Last Time

① Axioms of the integers
② Axioms of the natural numbers

Induction

$P(k)$ is a true/false statement depending only on k.

Thm

If

① $P(1)$ is true

② $P(k)$ is true implies $P(k+1)$ is true

then $P(k)$ is true for all $k \in \mathbb{N}$

Ex

For all integers $k \geq 2$, $k^2 < k^3$

Proof

Base Case: $2^2 = 4 < 8 = 2^3$

Assume that $k^2 < k^3$

then

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1 > 4k^2 + 3k + 1 > k^2 + 2k + 1 = (k+1)^2$$

$\square$

Well-Ordering Principle

Let $A \subseteq \mathbb{Z}$ be a nonempty subset. If there exists $b \in A$ such that for all $a \in A$, $b \leq a$, then b is the smallest element of A, written as $b = \min A$

Ex

Consider the integers divisible by 6, does this have a min?

What about the positive integers divisible by 6?

Thm

Every nonempty subset of $\mathbb{N}$ has a smallest element.

Proof

Consider the set

$$N = \left\{ K \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{every subset of } \mathbb{N} \text{ that} \\ \text{contains an integer} \leq K \text{ has a} \\ \text{smallest element} \end{array} \right\}$$

W.T.S. $N = \mathbb{N}$

Base Case $K = 1$ $\min(\mathbb{N}) = 1$ so if $1 \in A$, then 1 is the smallest element

Proof Cont.

If $n \in \mathbb{N}$ and $A \subseteq \mathbb{N}$ s.t. $m \in A$ for some $m \leq n$
$\Rightarrow A$ has a smallest subset.

Suppose $B \subseteq \mathbb{N}$ is a subset s.t. $\exists x \in B$, $x \leq n+1$.

Case I: $\exists y \in B$ s.t. $y \leq n \Rightarrow$ by inductive hypothesis B has a smallest element

Case II: $\forall y \in B$, $y \geq n+1 \Rightarrow \min(B) = n+1$ by definition
$\Rightarrow B$ has a smallest element

$\square$

Why do we care?

- Some people take this to be an axiom and use it to prove induction.

Def

Given two integers, m and n, we define

$$\gcd(m,n) = \min \left\{ k \in \mathbb{N} : k = mx + ny \text{ for some } x, y \in \mathbb{Z} \right\}$$

Exercises

1) Compute $\gcd(4,6)$, $\gcd(7,13)$, $\gcd(-4,10)$.

2) Given a nonzero integer n, compute $\gcd(0,n)$ and $\gcd(1,n)$

3) If $m, n \in \mathbb{Z}$, m, n are not both 0, then $S = \{ k \in \mathbb{N} \mid k = mx + ny \text{ for some } x, y \in \mathbb{Z} \}$ is not empty.

Introduction to Recursion

Def
Assume that for each $j \in \mathbb{N}$, we associate some $x_j \in \mathbb{Z}$. We call this list of integers a sequence and denote it $\{x_j\}_{j \in \mathbb{N}}$.

Examples
(1) $\{x_j\} = (x_1, x_2, \ldots, x_{500}, x_{501}, \ldots)$
(2) $\{x_j\} = (36, 41, 7, 63, 526, -46, \ldots)$
(3) $\{x_j\} = (1, 2, 4, 8, 16, \ldots)$
(4) $\{x_j\} = (1, 1, 2, 3, 5, 8, 13, 21, \ldots)$

Three Classes of Recursion

1) Given a sequence $(a_1, a_2, a_3, \ldots)$ we can define $x_j := \sum_{i=1}^{j} a_i$ with $x_1 = a_1$ and $x_{j+1} = x_j + a_{j+1}$

2) Given a sequence $(b_1, b_2, b_3, \ldots)$ we can define $x_j := \prod_{i=1}^{j} a_i$ with $x_1 = b_1$ and $x_{j+1} = x_j \, a_{j+1}$.

3) Special case of 2 is $n!$ where $1! = 1$ and $(n+1)! = n!(n+1)$.

Ex Let $m \in \mathbb{N}$, we define $x_1 = m$ and

$$x_{n+1} = \begin{cases} \frac{x_n}{2} & \text{if } n \text{ is even} \\ 3x_n + 1 & \text{if } n \text{ is odd} \end{cases}$$

Collatz Conjecture

$\forall m$, (x_i) eventually hits 1 (Still very open)

Geometric Series

Fix $m=3$ and consider $(1,3,9,27,81,\dots)$

$a_i = 3^i$

What if we sum the first j terms?

$1 = 1$

$1+3 = 4$

$1+3+9 = 13$

$1+3+9+27 = 40$

$1+3+9+27+81 = 121$

Def

$x_j = \sum_{i=0}^{j} r^i$ is called a "geometric series"

Question

Does there exist a formula for x_j?

Solution

$$1 + r + r^2 + r^3 + \dots + r^n$$

$$-r - r^2 - r^3 - \dots - r^n - r^{n+1} = -r(1 + \dots + r^n)$$

$= 1 - r^{n+1}$

$\Rightarrow \left(\sum_{i=0}^{n} r^i\right)(1-r) = 1 - r^{n+1}$

$\Rightarrow \sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$

Proof

Induct on n

Base Case: $\sum_{i=0}^{0} r^i = 1 = \frac{1-0}{1-0} = 1$

Inductive Hypothesis: Assume $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$

Note $\sum_{i=0}^{n+1} r^i = \sum_{i=0}^{n} r^i + r^{n+1}$

$$= \frac{1-r^{n+1}}{1-r} + \frac{r^{n+1}(1-r)}{1-r}$$

$$= \frac{1-r^{n+1}+r^{n+1}-r^{n+2}}{1-r} = \frac{1-r^{n+2}}{1-r}$$

$\square$

Exercises

1) Let $b \in \mathbb{Z}$ and $k, m \in \mathbb{Z}_{\geq 0}$, then

(i) If $b \in \mathbb{N}$, then $b^k \in \mathbb{N}$

(ii) $b^m b^k = b^{m+k}$

(iii) $(b^m)^k = b^{mk}$

2) Let $m \in \mathbb{Z}$ and let $\{x_j\}_{j=1}^{\infty}$ be a sequence in $\mathbb{Z}$. Then for all $k \in \mathbb{N}$

$$m\left(\sum_{j=1}^{k} x_j\right) = \sum_{j=1}^{k} (m x_j)$$

3) Let $\{x_j\}$ and $\{y_j\}$ be sequences in $\mathbb{Z}$ and let $a, b, c \in \mathbb{Z}$ be such that $a \leq b < c$

(i) $\sum_{j=a}^{c} x_j = \sum_{j=a}^{b} x_j + \sum_{j=b+1}^{c} x_j$

(ii) $\sum_{j=a}^{b} (x_j + y_j) = \left(\sum_{j=a}^{b} x_j\right) + \left(\sum_{j=a}^{b} y_j\right)$