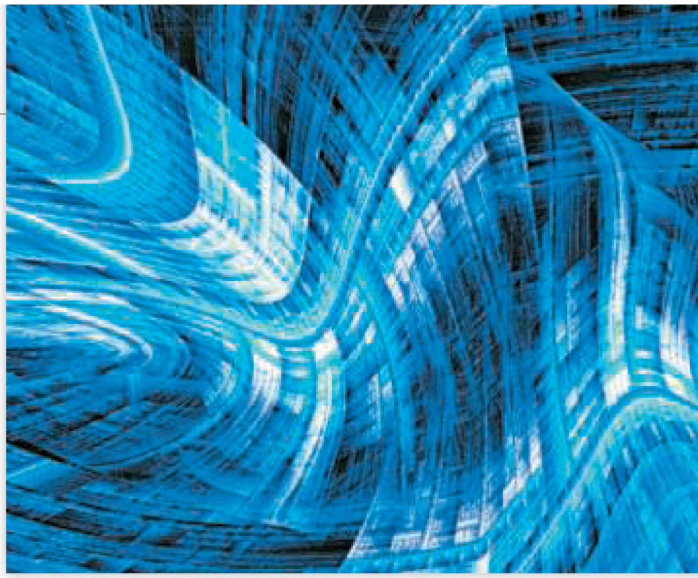
8

Techniques of Integration



OVERVIEW The Fundamental Theorem tells us how to evaluate a definite integral once we have an antiderivative for the integrand function. However, finding antiderivatives (or indefinite integrals) is not as straightforward as finding derivatives. In this chapter we study a number of important techniques that apply to finding integrals for specialized classes of functions such as trigonometric functions, products of certain functions, and rational functions. Since we cannot always find an antiderivative, we develop numerical methods for calculating definite integrals. We also study integrals whose domain or range are infinite, called *improper integrals*.

8.1 Using Basic Integration Formulas

Table 8.1 summarizes the indefinite integrals of many of the functions we have studied so far, and the substitution method helps us use the table to evaluate more complicated functions involving these basic ones. In this section we combine the Substitution Rules (studied in Chapter 5) with algebraic methods and trigonometric identities to help us use Table 8.1. A more extensive Table of Integrals is given at the back of the chapter, and we discuss its use in Section 8.6.

Sometimes we have to rewrite an integral to match it to a standard form of the type displayed in Table 8.1. We start with an example of this procedure.

EXAMPLE 1 Evaluate the integral

diff u.

$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx.$$

u

Solution We rewrite the integral and apply the Substitution Rule for Definite Integrals presented in Section 5.6, to find

$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx = \int_1^{11} \frac{du}{\sqrt{u}} \qquad u = x^2-3x+1,\ du = (2x-3)\,dx;\ u = 1 \text{ when } x = 3,\ u = 11 \text{ when } x = 5$$

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$$= \int_1^{11} u^{-1/2}\,du$$

$$= 2\sqrt{u}\,\Big]_1^{11} = 2\left(\sqrt{11}-1\right) \approx 4.63. \qquad \text{Table 8.1, Formula 2}$$ ■

TABLE 8.1 Basic integration formulas

1. $\displaystyle\int k\,dx = kx + C \qquad \text{(any number } k)$

2. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$

3. $\displaystyle\int \frac{dx}{x} = \ln|x| + C$

4. $\displaystyle\int e^x\,dx = e^x + C$

5. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad (a > 0,\ a \neq 1)$

6. $\displaystyle\int \sin x\,dx = -\cos x + C$

7. $\displaystyle\int \cos x\,dx = \sin x + C$

8. $\displaystyle\int \sec^2 x\,dx = \tan x + C$

8. $\displaystyle\int \csc^2 x\,dx = -\cot x + C$

10. $\displaystyle\int \sec x \tan x\,dx = \sec x + C$

11. $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$

12. $\displaystyle\int \tan x\,dx = \ln|\sec x| + C$

13. $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$

14. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$

15. $\displaystyle\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

16. $\displaystyle\int \sinh x\,dx = \cosh x + C$

17. $\displaystyle\int \cosh x\,dx = \sinh x + C$

18. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

19. $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

20. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

21. $\displaystyle\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \qquad (a > 0)$

22. $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \qquad (x > a > 0)$

EXAMPLE 2 Complete the square to evaluate

$$\int \frac{dx}{\sqrt{8x - x^2}}.$$

Solution We complete the square to simplify the denominator:

$$\begin{aligned} 8x - x^2 &= -(x^2 - 8x) = -(x^2 - 8x + 16 - 16) \\ &= -(x^2 - 8x + 16) + 16 = 16 - (x - 4)^2. \end{aligned}$$

Then

$$\begin{aligned} \int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x - 4)^2}} \\ &= \int \frac{du}{\sqrt{a^2 - u^2}} && a = 4,\ u = (x - 4),\ du = dx \\ &= \sin^{-1}\left(\frac{u}{a}\right) + C && \text{Table 8.1, Formula 18} \\ &= \sin^{-1}\left(\frac{x - 4}{4}\right) + C. \end{aligned}$$

EXAMPLE 3 Evaluate the integral

sin(x+2x) ; sin(A+B) = sin A cos B + sin B cos A

$$\int (\cos x \sin 2x + \sin x \cos 2x)\, dx.$$

Solution We can replace the integrand with an equivalent trigonometric expression using the Sine Addition Formula to obtain a simple substitution:

$$\begin{aligned}
\int (\cos x \sin 2x + \sin x \cos 2x)\, dx &= \int (\sin (x + 2x))\, dx \\
&= \int \sin 3x\, dx \\
&= \int \frac{1}{3} \sin u\, du && u = 3x,\ du = 3\,dx \\
&= -\frac{1}{3} \cos 3x + C. && \text{Table 8.1, Formula 6}
\end{aligned}$$

■

In Section 5.5 we found the indefinite integral of the secant function by multiplying it by a fractional form identically equal to one, and then integrating the equivalent result. We can use that same procedure in other instances as well, as we illustrate next.

EXAMPLE 4 Find $\displaystyle\int_0^{\pi/4} \frac{dx}{1 - \sin x}$.

Solution We multiply the numerator and denominator of the integrand by $1 + \sin x$. This procedure transforms the integral into one we can evaluate:

$$\begin{aligned}
\int_0^{\pi/4} \frac{dx}{1 - \sin x} &= \int_0^{\pi/4} \frac{1}{1 - \sin x} \cdot \underbrace{\boxed{\frac{1 + \sin x}{1 + \sin x}}}_{\text{Conjugate.}}\, dx && \text{Multiply and divide by conjugate.} \\
&= \int_0^{\pi/4} \frac{1 + \sin x}{1 - \sin^2 x}\, dx && \text{Simplify.} \\
&= \int_0^{\pi/4} \frac{1 + \sin x}{\cos^2 x}\, dx && 1 - \sin^2 x = \cos^2 x \\
&= \int_0^{\pi/4} (\sec^2 x + \sec x \tan x)\, dx && \text{Use Table 8.1, — } \textstyle\int \sec x \tan x\, dx = \sec x + C \\
& && \text{Formulas 8 and 10 — } \textstyle\int \sec^2 x\, dx = \tan x + C \\
&= \Big[\tan x + \sec x\Big]_0^{\pi/4} = \big(1 + \sqrt{2} - (0 + 1)\big) = \sqrt{2}.
\end{aligned}$$

■

EXAMPLE 5 Evaluate

$$\int \frac{3x^2 - 7x}{3x + 2}\, dx.$$

$$\begin{array}{r}
x \ -\ 3 \\
3x + 2\,\overline{)\,3x^2 - 7x} \\
\underline{3x^2 + 2x} \\
-9x \\
\underline{-9x - 6} \\
+\ 6
\end{array}$$

Solution The integrand is an improper fraction since the degree of the numerator is greater than the degree of the denominator. To integrate it, we perform long division to obtain a quotient plus a remainder that is a proper fraction:

$$\xleftarrow{\text{Solution.}}\ \frac{3x^2 - 7x}{3x + 2} = \underbrace{x - 3}_{\text{สมการเส้นตรง}} + \underbrace{\frac{6}{3x + 2}}_{\text{เศษ}}.$$

Therefore,

$6\int \frac{1}{u}\,du \longrightarrow 6\,\underbrace{(\ln|u|)}_{\ln|3x+2|}\left(\frac{du}{dx}\right)$ $\quad \frac{1}{3x+2}\,dx = \frac{1}{3}$

$$\int \frac{3x^2 - 7x}{3x + 2}\,dx = \int \left(x - 3 + \frac{6}{\underbrace{3x + 2}_{u}} \right) dx = \frac{x^2}{2} - 3x + 2\ln|3x + 2| + C. \qquad \blacksquare$$

Reducing an improper fraction by long division (Example 5) does not always lead to an expression we can integrate directly. We see what to do about that in Section 8.5.

EXAMPLE 6 Evaluate

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\,dx.$$

Solution We first separate the integrand to get

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\,dx = 3\int \frac{x\,dx}{\sqrt{1 - x^2}} + 2\int \frac{dx}{\sqrt{1 - x^2}}.$$

In the first of these new integrals, we substitute

$$u = 1 - x^2, \qquad du = -2x\,dx, \qquad \text{so} \qquad x\,dx = -\frac{1}{2}\,du.$$

Then we obtain

$$3\int \frac{x\,dx}{\sqrt{1 - x^2}} = 3\int \frac{(-1/2)\,du}{\sqrt{u}} = -\frac{3}{2}\int u^{-1/2}\,du$$

$$= -\frac{3}{2}\cdot\frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1 - x^2} + C_1.$$

The second of the new integrals is a standard form,

$$2\int \frac{dx}{\sqrt{1 - x^2}} = 2\sin^{-1} x + C_2. \qquad \text{Table 8.1, Formula 18}$$

$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\,dx = -3\sqrt{1 - x^2} + 2\sin^{-1} x + C. \qquad \blacksquare$$

The question of what to substitute for in an integrand is not always quite so clear. Sometimes we simply proceed by trial-and-error, and if nothing works out, we then try another method altogether. The next several sections of the text present some of these new methods, but substitution works in the following example.

EXAMPLE 7 Evaluate

$$\int \frac{dx}{(\underbrace{1 + \sqrt{x}}_{\text{แทนเป็น } u})^3}.$$

Solution We might try substituting for the term $\sqrt{x}$, but the derivative factor $1/\sqrt{x}$ is missing from the integrand, so this substitution will not help. The other possibility is to substitute for $(1 + \sqrt{x})$, and it turns out this works:

$$\int \frac{dx}{(\underbrace{1 + \sqrt{x}}_{u})^3} = \int \frac{2(u - 1)\,du}{u^3} \qquad u = 1 + \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx;$$

$$dx = 2\sqrt{x}\,du = 2(u - 1)\,du$$

$$= \int \left(\frac{2}{u^2} - \frac{2}{u^3} \right) du$$

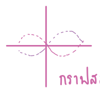
$$\text{จาก}\quad \frac{2}{u^2} + \frac{2}{u^3} = -\frac{2}{u} + \frac{1}{u^2} + C$$
$$= \frac{1 - 2u}{u^2} + C$$
$$= \frac{1 - 2\left(1 + \sqrt{x}\right)}{\left(1 + \sqrt{x}\right)^2} + C$$
$$= C - \frac{1 + 2\sqrt{x}}{\left(1 + \sqrt{x}\right)^2}.$$

When evaluating definite integrals, a property of the integrand may help us in calculating the result.

EXAMPLE 8 Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2} x^3 \cos x\, dx$.

Solution No substitution or algebraic manipulation is clearly helpful here. But we observe that the interval of integration is the symmetric interval $[-\pi/2, \pi/2]$. Moreover, the factor x^3 is an odd function, and $\cos x$ is an even function, so their product is odd. Therefore,

$$\int_{-\pi/2}^{\pi/2} x^3 \cos x\, dx = 0. \qquad \text{Theorem 8, Section 5.6}$$



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EXERCISES 8.1

Assorted Integrations

The integrals in Exercises 1–44 are in no particular order. Evaluate each integral using any algebraic method or trigonometric identity you think is appropriate. When necessary, use a substitution to reduce it to a standard form.

