Homework 1

November 2024

Problem 1

If m, n, p and q are integers, then

(i) $(m+n)(p+q) = (mp+np) + (mq+nq)$

(ii) $m + (n+p) = (p+m) + n$

(iii) $m(np) = p(mn)$

Problem 2

Let $x \in \mathbb{Z}$.

(a) If x has the property that for each integer m, $m + x = m$, then $x = 0$.

(b) If x has the property that for all $m \in \mathbb{Z}$, $mx = m$, then $x = 1$.

(c) If $x \cdot x = x$, then $x = 0$ or $x = 1$.

Problem 3

Read the following statements and show that they are *false* by giving a counter-example.

(a) Let $n, m \in \mathbb{N}$ be two natural numbers, such that n is even and m is odd. Then $n + m$ is even.

(b) Let $n \in \mathbb{N}$ be an even natural number, then there exists an even natural number $m \in \mathbb{N}$ such that their sum $n + m$ is odd.

(c) Let $a, b, c \in \mathbb{N}$ be three non-zero natural numbers such that $a^2 + b^2 = c^2$. Then the three numbers must be $a = 3$, $b = 4$, and $c = 5$.

Problem 4

Explain how the following two statements are different and prove that one of them is false.

- For all $n \in \mathbb{Z}$ there exists an integer m such that $n + m = 42$.
- There exists an integer m such that for all $n \in \mathbb{Z}$, $n + m = 42$.

Problem 5

Prove the following related statements using induction:

(a) Let $k \in \mathbb{N}$. Then $k \geq 1$.

(b) There exists no integer such that $0 < x < 1$.

(c) Let $n \in \mathbb{Z}$. There exists no integer x such that $n < x < n + 1$.

(d) For all $k \in \mathbb{N}$, $k^2 + 1 > k$.