discrepancy would encourage market participants to arbitrage the discrepancy to the also riskless result due to a short term interest rate. If the profit were greater than the short term interest rate, the very act of market buyers acting on the opportunity would reduce the opportunity to zero. That is, the markets are efficient! (CAPM)

In other words, the overall change in the value of the position we derived should equal the profit one could make from simply investing that money at the short term interest rate:

$$Interest = equity\ X\ interest\ rate\ X\ time$$

$$\Delta\ equity = \left(x - \frac{w}{w_1}\right) r \Delta t \qquad (18)$$

The last terms of (16) must equal (17), but negative since it is a hedged position and this balances inversely. Combining (16) and (17):

$$-\left(\frac{1}{2} w_{11} v^2 x^2 + w_2\right) \frac{\Delta t}{w_1} = \left(x - \frac{w}{w_1}\right) r \Delta t \qquad (19)$$

Divide by Δt:

$$-\left(\frac{1}{2} w_{11} v^2 x^2 + w_2\right) \frac{1}{w_1} = \left(x - \frac{w}{w_1}\right) r$$

Solve for differential equation of the value of the option’s second derivative, w_2:

$$-\left(\frac{1}{2} w_{11} v^2 x^2 + w_2\right) = (x w_1 - w) r$$

$$\frac{1}{2} w_{11} v^2 x^2 + w_2 = wr - x w_1 r$$

And we have our final modelled equation, the Black-Scholes-Merton Model

$$\boldsymbol{w_2 = rw - rxw_1 - \frac{1}{2} w_{11} v^2 x^2} \qquad \textbf{(20)}$$

But how to solve for w(x,t)? The authors cite previous boundary conditions. Recall that at option maturity t=t* with exercise price c, the value of the option call is:

$$w(x, t^*) = x - c, \qquad x \geq c$$

$$w(x, t^*) = 0, \qquad x < c$$

This makes perfect sense, for if the option exercise price is less than the stock price at expiration, it expires worthless. Otherwise, its value is the difference between the stock price and the option exercise price.

From here on out, most of the mathematics and substitution logic is beyond my capabilities. But of note processwise, the authors utilize extensive boundary conditions and a substitution for :

$$y_2 = y_{11}$$

$$y(u,s) = 1/\sqrt{2\pi} \int_{-u/\sqrt{2s}}^{\infty} c\left[e^{(u+q\sqrt{2s})\left(\frac{1}{2}v^2\right)/\left(r-\frac{1}{2}v^2\right)} - 1\right] e^{-q^2/2}\, dq. \tag{12}$$

Finally, with more substitutions we arrive at the final derivations:

$$\boldsymbol{w(x,t) = xN(d_1) - ce^{r(t-t*)}N(d_2)} \qquad \textbf{(21)}$$

$$d_1 = \frac{\ln\left(\frac{x}{c}\right) + (r + \frac{1}{2}v^2)(t^* - t)}{v(\sqrt{t^* - t}}$$

$$d_2 = \frac{\ln\left(\frac{x}{c}\right) + (r - \frac{1}{2}v^2)(t^* - t)}{v(\sqrt{t^* - t}}$$

Where $N(d_1)$ is the normal density function. Now, one can solve for the value of an option given stock price x, interest rate r, time remaining to expiration, variance v^2, and exercise price c.

Solving for the first derivative:

$$\boldsymbol{w_1(x,t) = N(d_1)} \qquad \textbf{(22)}$$

This leaves us with three final equations with which one can derive an option price, as well as its rate of change (first derivative) and its second derivative:

$$\boldsymbol{w(x,t) = xN(d_1) - ce^{r(t-t*)}N(d_2)} \qquad \textbf{(21)}$$

$$\boldsymbol{w_1(x,t) = N(d_1)} \qquad \textbf{(22)}$$

$$\boldsymbol{w_2(x,t) = rw - rxw_1 - \frac{1}{2}w_{11}v^2x^2} \qquad \textbf{(20)}$$

The authors perform another derivation of the model using the Capital Asset Pricing Model laid out by Merton (CAPM), but that is beyond the scope of this paper.