1. $\displaystyle\int_0^1 \frac{16x}{8x^2 + 2}\, dx$
2. $\displaystyle\int \frac{x^2}{x^2 + 1}\, dx$
3. $\displaystyle\int (\sec x - \tan x)^2\, dx$
4. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{dx}{\cos^2 x \tan x}$
5. $\displaystyle\int \frac{1 - x}{\sqrt{1 - x^2}}\, dx$
6. $\displaystyle\int \frac{dx}{x - \sqrt{x}}$
7. $\displaystyle\int \frac{e^{-\cot z}}{\sin^2 z}\, dz$
8. $\displaystyle\int \frac{2^{\ln z^3}}{16z}\, dz$
9. $\displaystyle\int \frac{dz}{e^z + e^{-z}}$
10. $\displaystyle\int_1^2 \frac{8\, dx}{x^2 - 2x + 2}$
11. $\displaystyle\int_{-1}^0 \frac{4\, dx}{1 + (2x + 1)^2}$
12. $\displaystyle\int_{-1}^3 \frac{4x^2 - 7}{2x + 3}\, dx$
13. $\displaystyle\int \frac{dt}{1 - \sec t}$
14. $\displaystyle\int \csc t \sin 3t\, dt$
15. $\displaystyle\int_0^{\pi/4} \frac{1 + \sin\theta}{\cos^2\theta}\, d\theta$
16. $\displaystyle\int \frac{d\theta}{\sqrt{2\theta - \theta^2}}$
17. $\displaystyle\int \frac{\ln y}{y + 4y \ln^2 y}\, dy$
18. $\displaystyle\int \frac{2^{\sqrt{y}}\, dy}{2\sqrt{y}}$
19. $\displaystyle\int \frac{d\theta}{\sec\theta + \tan\theta}$
20. $\displaystyle\int \frac{dt}{t\sqrt{3 + t^2}}$
21. $\displaystyle\int \frac{4t^3 - t^2 + 16t}{t^2 + 4}\, dt$
22. $\displaystyle\int \frac{x + 2\sqrt{x - 1}}{2x\sqrt{x - 1}}\, dx$
23. $\displaystyle\int_0^{\pi/2} \sqrt{1 - \cos\theta}\, d\theta$
24. $\displaystyle\int (\sec t + \cot t)^2\, dt$
25. $\displaystyle\int \frac{dy}{\sqrt{e^{2y} - 1}}$
26. $\displaystyle\int \frac{6\, dy}{\sqrt{y}\,(1 + y)}$
27. $\displaystyle\int \frac{2\, dx}{x\sqrt{1 - 4\ln^2 x}}$
28. $\displaystyle\int \frac{dx}{(x - 2)\sqrt{x^2 - 4x + 3}}$
29. $\displaystyle\int (\csc x - \sec x)(\sin x + \cos x)\, dx$
30. $\displaystyle\int 3\sinh\left(\frac{x}{2} + \ln 5\right) dx$
31. $\displaystyle\int_{\sqrt{2}}^3 \frac{2x^3}{x^2 - 1}\, dx$
32. $\displaystyle\int_{-1}^1 \sqrt{1 + x^2}\, \sin x\, dx$
33. $\displaystyle\int_{-1}^0 \sqrt{\frac{1 + y}{1 - y}}\, dy$
34. $\displaystyle\int e^{z + e^z}\, dz$

35. $\displaystyle\int \frac{7\,dx}{(x-1)\sqrt{x^2-2x-48}}$

36. $\displaystyle\int \frac{dx}{(2x+1)\sqrt{4x+4x^2}}$

37. $\displaystyle\int \frac{2\theta^3-7\theta^2+7\theta}{2\theta-5}\,d\theta$

38. $\displaystyle\int \frac{d\theta}{\cos\theta-1}$

39. $\displaystyle\int \frac{dx}{1+e^x}$

Hint: Use long division.

40. $\displaystyle\int \frac{\sqrt{x}}{1+x^3}\,dx$

Hint: Let $u = x^{3/2}$.

41. $\displaystyle\int \frac{e^{3x}}{e^x+1}\,dx$

42. $\displaystyle\int \frac{2^x-1}{3^x}\,dx$

43. $\displaystyle\int \frac{1}{\sqrt{x}\,(1+x)}\,dx$

44. $\displaystyle\int \frac{\tan\theta+3}{\sin\theta}\,d\theta$

Theory and Examples

45. Area Find the area of the region bounded above by $y = 2\cos x$ and below by $y = \sec x$, $-\pi/4 \le x \le \pi/4$.

46. Volume Find the volume of the solid generated by revolving the region in Exercise 45 about the x-axis.

47. Arc length Find the length of the curve $y = \ln(\cos x)$, $0 \le x \le \pi/3$.

48. Arc length Find the length of the curve $y = \ln(\sec x)$, $0 \le x \le \pi/4$.

49. Centroid Find the centroid of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

50. Centroid Find the centroid of the region bounded by the x-axis, the curve $y = \csc x$, and the lines $x = \pi/6$, $x = 5\pi/6$.

51. The functions $y = e^{x^3}$ and $y = x^3e^{x^3}$ do not have elementary antiderivatives, but $y = (1+3x^3)e^{x^3}$ does.

Evaluate

$$\int (1+3x^3)e^{x^3}\,dx.$$

52. Use the substitution $u = \tan x$ to evaluate the integral

$$\int \frac{dx}{1+\sin^2 x}.$$

53. Use the substitution $u = x^4+1$ to evaluate the integral

$$\int x^7\sqrt{x^4+1}\,dx.$$

54. Using different substitutions Show that the integral

$$\int ((x^2-1)(x+1))^{-2/3}dx$$

can be evaluated with any of the following substitutions.

a. $u = 1/(x+1)$

b. $u = ((x-1)/(x+1))^k$ for $k = 1, 1/2, 1/3, -1/3, -2/3$, and -1

c. $u = \tan^{-1} x$

d. $u = \tan^{-1}\sqrt{x}$

e. $u = \tan^{-1}((x-1)/2)$

f. $u = \cos^{-1} x$

g. $u = \cosh^{-1} x$

What is the value of the integral?

8.2 Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int u(x)\,v'(x)\,dx.$$

It is useful when u can be differentiated repeatedly and v' can be integrated repeatedly without difficulty. The integrals

$$\int x\cos x\,dx \qquad \text{and} \qquad \int x^2e^x\,dx$$

are such integrals because $u(x) = x$ or $u(x) = x^2$ can be differentiated repeatedly to become zero, and $v'(x) = \cos x$ or $v'(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int \ln x\,dx \qquad \text{and} \qquad \int e^x\cos x\,dx.$$

In the first case, the integrand $\ln x$ can be rewritten as $(\ln x)(1)$, and $u(x) = \ln x$ is easy to differentiate while $v'(x) = 1$ easily integrates to x. In the second case, each part of the integrand appears again after repeated differentiation or integration.

Product Rule in Integral Form

If u and v are differentiable functions of x, the Product Rule says that

$$\frac{d}{dx}\left[u(x)v(x)\right] = u'(x)v(x) + u(x)v'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[\,u(x)\,v(x)\,]\,dx = \int [\,u'(x)\,v(x) + u(x)\,v'(x)]\,dx$$

or

$$\int \frac{d}{dx}[\,u(x)\,v(x)\,]\,dx = \int u'(x)\,v(x)\,dx + \int u(x)\,v'(x)\,dx.$$

Rearranging the terms of this last equation, we get

$$\int u(x)\,v'(x)\,dx = \int \frac{d}{dx}[\,u(x)\,v(x)\,]\,dx - \int v(x)\,u'(x)\,dx,$$

leading to the **integration by parts** formula

Integration by Parts Formula

$$\int u(x)\,v'(x)\,dx = u(x)\,v(x) - \int v(x)\,u'(x)\,dx \tag{1}$$

This formula allows us to exchange the problem of computing the integral $\int u(x)\,v'(x)\,dx$ with the problem of computing a different integral, $\int v(x)\,u'(x)\,dx$. In many cases, we can choose the functions u and v so that the second integral is easier to compute than the first. There can be many choices for u and v, and it is not always clear which choice works best, so sometimes we need to try several.

The formula is often given in differential form. With $v'(x)\,dx = dv$ and $u'(x)\,dx = du$, the integration by parts formula becomes

Integration by Parts Formula—Differential Version

$$\int u\,dv = uv - \int v\,du \tag{2}$$

The next examples illustrate the technique.

EXAMPLE 1 Find

$$\int x\cos x\,dx.$$

Solution There is no obvious antiderivative of $x\cos x$, so we use the integration by parts formula

$$\int u(x)\,v'(x)\,dx = u(x)\,v(x) - \int v(x)\,u'(x)\,dx$$

to change this expression to one that is easier to integrate. We first decide how to choose the functions $u(x)$ and $v(x)$. In this case we factor the expression $x\cos x$ into

$$u(x) = x \quad \text{and} \quad v'(x) = \cos x.$$

Next we differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$$u'(x) = 1 \quad \text{and} \quad v(x) = \sin x.$$

When finding an antiderivative for $v'(x)$ we have a choice of how to pick a constant of integration C. We choose the constant $C = 0$, since that makes this antiderivative as simple as possible. We now apply the integration by parts formula:

$$\begin{aligned} \int \underbrace{x}_{u(x)} \underbrace{\cos x}_{v'(x)}\, dx &= \underbrace{x}_{u(x)} \underbrace{\sin x}_{v(x)} - \int \underbrace{\sin x}_{v(x)}\, \underbrace{(1)}_{u'(x)}\, dx && \text{Integration by parts formula} \\ &= x \sin x + \cos x + C && \text{Integrate and simplify.} \end{aligned}$$

and we have found the integral of the original function.

There are four apparent choices available for $u(x)$ and $v'(x)$ in Example 1:

1. Let $u(x) = 1$ and $v'(x) = x \cos x$. **2.** Let $u(x) = x$ and $v'(x) = \cos x$.

3. Let $u(x) = x \cos x$ and $v'(x) = 1$. **4.** Let $u(x) = \cos x$ and $v'(x) = x$.

Choice 2 was used in Example 1. The other three choices lead to integrals we don't know how to integrate. For instance, Choice 3, with $u'(x) = \cos x - x \sin x$, leads to the integral

$$\int (x \cos x - x^2 \sin x)\, dx.$$

The goal of integration by parts is to go from an integral $\int u(x)v'(x)\, dx$ that we don't see how to evaluate to an integral $\int v(x)u'(x)\, dx$ that we can evaluate. Generally, you choose $v'(x)$ first to be as much of the integrand as we can readily integrate; $u(x)$ is the leftover part. When finding $v(x)$ from $v'(x)$, any antiderivative will work, and we usually pick the simplest one; no arbitrary constant of integration is needed in $v(x)$ because it would simply cancel out of the right-hand side of Equation (2).

EXAMPLE 2 Find $\displaystyle\int \ln x\, dx$.

Solution We have not yet seen how to find an antiderivative for $\ln x$. If we set $u(x) = \ln x$, then $u'(x)$ is the simpler function $1/x$. It may not appear that a second function $v'(x)$ is multiplying $\ln x$, but we can choose $v'(x)$ to be the constant function $v'(x) = 1$. We use the integration by parts formula Equation (1) with

$$u(x) = \ln x \quad \text{and} \quad v'(x) = 1.$$

We differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$$u'(x) = \frac{1}{x} \quad \text{and} \quad v(x) = x.$$

Then

$$\begin{aligned} \int \underbrace{\ln x}_{u(x)} \cdot \underbrace{1}_{v'(x)}\, dx &= \underbrace{(\ln x)}_{u(x)}\, \underbrace{x}_{v(x)} - \int \underbrace{x}_{v(x)}\, \underbrace{\frac{1}{x}}_{u'(x)}\, dx && \text{Integration by parts formula} \\ &= x \ln x - x + C && \text{Simplify and integrate.} \end{aligned}$$

In the following examples we use the differential form to indicate the process of integration by parts. The computations are the same, with du and dv providing shorter expressions for $u'(x)\, dx$ and $v'(x)\, dx$. Sometimes we have to use integration by parts more than once, as in the next example.

EXAMPLE 3 Evaluate

$$\int \overset{u' = 2x}{x^2}\,\overset{V = e^x}{e^x}\,dx.$$

u v'

Solution We use the integration by parts formula Equation (1) with

$$u(x) = x^2 \quad \text{and} \quad v'(x) = e^x.$$

We differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$$u'(x) = 2x \quad \text{and} \quad v(x) = e^x.$$

We summarize this choice by setting $du = u'(x)\,dx$ and $dv = v'(x)\,dx$, so

$$du = 2x\,dx \quad \text{and} \quad dv = e^x\,dx.$$

We then have

$$\int \underbrace{x^2}_{u}\underbrace{e^x\,dx}_{dv} = \underbrace{x^2}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\,\underbrace{2x\,dx}_{du}.$$

Integration by parts formula

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int \underbrace{x}_{u}\underbrace{e^x\,dx}_{dv} = \underbrace{x}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\,\underbrace{dx}_{du} = xe^x - e^x + C.$$

Integration by parts Equation (2)
$u = x,\ dv = e^x\,dx$
$v = e^x, \quad du = dx$

Using this last evaluation, we then obtain

$$\begin{aligned}\int x^2e^x\,dx &= x^2e^x - 2\overbrace{\int xe^x\,dx}^{xe^x - e^x} \\ &= x^2e^x - 2xe^x + 2e^x + C,\end{aligned}$$

where the constant of integration is renamed after substituting for the integral on the right. ■

The technique of Example 3 works for any integral $\int x^n e^x\,dx$ in which n is a positive integer, because differentiating x^n will eventually lead to zero and integrating e^x is easy.

