8.2 Tests About a Population Mean

Tests About a Population Mean

- Confidence intervals for a population mean μ focused on three different cases.

- We now develop test procedures for these cases.

  Case I : Normal Population with Known σ
  Case II : Large-Sample Tests
  Case III : Normal Population Distribution

Case I: A Normal Population with Known σ

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Case I: A Normal Population with Known σ

- Although assumption that value of σ is known is rarely met in practice, this case provides a good starting point because of the ease with which general procedures and their properties can be developed.
- Null hypothesis in all three cases will state that μ has a particular numerical value, the ***null value***, which we will denote by μ_0 .
- Let $X_1, \ldots, X_n$ represent random sample of size $\boldsymbol{n}$ from normal population.

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Case I: A Normal Population with Known σ

- Then sample mean $\overline{X}$ has a normal distribution with expected value $\mu_{\overline{X}} = \mu$ and standard deviation $\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$
- When H_0 is true, $\mu_{\overline{X}} = \mu_0$
- Consider now the statistic Z obtained by standardizing $\overline{X}$ under assumption that H_0 is true:

$$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}}$$

- Substitution of computed sample mean $\overline{x}$ gives z, distance between $\overline{x}$ and μ_0 expressed in "standard deviation units."

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Case I: A Normal Population with Known σ

- For example, if null hypothesis is

$H_0 : \mu = 100$ $\quad \sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{25}} = 2.0$ and $\overline{x} = 103$

then the test statistic value is

$$z = \frac{\overline{x} - \mu}{\sigma_{\overline{X}}} \Rightarrow z = \frac{\overline{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \Rightarrow z = \frac{103 - 100}{\frac{10}{\sqrt{25}}} = 1.5$$

- That is, the observed value of $\overline{x}$ is 1.5 standard Deviations (of $\overline{X}$) larger than what we expect it to be when H_0 is true.

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Case I: A Normal Population with Known σ

Table A.3 Standard Normal Curve Areas *(cont.)* $\Phi(z) = P(Z \leq z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545

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Case I: A Normal Population with Known σ

- As we have discussed earlier, **cutoff value *c*** should be chosen to control probability of **type I error** at the **desired level α.**

- The required cutoff c is ***z* critical value** that captures upper-tail area α under the z curve.

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Case I: A Normal Population with Known σ

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985

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Case I: A Normal Population with Known σ

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Case I: A Normal Population with Known σ

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

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Case I: A Normal Population with Known σ

- **Test procedure** for **case I** is summarized in accompanying box, and corresponding rejection regions are illustrated in Figure 8.2.

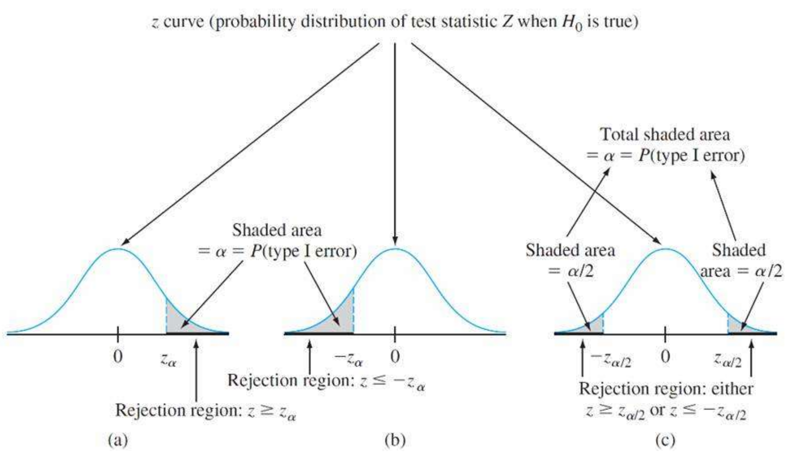

Figure 8.2 Rejection regions for ***z* tests**: (a) **upper-tailed test**; (b) **lower-tailed test**; (c) **two-tailed test**

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Case I: A Normal Population with Known σ

Null hypothesis: $H_0: \mu = \mu_0$

Test statistic value : $z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$

Alternative Hypothesis	Rejection Region for Level α Test
$H_a: \mu > \mu_0$	$z \geq z_\alpha$ (upper-tailed test)
$H_a: \mu < \mu_0$	$z \leq -z_\alpha$ (lower-tailed test)
$H_a: \mu \neq \mu_0$	either $z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$ (two-tailed test)

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Case I: A Normal Population with Known σ

- Use of the following sequence of steps is recommended when testing hypotheses about a parameter.