Model Success

With the birth of the BSM Model in 1973, the financial markets entered a new era of hedging, arbitrage, and valuation of assets. The model would propel options trading to practically a whole new asset class capable of curtailing risk in portfolios (as well as intensifying risk when used incorrectly or exogenic shocks would occur). The differentiation of the model led to a widespread adoption of “the Greeks”, variables which are used by traders to assess aspects of option valuation:

$$Delta: \frac{\partial C}{\partial x} \qquad Gamma: \frac{\partial^2 C}{\partial x^2} \qquad Vega: \frac{\partial C}{\partial \sigma} \qquad Theta: \frac{\partial C}{\partial t} \qquad Rho: \frac{\partial C}{\partial r}$$

In 1997, the contribution of this model to the quantitative financial world was recognized with the Nobel Prize in Economic Sciences being awarded to Merton and Scholes in 1997. Unfortunately, Fischer Black had passed away in 1995, and the Swedish Academy rendered the deceased as an “ineligible recipient”, designating Black merely as a “contributor” despite his co-authorship.

Shortcomings of the Model

Any model requires simplification, and many were made in ways which render this model inaccurate in certain respects. However it must be noted with hindsight that the model did finally offer an industry wide accepted model which did approximate option values in general, particularly when the market was in equilibrium. Whether this was also a self fulfilling prophecy is another discussion. Shortcomings are many, and I shall keep the list brief:

1) trading costs are not zero
2) All investors cannot borrow at the same interest rate
3) a stock’s volatility is not constant
4) markets are not efficient
5) market movement is not normally distributed
6) options are often exercised early
7) If markets are efficient and priced correctly, there would be no profits on hedged positions
8) Stock prices are not continuous

The misunderstanding of these and much subtler risks would contribute to multiple market crashes due to exogenic shocks to the markets. Events occur and volatility goes up; markets are not efficient, so when volatility creates market movement, options valuation by BSM would be drastically inadequate. An investment bank building a portfolio of millions in such an event could find itself completely insolvent due to the mismanagement of risk. For example, Long-Term Capital Management suffered precisely such a catastrophe, managing $7.5 billion in 1998 to liquidating for $250 million in August that year. Over the course of a few days, LTCM was actually losing $550 million dollars per day to being overleveraged on derivative positions. Such a catastrophe was a wakeup call for the financial world to manage risk better, reevaluate derivatives models, and regulate leverage. The “riskless hedge” of BSM is in the end an illusion for those who want to believe the markets are efficient.

Beyond BSM

The authors did continue to refine their model with additional papers, and later models were built by others to address shortcomings in the BSM Model. Some models of note include the Binomial Options Pricing Model which uses a “lattice based” numerical method that functions iteratively. The Black-Derman-Toy Model is a stochastic model which built further on this lattice model with a view to mean reversion. Of note, Black was also involved with this last model at Goldman Sachs. Modelling continues, and much of this work is done behind closed doors with proprietary models owned by the investment banks that build them. Black, Scholes, and Merton are the forefathers of a revolution in finance which continues to this day.

Partial Bibliography/Resources

Derman, E. (2004). *My Life as a Quant.* John Wiley and Sons.

Fischer, B. Scholes. (May-Jun., 1973).M. *The Pricing of Options and Corporate Liabilities.* The Journal of Political Economy. Vol 81, No 3.University of Chicago Press

Fontanills, G. (2005). *The Options Course.* John Wiley and Sons.

Hull, John C. (2008). *Options, Futures and Other Derivatives* (7 ed.). Prentice Hall.

Malkiel, B. (1973). *A Random Walk Down Wall Street*. W.W. Norton.

Merton, R.C. (1973). *Theory of Rational Option Pricing.* Bell Journal of Economics and Management Science, Vol 4, No 1. RAND Corporation

Morters, P. Peres, Y. (2008). *Brownian Motion.*

Szpiro, G.G. (2011). *Pricing the Future.*Perseus Books Group.

MIT courses: Advanced Stochastic Processes-fall 2013

Natenberg, S. (2015). *Option Volatility and Pricing*.McGraw Hill.

www.probabilitycourse.com