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

EXAMPLE 4 Evaluate

u v'

$$\int e^x \cos x\,dx.$$

Solution Let $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$, $v = \sin x$, and

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx. \qquad u(x) = e^x, \quad v(x) = \sin x$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x\,dx, \qquad v = -\cos x, \qquad du = e^x\,dx.$$

Then

$$\int e^x \cos x\, dx = e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x\, dx)\right) \qquad u(x) = e^x, \quad v(x) = -\cos x$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx.$$

The unknown integral now appears on both sides of the equation, but with opposite signs. Adding the integral to both sides and adding the constant of integration give

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^x \cos x\, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

EXAMPLE 5 Obtain a formula that expresses the integral

$$\int \cos^n x\, dx$$

in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \qquad \text{and} \qquad dv = \cos x\, dx,$$

so that

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$$du = (n-1)\cos^{n-2} x\,(-\sin x\, dx) \qquad \text{and} \qquad v = \sin x.$$

Integration by parts then gives

$$\int \cos^n x\, dx = \underbrace{\cos^{n-1} x}_{u}\, \underbrace{\sin x}_{v} + (n-1)\int \sin^2 x \cos^{n-2} x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int (1 - \cos^2 x)\cos^{n-2} x\, dx$$

คูณกัน

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x\, dx.$$

If we add

$$(n-1)\int \cos^n x\, dx$$

to both sides of this equation, we obtain

$$n\int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx.$$

We then divide through by n, and the final result is

$$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx.$$

The formula found in Example 5 is called a **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is easy to evaluate. For example, the result in Example 5 tells us that

$$\begin{aligned}\int \cos^3 x\, dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\, dx \\ &= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C.\end{aligned}$$

* มีขอบเขตแน่นอน แต่เป็นอะไรก็ได้

Evaluating Definite Integrals by Parts

The integration by parts formula in Equation (1) can be combined with Part 2 of the Fundamental Theorem in order to evaluate definite integrals by parts. Assuming that both u' and v' are continuous over the interval $[a, b]$, Part 2 of the Fundamental Theorem gives

Integration by Parts Formula for Definite Integrals

$$\int_a^b u(x)v'(x)\, dx = u(x)v(x)\Big]_a^b - \int_a^b v(x)u'(x)\, dx \qquad (3)$$

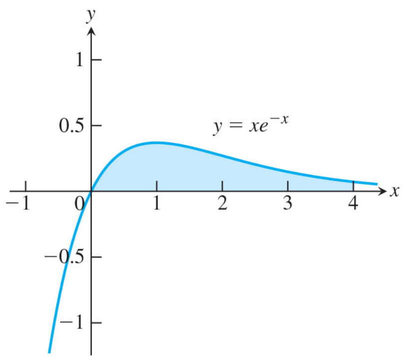


FIGURE 8.1 The region in Example 6.

EXAMPLE 6 Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

Solution The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x}\, dx.$$

Let $u = x$, $dv = e^{-x}\, dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\int_0^4 xe^{-x}\, dx = -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\, dx \qquad \text{Integration by parts Formula (3)}$$

ทำ by Part แล้ว
แทนเลขขอบเขต

$$\begin{aligned} &= [-4e^{-4} - (-0e^{-0})] + \int_0^4 e^{-x}\, dx \\ &= -4e^{-4} - e^{-x}\Big]_0^4 \\ &= -4e^{-4} - (e^{-4} - e^{-0}) = 1 - 5e^{-4} \approx 0.91. \end{aligned}$$

EXERCISES 8.2

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2}\, dx$
2. $\int \theta \cos \pi\theta\, d\theta$
3. $\int t^2 \cos t\, dt$
4. $\int x^2 \sin x\, dx$
5. $\int_1^2 x \ln x\, dx$
6. $\int_1^e x^3 \ln x\, dx$
7. $\int xe^x\, dx$
8. $\int xe^{3x}\, dx$
9. $\int x^2 e^{-x}\, dx$
10. $\int (x^2 - 2x + 1)e^{2x}\, dx$
11. $\int \tan^{-1} y\, dy$
12. $\int \sin^{-1} y\, dy$
13. $\int x \sec^2 x\, dx$
14. $\int 4x \sec^2 2x\, dx$
15. $\int x^3 e^x\, dx$
16. $\int p^4 e^{-p}\, dp$
17. $\int (x^2 - 5x)e^x\, dx$
18. $\int (r^2 + r + 1)e^r\, dr$

19. $\int x^5 e^x\,dx$

20. $\int t^2 e^{4t}\,dt$

21. $\int e^\theta \sin\theta\,d\theta$

22. $\int e^{-y}\cos y\,dy$

23. $\int e^{2x}\cos 3x\,dx$

24. $\int e^{-2x}\sin 2x\,dx$

Using Substitution

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\,ds$

26. $\int_0^1 x\sqrt{1-x}\,dx$

27. $\int_0^{\pi/3} x\tan^2 x\,dx$

28. $\int \ln(x+x^2)\,dx$

29. $\int \sin(\ln x)\,dx$

30. $\int z(\ln z)^2\,dz$

Evaluating Integrals

Evaluate the integrals in Exercises 31–56. Some integrals do not require integration by parts.

31. $\int x\sec x^2\,dx$

32. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

33. $\int x(\ln x)^2\,dx$

34. $\int \frac{1}{x(\ln x)^2}\,dx$

35. $\int \frac{\ln x}{x^2}\,dx$

36. $\int \frac{(\ln x)^3}{x}\,dx$

37. $\int x^3 e^{x^4}\,dx$

38. $\int x^5 e^{x^3}\,dx$

39. $\int x^3\sqrt{x^2+1}\,dx$

40. $\int x^2\sin x^3\,dx$

41. $\int \sin 3x\cos 2x\,dx$

42. $\int \sin 2x\cos 4x\,dx$

43. $\int \sqrt{x}\ln x\,dx$

44. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

45. $\int \cos\sqrt{x}\,dx$

46. $\int \sqrt{x}\,e^{\sqrt{x}}\,dx$

47. $\int_0^{\pi/2} \theta^2\sin 2\theta\,d\theta$

48. $\int_0^{\pi/2} x^3\cos 2x\,dx$

49. $\int_{2/\sqrt{3}}^{2} t\sec^{-1} t\,dt$

50. $\int_0^{1/\sqrt{2}} 2x\sin^{-1}(x^2)\,dx$

51. $\int x\tan^{-1} x\,dx$

52. $\int x^2\tan^{-1}\frac{x}{2}\,dx$

53. $\int (1+2x^2)e^{x^2}\,dx$

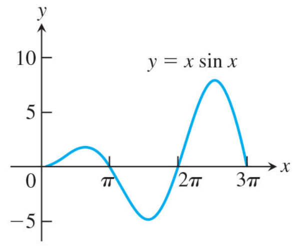
54. $\int \frac{xe^x}{(x+1)^2}\,dx$

55. $\int \sqrt{x}\left(\sin^{-1}\sqrt{x}\right)dx$

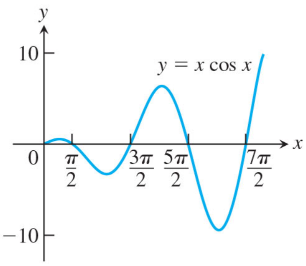
56. $\int \frac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}\,dx$

Theory and Examples

57. **Finding area** Find the area of the region enclosed by the curve $y = x\sin x$ and the x-axis (see the accompanying figure) for
   a. $0 \le x \le \pi$.
   b. $\pi \le x \le 2\pi$.
   c. $2\pi \le x \le 3\pi$.
   d. What pattern do you see here? What is the area between the curve and the x-axis for $n\pi \le x \le (n+1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.



58. **Finding area** Find the area of the region enclosed by the curve $y = x\cos x$ and the x-axis (see the accompanying figure) for
   a. $\pi/2 \le x \le 3\pi/2$.
   b. $3\pi/2 \le x \le 5\pi/2$.
   c. $5\pi/2 \le x \le 7\pi/2$.
   d. What pattern do you see? What is the area between the curve and the x-axis for
   $$\left(\frac{2n-1}{2}\right)\pi \le x \le \left(\frac{2n+1}{2}\right)\pi,$$
   n an arbitrary positive integer? Give reasons for your answer.



59. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

60. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$
   a. about the y-axis.
   b. about the line $x = 1$.

61. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about
   a. the y-axis.
   b. the line $x = \pi/2$.

62. Finding volume Find the volume of the solid generated by revolving the region bounded by the x-axis and the curve $y = x \sin x$, $0 \le x \le \pi$, about

a. the y-axis.

b. the line $x = \pi$.

(See Exercise 57 for a graph.)

63. Consider the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$.

a. Find the area of the region.

b. Find the volume of the solid formed by revolving this region about the x-axis.

c. Find the volume of the solid formed by revolving this region about the line $x = -2$.

d. Find the centroid of the region.

64. Consider the region bounded by the graphs of $y = \tan^{-1} x$, $y = 0$, and $x = 1$.

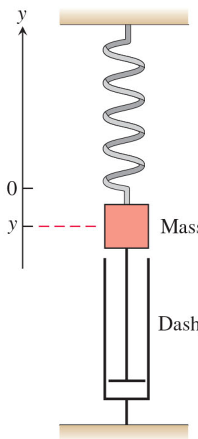
a. Find the area of the region.

b. Find the volume of the solid formed by revolving this region about the y-axis.

65. Average value A retarding force, symbolized by the dashpot in the accompanying figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t} \cos t, \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.



66. Average value In a mass-spring-dashpot system like the one in Exercise 65, the mass's position at time t is

$$y = 4e^{-t}(\sin t - \cos t), \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.

Reduction Formulas

In Exercises 67–71, use integration by parts to establish the reduction formula.

67. $\displaystyle\int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx$

68. $\displaystyle\int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx$

69. $\displaystyle\int x^n e^{ax} \, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx, \quad a \ne 0$

70. $\displaystyle\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx$

71. $\displaystyle\int x^m (\ln x)^n \, dx = \frac{x^{m+1}}{m+1} (\ln x)^n - \frac{n}{m+1} \int x^m (\ln x)^{n-1} \, dx, \quad m \ne -1$

72. $\displaystyle\int x^n \sqrt{x+1} \, dx = \frac{2x^n}{2n+3} (x+1)^{3/2} - \frac{2n}{2n+3} \int x^{n-1} \sqrt{x+1} \, dx$

73. $\displaystyle\int \frac{x^n}{\sqrt{x+1}} \, dx = \frac{2x^n}{2n+1} \sqrt{x+1} - \frac{2n}{2n+1} \int \frac{x^{n-1}}{\sqrt{x+1}} \, dx$

74. Use Example 5 to show that

$$\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx = \begin{cases} \left(\dfrac{\pi}{2}\right) \dfrac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n}, & n \text{ even} \\[2ex] \dfrac{2 \cdot 4 \cdot 6 \cdots (n-1)}{1 \cdot 3 \cdot 5 \cdots n}, & n \text{ odd} \end{cases}$$

75. Show that

$$\int_a^b \left(\int_x^b f(t) \, dt \right) dx = \int_a^b (x - a) f(x) \, dx.$$

76. Use integration by parts to obtain the formula

$$\int \sqrt{1 - x^2} \, dx = \frac{1}{2} x \sqrt{1 - x^2} + \frac{1}{2} \int \frac{1}{\sqrt{1 - x^2}} \, dx.$$

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\begin{aligned} \int f^{-1}(x) \, dx &= \int y f'(y) \, dy && y = f^{-1}(x), \quad x = f(y) \\ & && dx = f'(y) \, dy \\ &= y f(y) - \int f(y) \, dy && \text{Integration by parts with} \\ & && u = y, \, dv = f'(y) \, dy \\ &= x f^{-1}(x) - \int f(y) \, dy \end{aligned}$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\int \ln x\, dx = \int ye^y\, dy \qquad \begin{aligned} y &= \ln x, \quad x = e^y \\ dx &= e^y\, dy \end{aligned}$$
$$= ye^y - e^y + C$$
$$= x \ln x - x + C.$$

For the integral of $\cos^{-1} x$ we get

$$\int \cos^{-1} x\, dx = x \cos^{-1} x - \int \cos y\, dy \qquad y = \cos^{-1} x$$
$$= x\cos^{-1} x - \sin y + C$$
$$= x\cos^{-1} x - \sin(\cos^{-1} x) + C.$$

Use the formula

$$\int f^{-1}(x)\, dx = xf^{-1}(x) - \int f(y)\, dy \qquad y = f^{-1}(x) \tag{4}$$

to evaluate the integrals in Exercises 77–80. Express your answers in terms of x.

