Università della Svizzera italiana
Institute of Computing CI

Numerical Computing **2023**

Student: FULL NAME **Discussed with:** FULL NAME

Bonus assignment

Due date: Wednesday, 22 November 2023, 11:59 PM

Numerical Computing 2023 — Submission Instructions

(Please, notice that following instructions are mandatory: submissions that don't comply with, won't be considered)

- Assignments must be submitted to iCorsi (i.e. in electronic format).
- Provide both executable package and sources (e.g. C/C++ files, MATLAB). If you are using libraries, please add them in the file. Sources must be organized in directories called:
  Project_number_lastname_firstname
  and the file must be called:
  project_number_lastname_firstname.zip
  project_number_lastname_firstname.pdf
- The TAs will grade your project by reviewing your project write-up, and looking at the implementation you attempted, and benchmarking your code's performance.
- You are allowed to discuss all questions with anyone you like; however: (i) your submission must list anyone you discussed problems with and (ii) you must write up your submission independently.

Exercise 1: Inconsistent systems of equations [10 points]

Consider the following inconsistent systems of equations:

(a) $A_1 x = b_1$, where

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} b_1 = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$$

(b) $A_2 x = b_2$, where

$$A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} b_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Find the least squares solution x^* and compute the Euclidean norm of the residual, SE and RMSE.

solution:
Least Square solution x^* can be obtained by solving the following equation:

$$A^T A x = A^T b \tag{1}$$

Then from the x^* obtained, we can get the residual vector as following and from it, we can calculate the Euclidean norm and proceed to SE (Sum of Square Residuals) and RMSE (Root Mean Squared Error):

$$r = Ax^* - b \tag{2}$$

$$EuclideanNorm = ||r||_2 \tag{3}$$

$$SE = ||r||_2^2 \tag{4}$$

$$RMSE = \sqrt{\frac{SE}{m}} \tag{5}$$

Where m is the number of rows in residual vector.

(a)

$$\begin{bmatrix}1 & 1 & 1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 0\\1 & 0\end{bmatrix}x = \begin{bmatrix}1 & 1 & 1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}5\\2\\4\end{bmatrix}$$

$$\begin{bmatrix}3 & 0\\0 & 0\end{bmatrix}x = \begin{bmatrix}11\\0\end{bmatrix}$$

$$x^* = \begin{bmatrix}3.6667\\0\end{bmatrix}$$

$$r^* = \begin{bmatrix}1 & 0\\1 & 0\\1 & 0\end{bmatrix}\begin{bmatrix}3.6667\\0\end{bmatrix} - \begin{bmatrix}5\\2\\4\end{bmatrix} = \begin{bmatrix}-1.3333\\1.6667\\-0.3333\end{bmatrix}$$

$$EuclideanNorm = ||r||_2 = \sqrt{(-1.3333)^2 + (1.6667)^2 + (-0.3333)^2} = \sqrt{4.6667} \approx 2.1602$$

$$SE = ||r||_2^2 = (-1.3333)^2 + (1.6667)^2 + (-0.3333)^2 \approx 4.6667$$

$$RMSE = \sqrt{\frac{||r||_2^2}{m}} \approx \sqrt{\frac{4.6667}{3}} \approx 1.2472$$

(b)

$$\begin{bmatrix}1 & 0 & 1 & 1\\1 & 1 & 2 & 0\\0 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 2 & 1\\1 & 0 & 1\end{bmatrix}x = \begin{bmatrix}1 & 0 & 1 & 1\\1 & 1 & 2 & 0\\0 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}2\\2\\3\\4\end{bmatrix}$$

$$\begin{bmatrix}3 & 3 & 2\\3 & 6 & 3\\2 & 3 & 3\end{bmatrix}x = \begin{bmatrix}9\\10\\9\end{bmatrix}$$

If we re-arrange the formula for x^* we get:

$$x^* = (A_2^T A_2)^{-1} A_2^T b_2 \tag{6}$$

and the resulting x^* will be

$$x^* = \begin{bmatrix}2\\-0.3333\\2\end{bmatrix}$$

$$r = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 2 & 1\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}2\\-0.3333\\2\end{bmatrix} - \begin{bmatrix}2\\2\\3\\4\end{bmatrix} = \begin{bmatrix}-0.3333\\-0.3333\\0.3333\\0\end{bmatrix}$$

$$EuclideanNorm = ||r||_2 = \sqrt{(-0.3333)^2 + (-0.3333)^2 + (0.3333)^2 + (0)^2} = \sqrt{0.3333} \approx 0.5774$$

$$SE = ||r||_2^2 = (-0.3333)^2 + (-0.3333)^2 + (0.3333)^2 + (0)^2 = 0.3333$$

$$RMSE = \sqrt{\frac{||r||_2^2}{m}} \approx \sqrt{\frac{0.3333}{4}} \approx 0.2887$$

Exercise 2: Polynomials models for least squares [20 points]

Exercise 3: Analysis of periodic data [20 points]

Exercise 4: Data linearization and Levenberg-Marquardt method for the exponential model [20 points]

Exercise 5: Tikhonov regularization [15 points]