Math Circle 11/9/24

- Join Gradescope: Entry code Z3887V
- Red line grading method
- Using "art of proof" - available for free

- Idea is to start with definitions, axioms, etc. and use these things to build intuition
- The goal is that students can use this minimal set of tools to expand their toolbox.
- It's okay if this feels "dumb" or redundant, that's good! We need to build intuition first before fancier things.

We will start by looking at $\mathbb{Z}$, the integers with $+$ and $\cdot$

We define this such that if m, n, p are integers

(1) $m+n = n+m$
(2) $(m+n)+p = m+(n+p)$
(3) $m(n+p) = mn + mp$
(4) $mn = nm$
(5) $(mn)p = m(np)$

(6) There is an integer 0 s.t. $\forall m \in \mathbb{Z}$, $m+0=m$
(7) $\exists\ 1 \in \mathbb{Z}$ s.t. $1 \neq 0$, $m \cdot 1 = m$
(8) $\forall m \in \mathbb{Z}$, $\exists\ (-m)$ s.t. $m+(-m)=0$
(9) If $mn = mp$ and $m \neq 0$, then $n=p$

Prop 1.6: If $m, n, p \in \mathbb{Z}$, then

$$(m+n)p = mp + np$$

$$(m+n)p \overset{(4)}{=} p(m+n) \overset{(3)}{=} pm + pn \overset{(4)}{=} mp + np$$

Now that I did one, you try

(1.7) If $m \in \mathbb{Z}$, $0+m=m$ and $1 \cdot m = m$

(Pf) $0+m \overset{(1)}{=} m+0 \overset{(6)}{=} m \quad 1 \cdot m \overset{(4)}{=} m \cdot 1 \overset{(7)}{=} m$

(1.8) $(-m)+m=0$

(Pf) $(-m)+m \overset{(1)}{=} m+(-m) \overset{(8)}{=} 0$

(1.9) If $m+n=m+p$, then $n=p$

(Pf)
$$m+n=m+p$$
$$(-m)+(m+n)=(-m)+(m+p) \quad (2)$$
$$0+n=0+p \quad \text{Prop } 1.8$$
$$n=p \quad \text{Prop } 1.7$$

(1.10) (Showing uniqueness) If $m+x_1=0$ and $m+x_2=0$, then $x_1=x_2$

(Pf) $x_1=x_2$ (Prop 1.9)

(1.23) The x s.t. $m+x=n$ is unique.

(Pf) $m+x_1=n=m+x_2 \Rightarrow x_1=x_2$ (1.10)

(1.26) If $mn=0$, then $m=0$ or $n=0$

(Pf) $mn=m \cdot 0$ If $m=0$, done otherwise $n=0$

Section 2

Def $\mathbb{N} \subseteq \mathbb{Z}$ s.t.

(1) $m, n \in \mathbb{N}$, $m+n \in \mathbb{N}$
(2) $m, n \in \mathbb{N}$, $mn \in \mathbb{N}$
(3) $0 \notin \mathbb{N}$
(4) For every $m \in \mathbb{Z}$, then $m \in \mathbb{N}$, $m=0$ or $-m \in \mathbb{N}$

Prop 2.2
Exactly 1 of (4) is true.

(Pf) If $m=0$, then $m \notin \mathbb{N}$, $-m=0 \notin \mathbb{N}$

If $m, -m \in \mathbb{N}$, then $m+(-m)=0 \in \mathbb{N}$ a contradiction

So only one is true

Prop 2.3 $1 \in \mathbb{N}$

Theorem 2.17 (Induction)
Let $P(K)$ be a stament true or false dependent on K

If
(1) $P(1)$ is true
(2) $P(n)$ is true implies $P(n+1)$ is true
then $P(K)$ is true for all $K \in \mathbb{N}$