77. $\int \sin^{-1} x\, dx$

78. $\int \tan^{-1} x\, dx$

79. $\int \sec^{-1} x\, dx$

80. $\int \log_2 x\, dx$

Another way to integrate $f^{-1}(x)$ (when f^{-1} is integrable) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of f^{-1} as

$$\int f^{-1}(x)\, dx = xf^{-1}(x) - \int x\left(\frac{d}{dx} f^{-1}(x)\right) dx. \tag{5}$$

Exercises 81 and 82 compare the results of using Equations (4) and (5).

81. Equations (4) and (5) give different formulas for the integral of $\cos^{-1} x$:

a. $\int \cos^{-1} x\, dx = x\cos^{-1} x - \sin(\cos^{-1} x) + C$ Eq. (4)

b. $\int \cos^{-1} x\, dx = x\cos^{-1} x - \sqrt{1 - x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

82. Equations (4) and (5) lead to different formulas for the integral of $\tan^{-1} x$:

a. $\int \tan^{-1} x\, dx = x\tan^{-1} x - \ln \sec(\tan^{-1} x) + C$ Eq. (4)

b. $\int \tan^{-1} x\, dx = x\tan^{-1} x - \ln \sqrt{1 + x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

Evaluate the integrals in Exercises 83 and 84 with (**a**) Eq. (4) and (**b**) Eq. (5). In each case, check your work by differentiating your answer with respect to x.

83. $\int \sinh^{-1} x\, dx$

84. $\int \tanh^{-1} x\, dx$

8.3 Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions, as in the integral

$$\int \sec^2 x\, dx = \tan x + C.$$

The general idea is to use identities to transform the integrals we have to find into integrals that are easier to work with.

Products of Powers of Sines and Cosines

We begin with integrals of the form

$$\int \sin^m x \cos^n x\, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to m and n being odd or even.

Case 1 If ***m* is odd**, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \tag{1}$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x\, dx$ equal to $-d(\cos x)$.

Case 2 If ***n* is odd** in $\int \sin^m x \cos^n x\, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x\, dx$ equal to $d(\sin x)$.

Case 3 If **both *m* and *n* are even** in $\int \sin^m x \cos^n x\, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \tag{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Here are some examples illustrating each case.

EXAMPLE 1 Evaluate

$$\int \sin^3 x \cos^2 x\, dx.$$

Solution This is an example of Case 1.

$$\begin{aligned}
\int \sin^3 x \cos^2 x\, dx &= \int \sin^2 x \cos^2 x \sin x\, dx && m \text{ is odd.}\\
&= \int (1 - \cos^2 x)(\cos^2 x)(-d(\cos x)) && \sin x\, dx = -d(\cos x)\\
&= \int (1 - u^2)(u^2)(-du) && u = \cos x\\
&= \int (u^4 - u^2)\, du && \text{Multiply terms.}\\
&= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C
\end{aligned}$$

*กำลังเลขคู่เป็น + เสมอ

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EXAMPLE 2 Evaluate

$$\int \cos^5 x\, dx.$$

มี $\sin^2 x$ ด้วย

Solution This is an example of Case 2, where $m = 0$ is even and $n = 5$ is odd.

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$$\begin{aligned}
\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx &= \int (1 - \sin^2 x)^2\, d(\sin x) && \cos x\, dx = d(\sin x)\\
&= \int (1 - u^2)^2\, du && u = \sin x\\
&= \int (1 - 2u^2 + u^4)\, du && \text{Square } 1 - u^2.\\
&= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C
\end{aligned}$$

■

EXAMPLE 3 Evaluate

$$\int \sin^2 x \cos^4 x \, dx.$$

Solution This is an example of Case 3.

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx \qquad m \text{ and } n \text{ both even}$$

$$= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\boxed{\cos^2 2x} + \boxed{\cos^3 2x})\, dx\right]$$

For the term involving $\cos^2 2x$, we use

$$\int \cos^2 2x \, dx = \frac{1}{2}\int (1 + \cos 4x)\, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right). \qquad \text{Omit constant of integration until final result.}$$

* พิสูจน์ : $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$(1 - \cos^2\theta)$

$= 2\cos^2\theta - 1$

$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$

For the $\cos^3 2x$ term, we have

$$\int \cos^3 2x \, dx = \int (1 - \sin^2 2x)\cos 2x \, dx \qquad u = \sin 2x, \ du = 2\cos 2x \, dx$$

$$= \frac{1}{2}\int (1 - u^2)\, du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right). \qquad \text{Again omit } C.$$

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$

■

Eliminating Square Roots

In the next example, we use the identity $\cos^2\theta = (1 + \cos 2\theta)/2$ to eliminate a square root.

EXAMPLE 4 Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx.$$

Solution To eliminate the square root, we use the identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Therefore,

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx = \int_0^{\pi/4} \sqrt{2 \cos^2 2x}\, dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx$$

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$$= \sqrt{2} \int_0^{\pi/4} |\cos 2x|\, dx = \sqrt{2} \int_0^{\pi/4} \cos 2x\, dx$$

$\cos 2x \geq 0$ on $[0, \pi/4]$

$$= \sqrt{2} \left[\frac{\sin 2x}{2} \right]_0^{\pi/4} = \frac{\sqrt{2}}{2} [1 - 0] = \frac{\sqrt{2}}{2}.$$

Integrals of Powers of $\tan x$ and $\sec x$

We know how to integrate the tangent and secant functions and their squares. To integrate higher powers, we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

EXAMPLE 5 Evaluate

$$\int \tan^4 x\, dx.$$

*แยกให้เป็น $\tan^2 x$

Solution

$$\begin{aligned}
\int \tan^4 x\, dx &= \int \tan^2 x \cdot \tan^2 x\, dx = \int \tan^2 x \cdot (\sec^2 x - 1)\, dx \\
&= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx \\
&= \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)\, dx \\
&= \int \tan^2 x \sec^2 x\, dx - \int \sec^2 x\, dx + \int dx
\end{aligned}$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x\, dx$$

and have

$$\int u^2\, du = \frac{1}{3} u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x\, dx = \frac{1}{3} \tan^3 x - \tan x + x + C.$$

EXAMPLE 6 Evaluate

$$\int \sec^3 x\, dx.$$

Solution We integrate *by parts using

$$u = \sec x, \qquad dv = \sec^2 x\, dx, \qquad v = \tan x, \qquad du = \sec x \tan x\, dx.$$

Then

$$\int \sec^3 x\, dx = \sec x \tan x - \int (\tan x)(\sec x \tan x\, dx)$$

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$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x\, dx \qquad \tan^2 x = \sec^2 x - 1$$

ย้ายได้

$$= \sec x \tan x + \int \sec x\, dx - \int \sec^3 x\, dx.$$

* พยายามจัดรูปให้ย้ายข้างได้

Combining the two secant-cubed integrals gives

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

and

$$\int \sec^3 x\, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C.$$

■

EXAMPLE 7 Evaluate

$$\int \tan^4 x \sec^4 x\, dx.$$

Solution

$$\int (\tan^4 x)(\sec^4 x)\, dx = \int (\tan^4 x)(1 + \tan^2 x)(\sec^2 x)\, dx \qquad \sec^2 x = 1 + \tan^2 x$$

$$= \int (\tan^4 x + \tan^6 x)(\sec^2 x)\, dx$$

$$= \int (\tan^4 x)(\sec^2 x)\, dx + \int (\tan^6 x)(\sec^2 x)\, dx$$

$$= \int u^4\, du + \int u^6\, du = \frac{u^5}{5} + \frac{u^7}{7} + C \qquad u = \tan x,\ du = \sec^2 x\, dx$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$$

■

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx\, dx, \qquad \int \sin mx \cos nx\, dx, \qquad \text{and} \qquad \int \cos mx \cos nx\, dx$$

arise in many applications involving periodic functions. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities

$$\sin mx \sin nx = \frac{1}{2}[\cos (m - n)x - \cos (m + n)x], \tag{3}$$

$$\sin mx \cos nx = \frac{1}{2}[\sin (m - n)x + \sin (m + n)x], \tag{4}$$

$$\cos mx \cos nx = \frac{1}{2}[\cos (m - n)x + \cos (m + n)x]. \tag{5}$$

These identities come from the angle sum formulas for the sine and cosine functions (Section 1.3). They give functions whose antiderivatives are easily found.

EXAMPLE 8 Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$, we get

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \frac{1}{2}\int [\sin(-2x) + \sin 8x]\, dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$

EXERCISES 8.3

Powers of Sines and Cosines

Evaluate the integrals in Exercises 1–22.

1. $\int \cos 2x \, dx$
2. $\int_0^{\pi} 3 \sin \frac{x}{3} \, dx$
3. $\int \cos^3 x \sin x \, dx$
4. $\int \sin^4 2x \cos 2x \, dx$
5. $\int \sin^3 x \, dx$
6. $\int \cos^3 4x \, dx$
7. $\int \sin^5 x \, dx$
8. $\int_0^{\pi} \sin^5 \frac{x}{2} \, dx$
9. $\int \cos^3 x \, dx$
10. $\int_0^{\pi/6} 3 \cos^5 3x \, dx$
11. $\int \sin^3 x \cos^3 x \, dx$
12. $\int \cos^3 2x \sin^5 2x \, dx$
13. $\int \cos^2 x \, dx$
14. $\int_0^{\pi/2} \sin^2 x \, dx$
15. $\int_0^{\pi/2} \sin^7 y \, dy$
16. $\int 7 \cos^7 t \, dt$
17. $\int_0^{\pi} 8 \sin^4 x \, dx$
18. $\int 8 \cos^4 2\pi x \, dx$
19. $\int 16 \sin^2 x \cos^2 x \, dx$
20. $\int_0^{\pi} 8 \sin^4 y \cos^2 y \, dy$
21. $\int 8 \cos^3 2\theta \sin 2\theta \, d\theta$
22. $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta$

Integrating Square Roots

Evaluate the integrals in Exercises 23–32.

23. $\int_0^{2\pi} \sqrt{\frac{1 - \cos x}{2}} \, dx$
24. $\int_0^{\pi} \sqrt{1 - \cos 2x} \, dx$
25. $\int_0^{\pi} \sqrt{1 - \sin^2 t} \, dt$
26. $\int_0^{\pi} \sqrt{1 - \cos^2 \theta} \, d\theta$
27. $\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1 - \cos x}} \, dx$
28. $\int_0^{\pi/6} \sqrt{1 + \sin x} \, dx$

    $\left(\textit{Hint:} \text{ Multiply by } \sqrt{\frac{1 - \sin x}{1 - \sin x}}.\right)$
29. $\int_{5\pi/6}^{\pi} \frac{\cos^4 x}{\sqrt{1 - \sin x}} \, dx$
30. $\int_{\pi/2}^{3\pi/4} \sqrt{1 - \sin 2x} \, dx$
31. $\int_0^{\pi/2} \theta\sqrt{1 - \cos 2\theta} \, d\theta$
32. $\int_{-\pi}^{\pi} (1 - \cos^2 t)^{3/2} \, dt$

Powers of Tangents and Secants

Evaluate the integrals in Exercises 33–50.

33. $\int \sec^2 x \tan x \, dx$
34. $\int \sec x \tan^2 x \, dx$
35. $\int \sec^3 x \tan x \, dx$
36. $\int \sec^3 x \tan^3 x \, dx$
37. $\int \sec^2 x \tan^2 x \, dx$
38. $\int \sec^4 x \tan^2 x \, dx$
39. $\int_{-\pi/3}^{0} 2 \sec^3 x \, dx$
40. $\int e^x \sec^3 e^x \, dx$

41. $\int \sec^4 \theta \, d\theta$

42. $\int 3 \sec^4 3x \, dx$

43. $\int_{\pi/4}^{\pi/2} \csc^4 \theta \, d\theta$

44. $\int \sec^6 x \, dx$

45. $\int 4 \tan^3 x \, dx$

46. $\int_{-\pi/4}^{\pi/4} 6 \tan^4 x \, dx$

47. $\int \tan^5 x \, dx$

48. $\int \cot^6 2x \, dx$

49. $\int_{\pi/6}^{\pi/3} \cot^3 x \, dx$

50. $\int 8 \cot^4 t \, dt$

Products of Sines and Cosines

Evaluate the integrals in Exercises 51–56.

51. $\int \sin 3x \cos 2x \, dx$

52. $\int \sin 2x \cos 3x \, dx$

53. $\int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx$

54. $\int_{0}^{\pi/2} \sin x \cos x \, dx$

55. $\int \cos 3x \cos 4x \, dx$

56. $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x \, dx$

Exercises 57–62 require the use of various trigonometric identities before you evaluate the integrals.

57. $\int \sin^2 \theta \cos 3\theta \, d\theta$

58. $\int \cos^2 2\theta \sin \theta \, d\theta$

59. $\int \cos^3 \theta \sin 2\theta \, d\theta$

60. $\int \sin^3 \theta \cos 2\theta \, d\theta$

61. $\int \sin \theta \cos \theta \cos 3\theta \, d\theta$

62. $\int \sin \theta \sin 2\theta \sin 3\theta \, d\theta$

Assorted Integrations

Use any method to evaluate the integrals in Exercises 63–68.