1. Identify the parameter of interest and describe it in the context of the problem situation.
2. Determine the null value and state the null hypothesis.
3. State the appropriate alternative hypothesis.
4. Give the formula for the computed value of the test statistic (substituting the null value and the known values of any other parameters, but *not* those of any sample-based quantities).

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Case I: A Normal Population with Known σ

5. State the rejection region for the selected significance level α.

6. Compute any necessary sample quantities, substitute into the formula for the test statistic value, and compute that value.

7. Decide whether H_0 should be rejected, and state this conclusion in the problem context.

- The formulation of hypotheses (Steps 2 and 3) should be done before examining the data.

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Example 6

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- A manufacturer of sprinkler systems used for fire protection in office buildings claims that true average system-activation temperature is 130°.
- A sample of $n = 9$ systems, when tested, yields a sample average activation temperature of 131.08°F.
- If the distribution of activation times is normal with standard deviation 1.5°F, does the data contradict the manufacturer's claim at significance level $\alpha = 0.01$?

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Example 6

cont'd

1. Parameter of interest: μ = true average activation temperature.

2. Null hypothesis: H_0: $\mu = 130$ (null value = $\mu_0 = 130$).

3. Alternative hypothesis: H_a: $\mu \neq 130$ (a departure from the claimed value in *either* direction is of concern).

4. Test statistic value:

$\mu_0 = 130$

$\sigma = 1.5$

$n = 9$

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{\bar{x} - 130}{1.5/\sqrt{n}}$$

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Example 6

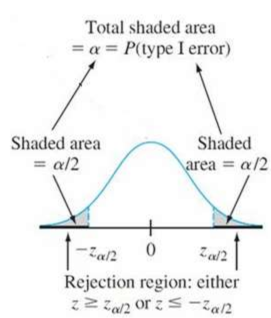
H_a: $\mu \neq 130$ either $z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$ (two-tailed test)

cont'd

5. Rejection region: The form of H_a implies use of two-tailed test with rejection region *either* $z \geq z_{0.005}$ *or* $z \leq -z_{0.005}$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0038
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952

- From Appendix Table A.3, $z_{0.005} = 2.58$,
  so we reject H_0
  if either $z \geq 2.58$ or $z \leq -2.58$



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Example 6

H_a: $\mu \neq 130$ either $z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$ (two-tailed test)

cont'd

6. Substituting $n = 9$ and $\overline{x} = 131.08$,

$$z = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{\overline{x} - 130}{1.5/\sqrt{n}} \Rightarrow z = \frac{131.08 - 130}{1.5/\sqrt{9}} = \frac{1.08}{0.5} = 2.16$$

- That is, the observed sample mean is a bit more than 2 standard deviations above what would have been expected were H_0 true.

7. The computed value $z = 2.16$ does not fall in rejection region ($-2.58 < 2.16 < 2.58$), so H_0 cannot be rejected at significance level 0.01.

Data does not give strong support to the claim that the true average differs from the design value of 130.

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Case II: Large-Sample Tests

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Case II: Large-Sample Tests

- When sample size is large, the z tests for case I are easily modified to yield valid test procedures without requiring either normal population distribution or known σ.
- Earlier we used key result to justify large-sample confidence intervals:
- Large n implies that standardized variable

$$Z = \frac{\overline{X} - \mu}{\boldsymbol{S}/\sqrt{n}}$$

- has *approximately* a standard normal distribution.

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Case II: Large-Sample Tests

Substitution of null value μ_0 in place of μ yields **test statistic**

$$Z = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$

which has approximately standard normal distribution when H_0 is true.

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Case II: Large-Sample Tests

- The use of rejection regions given previously for case I (e.g., $z \geq z_{\alpha}$ when alternative hypothesis is H_a: $\mu > \mu_0$) then results in test procedures for which significance level is approximately (rather than exactly) α.

- The rule of thumb $\boldsymbol{n > 40}$ will again be used to characterize large sample size.

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Example 8

- Dynamic cone penetrometer (DCP) is used for measuring material resistance to penetration (mm/blow) as a cone is driven into pavement or subgrade.

- Suppose that for a particular application it is required that the true average DCP value for a certain type of pavement be less than 30.

- The pavement will not be used unless there is conclusive evidence that specification has been met.