63. $\int \dfrac{\sec^3 x}{\tan x} \, dx$

64. $\int \dfrac{\sin^3 x}{\cos^4 x} \, dx$

65. $\int \dfrac{\tan^2 x}{\csc x} \, dx$

66. $\int \dfrac{\cot x}{\cos^2 x} \, dx$

67. $\int x \sin^2 x \, dx$

68. $\int x \cos^3 x \, dx$

Applications

69. Arc length Find the length of the curve

$$y = \ln(\sin x), \quad \frac{\pi}{6} \le x \le \frac{\pi}{2}$$

70. Center of gravity Find the center of gravity of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

71. Volume Find the volume generated by revolving one arch of the curve $y = \sin x$ about the x-axis.

72. Area Find the area between the x-axis and the curve $y = \sqrt{1 + \cos 4x}$, $0 \le x \le \pi$.

73. Centroid Find the centroid of the region bounded by the graphs of $y = x + \cos x$ and $y = 0$ for $0 \le x \le 2\pi$.

74. Volume Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sin x + \sec x$, $y = 0$, $x = 0$, and $x = \pi/3$ about the x-axis.

75. Volume Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \tan^{-1} x$, $x = 0$, and $y = \pi/4$ about the y-axis.

76. Average Value Find the average value of the function $f(x) = \dfrac{1}{1 - \sin \theta}$ on $[0, \pi/6]$.

8.4 Trigonometric Substitutions

Trigonometric substitutions occur when we replace the variable of integration by a trigonometric function. The most common substitutions are $x = a \tan \theta$, $x = a \sin \theta$, and $x = a \sec \theta$. These substitutions are effective in transforming integrals involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, and $\sqrt{x^2 - a^2}$ into integrals we can evaluate directly since they come from the reference right triangles in Figure 8.2.

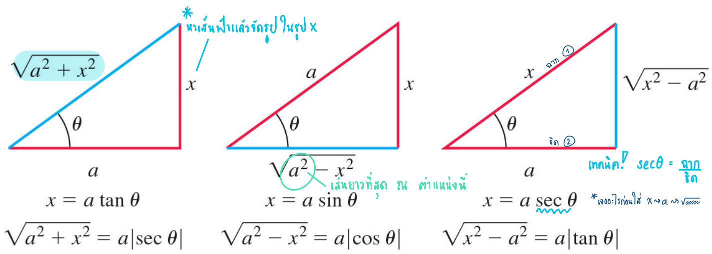

FIGURE 8.2 Reference triangles for the three basic substitutions identifying the sides labeled x and a for each substitution.

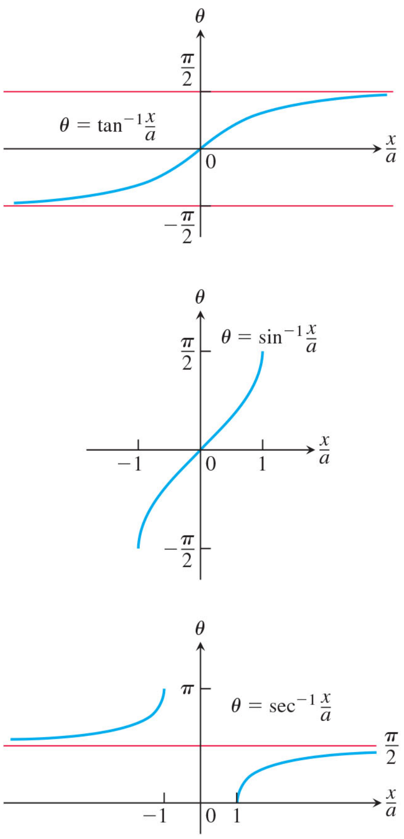

FIGURE 8.3 The arctangent, arcsine, and arcsecant of x/a, graphed as functions of x/a.

With $x = a \tan\theta$,

$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta) = a^2\sec^2\theta.$$

With $x = a\sin\theta$,

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1 - \sin^2\theta) = a^2\cos^2\theta.$$

With $x = a\sec\theta$,

$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta.$$

We want any substitution we use in an integration to be reversible so that we can change back to the original variable afterward. For example, if $x = a\tan\theta$, we want to be able to set $\theta = \tan^{-1}(x/a)$ after the integration takes place. If $x = a\sin\theta$, we want to be able to set $\theta = \sin^{-1}(x/a)$ when we're done, and similarly for $x = a\sec\theta$.

As we know from Section 1.6, the functions in these substitutions have inverses only for selected values of θ (Figure 8.3). For reversibility,

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$$x = a\tan\theta \quad \text{requires} \quad \theta = \tan^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a\sin\theta \quad \text{requires} \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

$$x = a\sec\theta \quad \text{requires} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad \begin{cases} 0 \le \theta < \dfrac{\pi}{2} & \text{if } \dfrac{x}{a} \ge 1, \\[2ex] \dfrac{\pi}{2} < \theta \le \pi & \text{if } \dfrac{x}{a} \le -1. \end{cases}$$

To simplify calculations with the substitution $x = a\sec\theta$, we will restrict its use to integrals in which $x/a \ge 1$. This will place θ in $[0, \pi/2)$ and make $\tan\theta \ge 0$. We will then have $\sqrt{x^2 - a^2} = \sqrt{a^2\tan^2\theta} = |a\tan\theta| = a\tan\theta$, free of absolute values, provided $a > 0$.

Procedure for a Trigonometric Substitution

1. Write down the substitution for x, calculate the differential dx, and specify the selected values of θ for the substitution.
2. Substitute the trigonometric expression and the calculated differential into the integrand, and then simplify the results algebraically.
3. Integrate the trigonometric integral, keeping in mind the restrictions on the angle θ for reversibility.
4. Draw an appropriate reference triangle to reverse the substitution in the integration result and convert it back to the original variable x.

EXAMPLE 1 Evaluate

$$\int \frac{dx}{\sqrt{4 + x^2}}.$$

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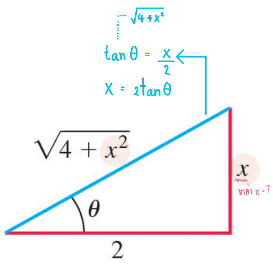


Solution We set

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$4 + x^2 = 4 + 4\tan^2\theta = 4(1 + \tan^2\theta) = 4\sec^2\theta.$$

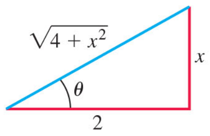


FIGURE 8.4 Reference triangle for $x = 2\tan\theta$ (Example 1):

$$\tan\theta = \frac{x}{2}$$

and

$$\sec\theta = \frac{\sqrt{4+x^2}}{2}.$$

Then

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} \qquad \sqrt{\sec^2\theta} = |\sec\theta|$$

$$= \int \sec\theta\, d\theta \qquad \sec\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C. \qquad \text{From Fig. 8.4}$$

Notice how we expressed $\ln|\sec\theta + \tan\theta|$ in terms of x: We drew a reference triangle for the original substitution $x = 2\tan\theta$ (Figure 8.4) and read the ratios from the triangle. ■

EXAMPLE 2 Here we find an expression for the inverse hyperbolic sine function in terms of the natural logarithm. Following the same procedure as in Example 1, we find that

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \int \sec\theta\, d\theta \qquad x = a\tan\theta,\ dx = a\sec^2\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right| + C \qquad \text{Fig. 8.2}$$

From Table 7.9, $\sinh^{-1}(x/a)$ is also an antiderivative of $1/\sqrt{a^2+x^2}$, so the two antiderivatives differ by a constant, giving

$$\sinh^{-1}\frac{x}{a} = \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right| + C.$$

Setting $x = 0$ in this last equation, we find $0 = \ln|1| + C$, so $C = 0$. Since $\sqrt{a^2+x^2} > |x|$, we conclude that

$$\boxed{\sinh^{-1}\frac{x}{a} = \ln\left(\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right)}$$

(See also Exercise 76 in Section 7.3.) ■

EXAMPLE 3 Evaluate

$$\int \frac{x^2\, dx}{\sqrt{9-x^2}}.$$

Solution We set

$$x = 3\sin\theta, \qquad dx = 3\cos\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$9 - x^2 = 9 - 9\sin^2\theta = 9(1 - \sin^2\theta) = 9\cos^2\theta.$$

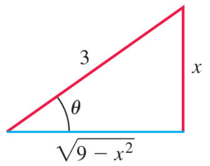


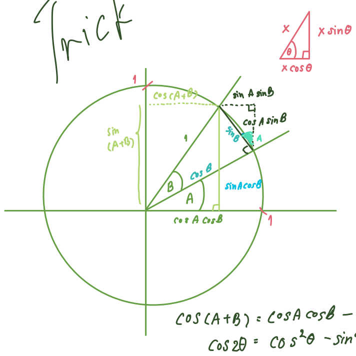
FIGURE 8.5 Reference triangle for $x = 3\sin\theta$ (Example 3):

$$\sin\theta = \frac{x}{3}$$

and

$$\cos\theta = \frac{\sqrt{9 - x^2}}{3}.$$

Trick



cos(A+B) = cosA cosB − sin A sin B
cos 2θ = cos²θ − sin²θ
= cos²θ − (1−cos²θ)
cos 2θ = 2cos²θ − 1
cos²θ = (cos2θ + 1)/2

Then

X   dx

$$\int \frac{x^2\,dx}{\sqrt{9 - x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\,d\theta}{|3\cos\theta|}$$

$$= 9\int \sin^2\theta\,d\theta \qquad \cos\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= 9\int \frac{1 - \cos 2\theta}{2}\,d\theta \quad ;\ \sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

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$$= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

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$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

ให้เป็นเทอม x

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9 - x^2}}{3}\right) + C \qquad \text{From Fig. 8.5}$$

$$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9 - x^2} + C.$$

EXAMPLE 4 Evaluate

$$\int \frac{dx}{\sqrt{25x^2 - 4}}, \qquad x > \frac{2}{5}.$$

Solution We first rewrite the radical as

$$\sqrt{25x^2 - 4} = \sqrt{25\left(x^2 - \frac{4}{25}\right)}$$

$$= 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2} \qquad \sqrt{x^2 - a^2} \text{ with } a = \frac{2}{5}$$

to put the radicand in the form $x^2 - a^2$. We then substitute

$$x = \frac{2}{5}\sec\theta, \qquad dx = \frac{2}{5}\sec\theta\tan\theta\,d\theta, \qquad 0 < \theta < \frac{\pi}{2}.$$

We then get

$$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25} = \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta$$

and

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta. \qquad \tan\theta > 0 \text{ for } 0 < \theta < \pi/2$$

With these substitutions, we have

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \int \frac{dx}{5\sqrt{x^2 - (4/25)}} = \int \frac{(2/5)\sec\theta\tan\theta\,d\theta}{5\cdot(2/5)\tan\theta}$$

$$= \frac{1}{5}\int \sec\theta\,d\theta = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}\right| + C. \qquad \text{From Fig. 8.6}$$

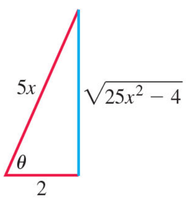

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FIGURE 8.6 If $x = (2/5)\sec\theta$, $0 < \theta < \pi/2$, then $\theta = \sec^{-1}(5x/2)$, and we can read the values of the other trigonometric functions of θ from this right triangle (Example 4).

EXERCISES 8.4

Using Trigonometric Substitutions

Evaluate the integrals in Exercises 1–14.

1. $\displaystyle\int \frac{dx}{\sqrt{9+x^2}}$

2. $\displaystyle\int \frac{3\,dx}{\sqrt{1+9x^2}}$

3. $\displaystyle\int_{-2}^{2} \frac{dx}{4+x^2}$

4. $\displaystyle\int_{0}^{2} \frac{dx}{8+2x^2}$

5. $\displaystyle\int_{0}^{3/2} \frac{dx}{\sqrt{9-x^2}}$

6. $\displaystyle\int_{0}^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$

7. $\displaystyle\int \sqrt{25-t^2}\,dt$

8. $\displaystyle\int \sqrt{1-9t^2}\,dt$

9. $\displaystyle\int \frac{dx}{\sqrt{4x^2-49}}, \quad x > \frac{7}{2}$

10. $\displaystyle\int \frac{5\,dx}{\sqrt{25x^2-9}}, \quad x > \frac{3}{5}$

11. $\displaystyle\int \frac{\sqrt{y^2-49}}{y}\,dy, \quad y > 7$

12. $\displaystyle\int \frac{\sqrt{y^2-25}}{y^3}\,dy, \quad y > 5$

13. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1}}, \quad x > 1$

14. $\displaystyle\int \frac{2\,dx}{x^3\sqrt{x^2-1}}, \quad x > 1$

Assorted Integrations

Use any method to evaluate the integrals in Exercises 15–34. Most will require trigonometric substitutions, but some can be evaluated by other methods.

15. $\displaystyle\int \frac{x}{\sqrt{9-x^2}}\,dx$

16. $\displaystyle\int \frac{x^2}{4+x^2}\,dx$

17. $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+4}}$

18. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}}$

19. $\displaystyle\int \frac{8\,dw}{w^2\sqrt{4-w^2}}$

20. $\displaystyle\int \frac{\sqrt{9-w^2}}{w^2}\,dw$

21. $\displaystyle\int \sqrt{\frac{x+1}{1-x}}\,dx$

22. $\displaystyle\int x\sqrt{x^2-4}\,dx$

23. $\displaystyle\int_{0}^{\sqrt{3}/2} \frac{4x^2\,dx}{(1-x^2)^{3/2}}$

24. $\displaystyle\int_{0}^{1} \frac{dx}{(4-x^2)^{3/2}}$

25. $\displaystyle\int \frac{dx}{(x^2-1)^{3/2}}, \quad x > 1$

26. $\displaystyle\int \frac{x^2\,dx}{(x^2-1)^{5/2}}, \quad x > 1$

27. $\displaystyle\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$

28. $\displaystyle\int \frac{(1-x^2)^{1/2}}{x^4}\,dx$

29. $\displaystyle\int \frac{8\,dx}{(4x^2+1)^2}$

30. $\displaystyle\int \frac{6\,dt}{(9t^2+1)^2}$

31. $\displaystyle\int \frac{x^3\,dx}{x^2-1}$

32. $\displaystyle\int \frac{x\,dx}{25+4x^2}$

33. $\displaystyle\int \frac{v^2\,dv}{(1-v^2)^{5/2}}$

34. $\displaystyle\int \frac{(1-r^2)^{5/2}}{r^8}\,dr$

In Exercises 35–48, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

35. $\displaystyle\int_{0}^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$

36. $\displaystyle\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}}$

37. $\displaystyle\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}}$

38. $\displaystyle\int_{1}^{e} \frac{dy}{y\sqrt{1+(\ln y)^2}}$

39. $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$

40. $\displaystyle\int \frac{dx}{1+x^2}$

41. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-1}}$

42. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$

43. $\displaystyle\int \frac{x\,dx}{\sqrt{1+x^4}}$

44. $\displaystyle\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}\,dx$

45. $\displaystyle\int \sqrt{\frac{4-x}{x}}\,dx$
   (*Hint:* Let $x = u^2$.)

46. $\displaystyle\int \sqrt{\frac{x}{1-x^3}}\,dx$
   (*Hint:* Let $u = x^{3/2}$.)

47. $\displaystyle\int \sqrt{x}\,\sqrt{1-x}\,dx$

48. $\displaystyle\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx$

Complete the Square Before Using Trigonometric Substitutions

For Exercises 49–52, complete the square before using an appropriate trigonometric substitution.

49. $\displaystyle\int \sqrt{8-2x-x^2}\,dx$

50. $\displaystyle\int \frac{1}{\sqrt{x^2-2x+5}}\,dx$

51. $\displaystyle\int \frac{\sqrt{x^2+4x+3}}{x+2}\,dx$

52. $\displaystyle\int \frac{\sqrt{x^2+2x+2}}{x^2+2x+1}\,dx$

Initial Value Problems

Solve the initial value problems in Exercises 53–56 for y as a function of x.

53. $\displaystyle x\frac{dy}{dx} = \sqrt{x^2-4}, \quad x \ge 2, \quad y(2) = 0$

54. $\displaystyle \sqrt{x^2-9}\,\frac{dy}{dx} = 1, \quad x > 3, \quad y(5) = \ln 3$

55. $\displaystyle (x^2+4)\frac{dy}{dx} = 3, \quad y(2) = 0$

56. $\displaystyle (x^2+1)^2\frac{dy}{dx} = \sqrt{x^2+1}, \quad y(0) = 1$

Applications and Examples

57. **Area** Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve $y = \sqrt{9-x^2}/3$.

58. **Area** Find the area enclosed by the ellipse

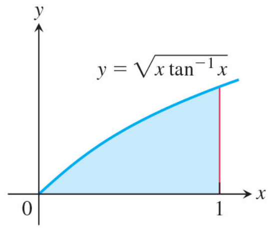
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

59. Consider the region bounded by the graphs of $y = \sin^{-1} x$, $y = 0$, and $x = 1/2$.

a. Find the area of the region.

b. Find the centroid of the region.

60. Consider the region bounded by the graphs of $y = \sqrt{x \tan^{-1} x}$ and $y = 0$ for $0 \le x \le 1$. Find the volume of the solid formed by revolving this region about the x-axis (see accompanying figure).



61. Evaluate $\int x^3 \sqrt{1 - x^2}\, dx$ using

a. integration by parts.

b. a u-substitution.

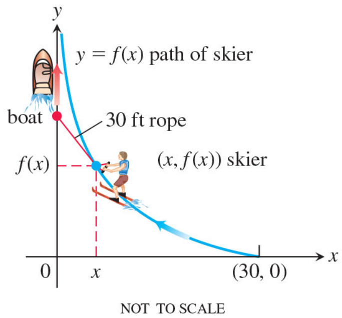
c. a trigonometric substitution.

62. Path of a water skier Suppose that a boat is positioned at the origin with a water skier tethered to the boat at the point $(30, 0)$ on a rope 30 ft long. As the boat travels along the positive y-axis, the skier is pulled behind the boat along an unknown path $y = f(x)$, as shown in the accompanying figure.

a. Show that $f'(x) = \dfrac{-\sqrt{900 - x^2}}{x}$.

(*Hint*: Assume that the skier is always pointed directly at the boat and the rope is on a line tangent to the path $y = f(x)$.)

b. Solve the equation in part (a) for $f(x)$, using $f(30) = 0$.



63. Find the average value of $f(x) = \dfrac{\sqrt{x+1}}{\sqrt{x}}$ on the interval $[1, 3]$.

64. Find the length of the curve $y = 1 - e^{-x}$, $0 \le x \le 1$.

8.5 Integration of Rational Functions by Partial Fractions

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called *partial fractions*, which are easily integrated. For instance, the rational function $(5x - 3)/(x^2 - 2x - 3)$ can be rewritten as

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{2}{x + 1} + \frac{3}{x - 3}.$$

You can verify this equation algebraically by placing the fractions on the right side over a common denominator $(x + 1)(x - 3)$. The skill acquired in writing rational functions as such a sum is useful in other settings as well (for instance, when using certain transform methods to solve differential equations). To integrate the rational function $(5x - 3)/(x^2 - 2x - 3)$ on the left side of our previous expression, we simply sum the integrals of the fractions on the right side:

$$\int \frac{5x - 3}{(x + 1)(x - 3)}\, dx = \int \frac{2}{x + 1}\, dx + \int \frac{3}{x - 3}\, dx$$
$$= 2 \ln|x + 1| + 3 \ln|x - 3| + C.$$

The method for rewriting rational functions as a sum of simpler fractions is called **the method of partial fractions**. In the case of the preceding example, it consists of finding constants A and B such that

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}. \qquad (1)$$

(Pretend for a moment that we do not know that $A = 2$ and $B = 3$ will work.) We call the fractions $A/(x + 1)$ and $B/(x - 3)$ **partial fractions** because their denominators are only part of the original denominator $x^2 - 2x - 3$. We call A and B **undetermined coefficients** until suitable values for them have been found.

To find A and B, we first clear Equation (1) of fractions and regroup in powers of x, obtaining

$$5x - 3 = A(x - 3) + B(x + 1) = (A + B)x - 3A + B.$$

This will be an identity in x if and only if the coefficients of like powers of x on the two sides are equal:

$$A + B = 5, \qquad -3A + B = -3.$$

Solving these equations simultaneously gives $A = 2$ and $B = 3$.

General Description of the Method

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

- *The degree of $f(x)$ must be less than the degree of $g(x)$.* That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term. Example 3 of this section illustrates such a case.
- We must know the factors of $g(x)$. In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

Here is how we find the partial fractions of a proper fraction $f(x)/g(x)$ when the factors of g are known. A quadratic polynomial (or factor) is **irreducible** if it cannot be written as the product of two linear factors with real coefficients. That is, the polynomial has no real roots.

Method of Partial Fractions When $f(x)/g(x)$ Is Proper

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{(x^2 + px + q)} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

EXAMPLE 1 Use partial fractions to evaluate

$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}\,dx.$$

Solution Note that each of the factors $(x-1)$, $(x+1)$, and $(x+3)$ is raised only to the first power. Therefore, the partial fraction decomposition has the form

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}.$$

To find the values of the undetermined coefficients A, B, and C, we clear fractions and get

$$\begin{aligned} x^2+4x+1 &= A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1) \\ &= A(x^2+4x+3) + B(x^2+2x-3) + C(x^2-1) \\ &= (A+B+C)x^2 + (4A+2B)x + (3A-3B-C). \end{aligned}$$

The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of x, obtaining

$$\begin{array}{ll} \text{Coefficient of } x^2: & A + B + C = 1 \\ \text{Coefficient of } x^1: & 4A + 2B = 4 \\ \text{Coefficient of } x^0: & 3A - 3B - C = 1 \end{array}$$

There are several ways of solving such a system of linear equations for the unknowns A, B, and C, including elimination of variables or the use of a calculator or computer. The solution is $A = 3/4$, $B = 1/2$, and $C = -1/4$. Hence we have

$$\begin{aligned} \int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx &= \int \left[\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3}\right] dx \\ &= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K, \end{aligned}$$

where K is the arbitrary constant of integration (we call it K here to avoid confusion with the undetermined coefficient we labeled as C). ■

EXAMPLE 2 Use partial fractions to evaluate

$$\int \frac{6x+7}{(x+2)^2}\,dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} \qquad \text{Two terms because } (x+2) \text{ is squared}$$

$$\begin{aligned} 6x + 7 &= A(x+2) + B \qquad \text{Multiply both sides by } (x+2)^2. \\ &= Ax + (2A + B) \end{aligned}$$

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\begin{aligned} \int \frac{6x+7}{(x+2)^2}\,dx &= \int \left(\frac{6}{x+2} - \frac{5}{(x+2)^2}\right) dx \\ &= 6\int \frac{dx}{x+2} - 5\int (x+2)^{-2}\,dx \\ &= 6\ln|x+2| + 5(x+2)^{-1} + C. \end{aligned}$$ ■

The next example shows how to handle the case when $f(x)/g(x)$ is an improper fraction. It is a case where the degree of f is larger than the degree of g.

EXAMPLE 3 Use partial fractions to evaluate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\, dx. \quad \text{······ หารยาวได้}$$

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r l}
 & \quad 2x \\
x^2 - 2x - 3 \,\big) & \overline{2x^3 - 4x^2 - x - 3} \\
 & \underline{2x^3 - 4x^2 - 6x - 3} \\
 & \qquad\qquad\quad 5x - 3
\end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\frac{5X-3}{(X+1)(X-3)} \cdots\cdots \frac{A}{(X+1} + \frac{B}{(X-3)}$$

$$= Ax - 3A + Bx + B$$

$$\begin{aligned}
\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\, dx &= \int 2x\, dx + \int \frac{5x - 3}{x^2 - 2x - 3}\, dx \\
&= \int 2x\, dx + \int \frac{2}{x + 1}\, dx + \int \frac{3}{x - 3}\, dx \\
&= x^2 + 2 \ln |x + 1| + 3 \ln |x - 3| + C.
\end{aligned}$$

EXAMPLE 4 Use partial fractions to evaluate

$$\int \frac{-2x + 4}{(x^2 + 1)\underbrace{(x - 1)^2}_{(x-1)(x-1)}}\, dx.$$

Solution The denominator has an irreducible quadratic factor $x^2 + 1$ as well as a repeated linear factor $(x - 1)^2$, so we write

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}. \tag{2}$$

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Clearing the equation of fractions gives

$$\begin{aligned}
-2x + 4 &= (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1) \\
&= (A + C)x^3 + (-2A + B - C + D)x^2 \\
&\quad + (A - 2B + C)x + (B - C + D).
\end{aligned}$$

Equating coefficients of like terms gives

Coefficients of x^3:	$0 = A + C$
Coefficients of x^2:	$0 = -2A + B - C + D$
Coefficients of x^1:	$-2 = A - 2B + C$
Coefficients of x^0:	$4 = B - C + D$

We solve these equations simultaneously to find the values of A, B, C, and D:

$-4 = -2A, \qquad A = 2$	Subtract fourth equation from second.
$C = -A = -2$	From the first equation
$B = (A + C + 2)/2 = 1$	From the third equation and $C = -A$
$D = 4 - B + C = 1.$	From the fourth equation

We substitute these values into Equation (2), obtaining

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}.$$

Finally, using the expansion above we can integrate:

$$\begin{aligned}
\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx &= \int \left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \\
&= \int \left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \\
&= \ln(x^2+1) + \tan^{-1} x - 2\ln|x-1| - \frac{1}{x-1} + C. \quad \blacksquare
\end{aligned}$$

EXAMPLE 5 Use partial fractions to evaluate

$$\int \frac{dx}{x(x^2+1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

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Multiplying by $x(x^2+1)^2$, we have

$$\begin{aligned}
1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\
&= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\
&= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A.
\end{aligned}$$

If we equate coefficients, we get the system

$$A+B=0, \qquad C=0, \qquad 2A+B+D=0, \qquad C+E=0, \qquad A=1.$$

Solving this system gives $A = 1$, $B = -1$, $C = 0$, $D = -1$, and $E = 0$. Thus,

$$\begin{aligned}
\int \frac{dx}{x(x^2+1)^2} &= \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right] dx \\
&= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2} \\
&= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2} \qquad \begin{array}{l} u = x^2+1, \\ du = 2x\,dx \end{array} \\
&= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + K \\
&= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K \\
&= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K. \quad \blacksquare
\end{aligned}$$

HISTORICAL BIOGRAPHY
Oliver Heaviside
(1850–1925)
www.goo.gl/5rnavZ

When the degree of the polynomial $f(x)$ is less than the degree of $g(x)$ and

$$g(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$$

is a product of n distinct linear factors, each raised to the first power, there is a quick way to expand $f(x)/g(x)$ by partial fractions.