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Example 8

cont'd

Let's state and test appropriate hypotheses using the following data ("Probabilistic Model for the Analysis of Dynamic Cone Penetrometer Test Values in Pavement Structure Evaluation," *J. of Testing and Evaluation*, 1999: 7–14):

14.1	14.5	15.5	16.0	16.0	16.7	16.9	17.1	17.5	17.8
17.8	18.1	18.2	18.3	18.3	19.0	19.2	19.4	20.0	20.0
20.8	20.8	21.0	21.5	23.5	27.5	27.5	28.0	28.3	30.0
30.0	31.6	31.7	31.7	32.5	33.5	33.9	35.0	35.0	35.0
36.7	40.0	40.0	41.3	41.7	47.5	50.0	51.0	51.8	54.4
55.0	57.0								

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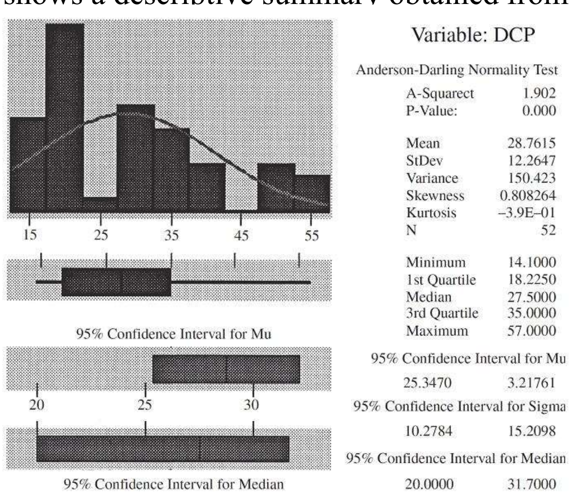
Example 8

cont'd

Figure 8.3 shows a descriptive summary obtained from Minitab.



Descriptive Statistics

Figure 8.3 Minitab descriptive summary for the DCP data of Example 8

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Example 8

cont'd

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Example 8

cont'd

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455

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Example 8

cont'd

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Example 8

cont'd

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Example 8

cont'd

1. μ = true average DCP value

2. H_0: $\mu = 30$

3. Ha: $\mu < 30$(so the pavement will not be used unless the null hypothesis is rejected)

4. $z = \dfrac{\bar{x} - 30}{s/\sqrt{n}}$

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Example 8

cont'd

5. A test with significance level .05 rejects H_0 when $z \leq -1.645$ (a lower-tailed test).

6. With $n = 52$, $\bar{x} = 28.76$, and $s = 12.2647$,

$$z = \frac{28.76 - 30}{12.2647/\sqrt{52}} = \frac{-1.24}{1.701} = -.73$$

7. Since $-.73 > -1.645$, H_0 cannot be rejected. We do not have compelling evidence for concluding that $\mu < 30$; use of the pavement is not justified.

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Case III: A Normal Population Distribution

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Case III: A Normal Population Distribution

- The key result on which tests for normal population mean are based was used to derive the one-sample *t* CI:

- If $X_1, X_2, \ldots, X_n$ is a random sample from normal distribution, standardized variable

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

- has *t* distribution with $n - 1$ degrees of freedom (df).

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Case III: A Normal Population Distribution

- Consider testing against H_0: $\mu = \mu_0$ against H_a: $\mu > \mu_0$ by using test statistic

$$T = (\overline{X} - \mu_0)/(S/\sqrt{n}).$$

- That is, test statistic results from standardizing $\overline{X}$ under the assumption that H_0 is true (using $\overline{X}$ the estimated standard deviation of $S/\sqrt{n}$, rather than $\sigma/\sqrt{n}$).

- When H_0 is true, test statistic has a t distribution with $n - 1$ df.

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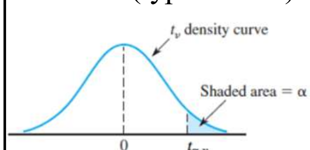
Case III: A Normal Population Distribution

- Knowledge of the test statistic's distribution when H_0 is true (the "null distribution") allows us to construct rejection region for which the type I error probability is controlled at the desired level.

- In particular, use of upper-tail t critical value $t_{\alpha,n-1}$ to specify the rejection region $t \geq t_{\alpha,n-1}$ implies that

P(type I error) = P(H_0 is rejected when it is true)



$= P(T \geq t_{\alpha,n-1}$ when T has t distribution with $n - 1$ df)

$= \alpha$

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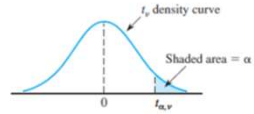
Case III: A Normal Population Distribution

Table A.5 Critical Values for t Distributions



ν \ α	.10	.05	.025	.01	.005	.001	.0005
1	3.078	6.314	12.706	31.821	63.657	318.31	636.62
2	1.886	2.920	4.303	6.965	9.925	22.326	31.598
3	1.638	2.353	3.182	4.541	5.841	10.213	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
				. . .			
30	1.310	1.697	2.042	2.457	2.750	3.385	3.646
32	1.309	1.694	2.037	2.449	2.738	3.365	3.622
34	1.307	1.691	2.032	2.441	2.728	3.348	3.601
36	1.306	1.688	2.028	2.434	2.719	3.333	3.582
38	1.304	1.686	2.024	2.429	2.712	3.319	3.566
40	1.303	1.684	2.021	2.423	2.704	3.307	3.551
50	1.299	1.676	2.009	2.403	2.678	3.262	3.496
60	1.296	1.671	2.000	2.390	2.660	3.232	3.460
120	1.289	1.658	1.980	2.358	2.617	3.160	3.373
∞	1.282	1.645	1.960	2.326	2.576	3.090	3.291

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Case III: A Normal Population Distribution

- Test statistic is really the same here as in the large-sample case but is labeled T to emphasize that its null distribution is t distribution with $n - 1$ df rather than standard normal (z) distribution.