EXAMPLE 6 Find A, B, and C in the partial fraction expansion

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3}. \tag{3}$$

Solution If we multiply both sides of Equation (3) by $(x - 1)$ to get

คูณ (x-1) ทั้งสอง เพื่อหาค่า A* ;

$$\frac{x^2 + 1}{(x - 2)(x - 3)} = A + \frac{B(x - 1)}{x - 2} + \frac{C(x - 1)}{x - 3}$$

and set $x = 1$, the resulting equation gives the value of A:

$$\frac{(1)^2 + 1}{(1 - 2)(1 - 3)} = A + 0 + 0,$$

$$A = 1.$$

In exactly the same way, we can multiply both sides by $(x - 2)$ and then substitute in *$x = 2$. This gives

$$\frac{(2)^2 + 1}{(2 - 1)(2 - 3)} = B.$$

So $B = -5$. Finally, we multiply both sides by $(x - 3)$ and then substitute in *$x = 3$, which yields

$$\frac{(3)^2 + 1}{(3 - 1)(3 - 2)} = C,$$

and $C = 5$. ■

Other Ways to Determine the Coefficients

Another way to determine the constants that appear in partial fractions is to differentiate, as in the next example. Still another is to assign selected numerical values to x.

EXAMPLE 7 Find A, B, and C in the equation

$$\frac{x - 1}{(x + 1)^3} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{(x + 1)^3}$$

by clearing fractions, differentiating the result, and substituting $x = -1$.

Solution We first clear fractions:

diff

คูณ $(x+1)^3$ ทั้งสองข้าง ; $$x - 1 = A(x + 1)^2 + B(x + 1) + C.$$ * diffทั้งสองข้าง

Substituting $x = -1$ shows $C = -2$. We then differentiate both sides with respect to x, obtaining

$$1 = 2A(x + 1) + B.$$

Substituting $x = -1$ shows $B = 1$. We differentiate again to get $0 = 2A$, which shows $A = 0$. Hence,

$$\frac{x - 1}{(x + 1)^3} = \frac{1}{(x + 1)^2} - \frac{2}{(x + 1)^3}.$$ ■

In some problems, assigning small values to x, such as $x = 0, \pm 1, \pm 2$, to get equations in A, B, and C provides a fast alternative to other methods.

EXAMPLE 8 Find A, B, and C in the expression

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

by assigning numerical values to x.

Solution Clear fractions to get

$$x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$

Then let $x = 1, 2, 3$ successively to find A, B, and C:

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$$\begin{aligned}
x = 1:\quad (1)^2 + 1 &= A(-1)(-2) + B(0) + C(0)\\
2 &= 2A\\
A &= 1\\
x = 2:\quad (2)^2 + 1 &= A(0) + B(1)(-1) + C(0)\\
5 &= -B\\
B &= -5\\
x = 3:\quad (3)^2 + 1 &= A(0) + B(0) + C(2)(1)\\
10 &= 2C\\
C &= 5.
\end{aligned}$$

Conclusion:

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}.$$

EXERCISES 8.5

Expanding Quotients into Partial Fractions

Expand the quotients in Exercises 1–8 by partial fractions.

1. $\dfrac{5x-13}{(x-3)(x-2)}$
2. $\dfrac{5x-7}{x^2-3x+2}$
3. $\dfrac{x+4}{(x+1)^2}$
4. $\dfrac{2x+2}{x^2-2x+1}$
5. $\dfrac{z+1}{z^2(z-1)}$
6. $\dfrac{z}{z^3-z^2-6z}$
7. $\dfrac{t^2+8}{t^2-5t+6}$
8. $\dfrac{t^4+9}{t^4+9t^2}$

Nonrepeated Linear Factors

In Exercises 9–16, express the integrand as a sum of partial fractions and evaluate the integrals.

9. $\displaystyle\int \frac{dx}{1-x^2}$
10. $\displaystyle\int \frac{dx}{x^2+2x}$
11. $\displaystyle\int \frac{x+4}{x^2+5x-6}\,dx$
12. $\displaystyle\int \frac{2x+1}{x^2-7x+12}\,dx$
13. $\displaystyle\int_4^8 \frac{y\,dy}{y^2-2y-3}$
14. $\displaystyle\int_{1/2}^1 \frac{y+4}{y^2+y}\,dy$
15. $\displaystyle\int \frac{dt}{t^3+t^2-2t}$
16. $\displaystyle\int \frac{x+3}{2x^3-8x}\,dx$

Repeated Linear Factors

In Exercises 17–20, express the integrand as a sum of partial fractions and evaluate the integrals.

17. $\displaystyle\int_0^1 \frac{x^3\,dx}{x^2+2x+1}$
18. $\displaystyle\int_{-1}^0 \frac{x^3\,dx}{x^2-2x+1}$
19. $\displaystyle\int \frac{dx}{(x^2-1)^2}$
20. $\displaystyle\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$

Irreducible Quadratic Factors

In Exercises 21–32, express the integrand as a sum of partial fractions and evaluate the integrals.

21. $\displaystyle\int_0^1 \frac{dx}{(x+1)(x^2+1)}$
22. $\displaystyle\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt$
23. $\displaystyle\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy$
24. $\displaystyle\int \frac{8x^2+8x+2}{(4x^2+1)^2}\,dx$
25. $\displaystyle\int \frac{2s+2}{(s^2+1)(s-1)^3}\,ds$
26. $\displaystyle\int \frac{s^4+81}{s(s^2+9)^2}\,ds$
27. $\displaystyle\int \frac{x^2-x+2}{x^3-1}\,dx$
28. $\displaystyle\int \frac{1}{x^4+x}\,dx$
29. $\displaystyle\int \frac{x^2}{x^4-1}\,dx$
30. $\displaystyle\int \frac{x^2+x}{x^4-3x^2-4}\,dx$
31. $\displaystyle\int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2}\,d\theta$
32. $\displaystyle\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}\,d\theta$

Improper Fractions

In Exercises 33–38, perform long division on the integrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

33. $\displaystyle\int \frac{2x^3 - 2x^2 + 1}{x^2 - x}\,dx$

34. $\displaystyle\int \frac{x^4}{x^2 - 1}\,dx$

35. $\displaystyle\int \frac{9x^3 - 3x + 1}{x^3 - x^2}\,dx$

36. $\displaystyle\int \frac{16x^3}{4x^2 - 4x + 1}\,dx$

37. $\displaystyle\int \frac{y^4 + y^2 - 1}{y^3 + y}\,dy$

38. $\displaystyle\int \frac{2y^4}{y^3 - y^2 + y - 1}\,dy$

Evaluating Integrals

Evaluate the integrals in Exercises 39–54.

39. $\displaystyle\int \frac{e^t\,dt}{e^{2t} + 3e^t + 2}$

40. $\displaystyle\int \frac{e^{4t} + 2e^{2t} - e^t}{e^{2t} + 1}\,dt$

41. $\displaystyle\int \frac{\cos y\,dy}{\sin^2 y + \sin y - 6}$

42. $\displaystyle\int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2}$

43. $\displaystyle\int \frac{(x-2)^2\tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2}\,dx$

44. $\displaystyle\int \frac{(x+1)^2\tan^{-1}(3x) + 9x^3 + x}{(9x^2+1)(x+1)^2}\,dx$

45. $\displaystyle\int \frac{1}{x^{3/2} - \sqrt{x}}\,dx$

46. $\displaystyle\int \frac{1}{(x^{1/3} - 1)\sqrt{x}}\,dx$
(*Hint:* Let $x = u^6$.)

47. $\displaystyle\int \frac{\sqrt{x+1}}{x}\,dx$
(*Hint:* Let $x + 1 = u^2$.)

48. $\displaystyle\int \frac{1}{x\sqrt{x+9}}\,dx$

49. $\displaystyle\int \frac{1}{x(x^4+1)}\,dx$
$\left(\textit{Hint: } \text{Multiply by } \dfrac{x^3}{x^3}.\right)$

50. $\displaystyle\int \frac{1}{x^6(x^5+4)}\,dx$

51. $\displaystyle\int \frac{1}{\cos 2\theta\,\sin\theta}\,d\theta$

52. $\displaystyle\int \frac{1}{\cos\theta + \sin 2\theta}\,d\theta$

53. $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx$

54. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{2-\sqrt{x}} + \sqrt{x}}\,dx$

Use any method to evaluate the integrals in Exercises 55–66.

55. $\displaystyle\int \frac{x^3 - 2x^2 - 3x}{x+2}\,dx$

56. $\displaystyle\int \frac{x+2}{x^3 - 2x^2 - 3x}\,dx$

57. $\displaystyle\int \frac{2^x - 2^{-x}}{2^x + 2^{-x}}\,dx$

58. $\displaystyle\int \frac{2^x}{2^{2x} + 2^x - 2}\,dx$

59. $\displaystyle\int \frac{1}{x^4 - 1}\,dx$

60. $\displaystyle\int \frac{x^4 - 1}{x^5 - 5x + 1}\,dx$

61. $\displaystyle\int \frac{\ln x + 2}{x(\ln x + 1)(\ln x + 3)}\,dx$

62. $\displaystyle\int \frac{2}{x(\ln x - 2)^3}\,dx$

63. $\displaystyle\int \frac{1}{\sqrt{x^2 - 1}}\,dx$

64. $\displaystyle\int \frac{x}{x + \sqrt{x^2 + 2}}\,dx$

65. $\displaystyle\int x^5\sqrt{x^3 + 1}\,dx$

66. $\displaystyle\int x^2\sqrt{1 - x^2}\,dx$

Initial Value Problems

Solve the initial value problems in Exercises 67–70 for x as a function of t.

67. $(t^2 - 3t + 2)\dfrac{dx}{dt} = 1 \quad (t > 2), \quad x(3) = 0$

68. $(3t^4 + 4t^2 + 1)\dfrac{dx}{dt} = 2\sqrt{3}, \quad x(1) = -\pi\sqrt{3}/4$

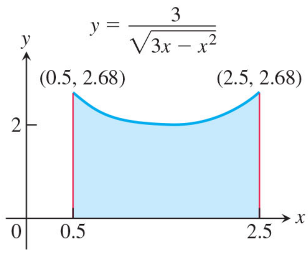
69. $(t^2 + 2t)\dfrac{dx}{dt} = 2x + 2 \quad (t, x > 0), \quad x(1) = 1$

70. $(t + 1)\dfrac{dx}{dt} = x^2 + 1 \quad (t > -1), \quad x(0) = 0$

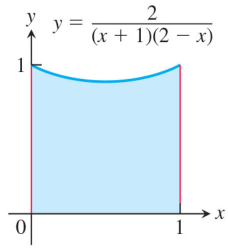
Applications and Examples

In Exercises 71 and 72, find the volume of the solid generated by revolving the shaded region about the indicated axis.

71. The x-axis



72. The y-axis



73. Find the length of the curve $y = \ln(1 - x^2)$, $0 \le x \le \frac{1}{2}$.

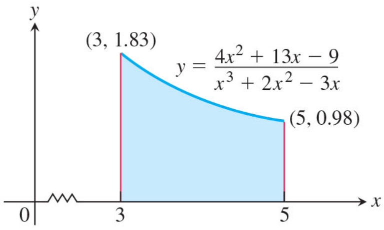
74. Integrate $\displaystyle\int \sec\theta\,d\theta$ by

a. multiplying by $\dfrac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}$ and then using a u-substitution.

b. writing the integral as $\displaystyle\int \frac{1}{\cos\theta}\,d\theta$. Then multiply by $\dfrac{\cos\theta}{\cos\theta}$, use a trigonometric identity and a u-substitution, and finally integrate using partial fractions.