- Rejection region for the t test differs from that for z test only in that t critical value $t_{\alpha,n-1}$ replaces the z critical value z_α .

- Similar comments apply to alternatives for which a lower-tailed or two-tailed test is appropriate.

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Case III: A Normal Population Distribution

The One-Sample t Test

Null hypothesis: H_0: $\mu = \mu_0$

Test statistic value: $t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$

Alternative Hypothesis	**Rejection Region for a Level α Test**
H_a: $\mu > \mu_0$	$t \geq t_{\alpha,n-1}$ (upper-tailed)
H_a: $\mu < \mu_0$	$t \leq -t_{\alpha,n-1}$ (lower-tailed)
H_a: $\mu \neq \mu_0$	either $t \geq t_{\alpha/2,n-1}$ or $t \leq -t_{\alpha/2,n-1}$ (two-tailed)

Example 9

- Glycerol is a major by-product of ethanol fermentation in wine production and contributes to the sweetness, body, and fullness of wines.
- The article "A Rapid and Simple Method for Simultaneous Determination of Glycerol, Fructose, and Glucose in Wine" (*American J. of Enology and Viticulture,* 2007: 279–283) includes the following observations on glycerol concentration (mg/mL) for samples of standard-quality (uncertified) white wines: 2.67, 4.62, 4.14, 3.81, 3.83.
- Suppose the desired concentration value is 4.
- Does the sample data suggest that true average concentration is something other than the desired value?

Example 9

cont'd

- The accompanying normal probability plot from Minitab provides strong support for assuming that the population distribution of glycerol concentration is normal.

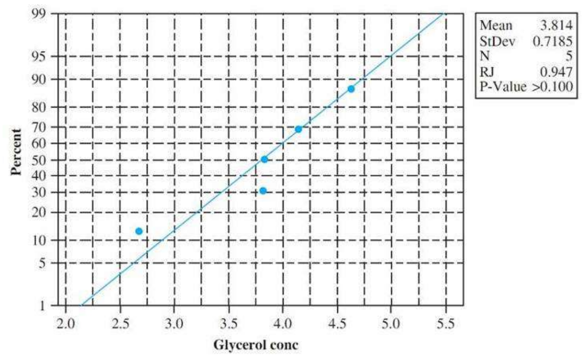

Figure 8.4 Normal probability plot for the data of Example 9

- Let's carry out test of appropriate hypotheses using the one-sample *t* test with a significance level of 0.05.

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Example 9

cont'd

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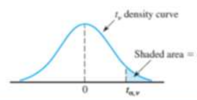
Example 9

cont'd

Table A.5 Critical Values for t Distributions



	α						
ν	.10	.05	.025	.01	.005	.001	.0005
1	3.078	6.314	12.706	31.821	63.657	318.31	636.62
2	1.886	2.920	4.303	6.965	9.925	22.326	31.598
3	1.638	2.353	3.182	4.541	5.841	10.213	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610

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Example 9

cont'd

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Example 9

cont'd

Example 9

cont'd

1. μ = true average glycerol concentration

2. H_0: $\mu = 4$

3. H_a: $\mu \neq 4$

4. $t = \dfrac{\overline{x} - 4}{s/\sqrt{n}}$

5. The inequality in H_a implies that a two-tailed test is appropriate, which requires $t = (3.814 - 4)/0.321 = -0.58$. Thus H_0 will be rejected if either $t \geq 2.776$ or $t \leq -2.776$.

Example 9

cont'd

6. $\sum x_i = 19.07$, and $\sum x_i^2 = 74.7979$, from which $\overline{x} = 3.814$
$s = .718$, and the estimated standard error of the mean is $s/\sqrt{n} = .321$. The test statistic value is then $t = (3.814 - 4)/0.321 = -0.58$.

7. Clearly $t = -0.58$ does not lie in the rejection region for a significance level of 0.05.

It is still plausible that $\mu = 4$. The deviation of the sample mean 3.814 from its expected value 4 when H_0 is true can be attributed just to sampling variability rather than to H_0 being false.

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