T **75.** Find, to two decimal places, the x-coordinate of the centroid of the region in the first quadrant bounded by the x-axis, the curve $y = \tan^{-1} x$, and the line $x = \sqrt{3}$.

T **76.** Find the x-coordinate of the centroid of this region to two decimal places.



T **77. Social diffusion** Sociologists sometimes use the phrase "social diffusion" to describe the way information spreads through a population. The information might be a rumor, a cultural fad, or news about a technical innovation. In a sufficiently large population, the number of people x who have the information is treated as a differentiable function of time t, and the rate of diffusion, dx/dt, is assumed to be proportional to the number of people who have the information times the number of people who do not. This leads to the equation

$$\frac{dx}{dt} = kx(N - x),$$

where N is the number of people in the population.

Suppose t is in days, $k = 1/250$, and two people start a rumor at time $t = 0$ in a population of $N = 1000$ people.

a. Find x as a function of t.

b. When will half the population have heard the rumor? (This is when the rumor will be spreading the fastest.)

T **78. Second-order chemical reactions** Many chemical reactions are the result of the interaction of two molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If a is the amount of substance A and b is the amount of substance B at time $t = 0$, and if x is the amount of product at time t, then the rate of formation of x may be given by the differential equation

$$\frac{dx}{dt} = k(a - x)(b - x),$$

or

$$\frac{1}{(a - x)(b - x)} \frac{dx}{dt} = k,$$

where k is a constant for the reaction. Integrate both sides of this equation to obtain a relation between x and t **(a)** if $a = b$, and **(b)** if $a \neq b$. Assume in each case that $x = 0$ when $t = 0$.

8.6 Integral Tables and Computer Algebra Systems

In this section we discuss how to use tables and computer algebra systems (CAS) to evaluate integrals.

Integral Tables

A Brief Table of Integrals is provided at the back of the text, after the index. (More extensive tables appear in compilations such as *CRC Mathematical Tables*, which contain thousands of integrals.) The integration formulas are stated in terms of constants a, b, c, m, n, and so on. These constants can usually assume any real value and need not be integers. Occasional limitations on their values are stated with the formulas. Formula 21 requires $n \neq -1$, for example, and Formula 27 requires $n \neq -2$.

The formulas also assume that the constants do not take on values that require dividing by zero or taking even roots of negative numbers. For example, Formula 24 assumes that $a \neq 0$, and Formulas 29a and 29b cannot be used unless b is positive.

EXAMPLE 1 Find

$$\int x(2x + 5)^{-1}\, dx.$$

Solution We use Formula 24 at the back of the book (not 22, which requires $n \neq -1$):

$$\int x(ax + b)^{-1}\, dx = \frac{x}{a} - \frac{b}{a^2} \ln |ax + b| + C.$$

With $a = 2$ and $b = 5$, we have

$$\int x(2x + 5)^{-1}\, dx = \frac{x}{2} - \frac{5}{4} \ln |2x + 5| + C.$$ ■

EXAMPLE 2 Find

$$\int \frac{dx}{x\sqrt{2x-4}}.$$

Solution We use Formula 29b:

$$\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}} \tan^{-1} \sqrt{\frac{ax-b}{b}} + C.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x-4}} = \frac{2}{\sqrt{4}} \tan^{-1} \sqrt{\frac{2x-4}{4}} + C = \tan^{-1} \sqrt{\frac{x-2}{2}} + C.$$

EXAMPLE 3 Find

$$\int x \sin^{-1} x \, dx.$$

Solution We begin by using Formula 106:

$$\int x^n \sin^{-1} ax \, dx = \frac{x^{n+1}}{n+1} \sin^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1}\, dx}{\sqrt{1-a^2x^2}}, \qquad n \neq -1.$$

With $n = 1$ and $a = 1$, we have

$$\int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \frac{x^2\, dx}{\sqrt{1-x^2}}.$$

Next we use Formula 49 to find the integral on the right:

$$\int \frac{x^2}{\sqrt{a^2-x^2}}\, dx = \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) - \frac{1}{2} x\sqrt{a^2-x^2} + C.$$

With $a = 1$,

$$\int \frac{x^2\, dx}{\sqrt{1-x^2}} = \frac{1}{2} \sin^{-1} x - \frac{1}{2} x\sqrt{1-x^2} + C.$$

The combined result is

$$\begin{aligned} \int x \sin^{-1} x \, dx &= \frac{x^2}{2} \sin^{-1} x - \frac{1}{2}\left(\frac{1}{2} \sin^{-1} x - \frac{1}{2} x\sqrt{1-x^2} + C\right) \\ &= \left(\frac{x^2}{2} - \frac{1}{4}\right)\sin^{-1} x + \frac{1}{4} x\sqrt{1-x^2} + C'. \end{aligned}$$

Reduction Formulas

The time required for repeated integrations by parts can sometimes be shortened by applying reduction formulas like

$$\int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx \tag{1}$$

$$\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx \tag{2}$$

$$\int \sin^n x \cos^m x \, dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n} \int \sin^{n-2} x \cos^m x \, dx \qquad (n \neq -m). \tag{3}$$

By applying such a formula repeatedly, we can eventually express the original integral in terms of a power low enough to be evaluated directly. The next example illustrates this procedure.

EXAMPLE 4 Find

$$\int \tan^5 x\,dx.$$

Solution We apply Equation (1) with $n = 5$ to get

$$\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \int \tan^3 x\,dx.$$

We then apply Equation (1) again, with $n = 3$, to evaluate the remaining integral:

$$\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x - \int \tan x\,dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C.$$

The combined result is

$$\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C'.$$ ■

As their form suggests, reduction formulas are derived using integration by parts. (See Example 5 in Section 8.2.)

Integration with a CAS

A powerful capability of computer algebra systems is their ability to integrate symbolically. This is performed with the **integrate command** specified by the particular system (for example, **int** in Maple, **Integrate** in Mathematica).

EXAMPLE 5 Suppose that you want to evaluate the indefinite integral of the function

$$f(x) = x^2\sqrt{a^2 + x^2}.$$

Using Maple, you first define or name the function:

$$> f := x\hat{\ }2 * \text{sqrt}\,(a\hat{\ }2 + x\hat{\ }2);$$

Then you use the integrate command on f, identifying the variable of integration:

$$> \text{int}(f, x);$$

Maple returns the answer

$$\frac{1}{4}x(a^2 + x^2)^{3/2} - \frac{1}{8}a^2x\sqrt{a^2 + x^2} - \frac{1}{8}a^4\ln\left(x + \sqrt{a^2 + x^2}\right).$$

If you want to see whether the answer can be simplified, enter

$$> \text{simplify}(\%);$$

Maple returns

$$\frac{1}{8}a^2x\sqrt{a^2 + x^2} + \frac{1}{4}x^3\sqrt{a^2 + x^2} - \frac{1}{8}a^4\ln\left(x + \sqrt{a^2 + x^2}\right).$$

If you want the definite integral for $0 \le x \le \pi/2$, you can use the format

$$> \text{int}(f, x = 0..\text{Pi}/2);$$

Maple will return the expression

$$\frac{1}{64}\pi(4a^2+\pi^2)^{(3/2)} - \frac{1}{32}a^2\pi\sqrt{4a^2+\pi^2} + \frac{1}{8}a^4\ln(2)$$
$$-\frac{1}{8}a^4\ln\left(\pi+\sqrt{4a^2+\pi^2}\right) + \frac{1}{16}a^4\ln\left(a^2\right).$$

You can also find the definite integral for a particular value of the constant a:

$$> a := 1;$$
$$> \operatorname{int}(f, x = 0..1);$$

Maple returns the numerical answer

$$\frac{3}{8}\sqrt{2} + \frac{1}{8}\ln\left(\sqrt{2}-1\right).$$

EXAMPLE 6 Use a CAS to find

$$\int \sin^2 x \cos^3 x\, dx.$$

Solution With Maple, we have the entry

$$> \operatorname{int}((\sin\text{^}2)(x) * (\cos\text{^}3)(x), x);$$

with the immediate return

$$-\frac{1}{5}\sin(x)\cos(x)^4 + \frac{1}{15}\cos(x)^2\sin(x) + \frac{2}{15}\sin(x).$$

Computer algebra systems vary in how they process integrations. We used Maple in Examples 5 and 6. Mathematica would have returned somewhat different results:

1. In Example 5, given

$$In[1] := \text{Integrate}[x\text{^}2 * \text{Sqrt}[a\text{^}2 + x\text{^}2], x]$$

Mathematica returns

$$Out[1] = \sqrt{a^2+x^2}\left(\frac{a^2x}{8}+\frac{x^3}{4}\right) - \frac{1}{8}a^4\,\text{Log}\left[x+\sqrt{a^2+x^2}\right]$$

without having to simplify an intermediate result. The answer is close to Formula 22 in the integral tables.

2. The Mathematica answer to the integral

$$In[2] := \text{Integrate}[\text{Sin}[x]\text{^}2 * \text{Cos}[x]\text{^}3, x]$$

in Example 6 is

$$Out[2] = \frac{\text{Sin}[x]}{8} - \frac{1}{48}\text{Sin}[3x] - \frac{1}{80}\text{Sin}[5x]$$

differing from the Maple answer. Both answers are correct.

Although a CAS is very powerful and can aid us in solving difficult problems, each CAS has its own limitations. There are even situations where a CAS may further complicate a problem (in the sense of producing an answer that is extremely difficult to use or interpret). Note, too, that neither Maple nor Mathematica returns an arbitrary constant $+C$. On the other hand, a little mathematical thinking on your part may reduce the problem to one that is quite easy to handle. We provide an example in Exercise 67.

Many hardware devices have an availability to integration applications, based on software (like Maple or Mathematica), that provide for symbolic input of the integrand to return symbolic output of the indefinite integral. Many of these software applications calculate definite integrals as well. These applications give another tool for finding integrals, aside from using integral tables. However, in some instances, the integration software may not provide an output answer at all.

Nonelementary Integrals

Many functions have antiderivatives that cannot be expressed using the standard functions that we have encountered, such as polynomials, trigonometric functions, and exponential functions. Integrals of functions that do not have elementary antiderivatives are called **nonelementary** integrals. These integrals can sometimes be expressed with infinite series (Chapter 10) or approximated using numerical methods (Section 8.7). Examples of nonelementary integrals include the error function (which measures the probability of random errors)

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

and integrals such as

$$\int \sin x^2\,dx \qquad \text{and} \qquad \int \sqrt{1+x^4}\,dx$$

that arise in engineering and physics. These and a number of others, such as

$$\int \frac{e^x}{x}\,dx, \qquad \int e^{(e^x)}\,dx, \qquad \int \frac{1}{\ln x}\,dx, \qquad \int \ln(\ln x)\,dx, \qquad \int \frac{\sin x}{x}\,dx,$$

$$\int \sqrt{1-k^2\sin^2 x}\,dx, \qquad 0 < k < 1,$$

look so easy they tempt us to try them just to see how they turn out. It can be proved, however, that there is no way to express any of these integrals as finite combinations of elementary functions. The same applies to integrals that can be changed into these by substitution. The functions in these integrals all have antiderivatives, as a consequence of the Fundamental Theorem of Calculus, Part 1, because they are continuous. However, none of the antiderivatives are elementary. The integrals you are asked to evaluate in this chapter have elementary antiderivatives.

EXERCISES 8.6

Using Integral Tables

Use the table of integrals at the back of the book to evaluate the integrals in Exercises 1–26.

1. $\displaystyle\int \frac{dx}{x\sqrt{x-3}}$

2. $\displaystyle\int \frac{dx}{x\sqrt{x+4}}$

3. $\displaystyle\int \frac{x\,dx}{\sqrt{x-2}}$

4. $\displaystyle\int \frac{x\,dx}{(2x+3)^{3/2}}$

5. $\displaystyle\int x\sqrt{2x-3}\,dx$

6. $\displaystyle\int x(7x+5)^{3/2}\,dx$

7. $\displaystyle\int \frac{\sqrt{9-4x}}{x^2}\,dx$

8. $\displaystyle\int \frac{dx}{x^2\sqrt{4x-9}}$

9. $\displaystyle\int x\sqrt{4x-x^2}\,dx$

10. $\displaystyle\int \frac{\sqrt{x-x^2}}{x}\,dx$

11. $\displaystyle\int \frac{dx}{x\sqrt{7+x^2}}$

12. $\displaystyle\int \frac{dx}{x\sqrt{7-x^2}}$

13. $\displaystyle\int \frac{\sqrt{4-x^2}}{x}\,dx$

14. $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$

15. $\displaystyle\int e^{2t}\cos 3t\,dt$

16. $\displaystyle\int e^{-3t}\sin 4t\,dt$

17. $\displaystyle\int x\cos^{-1}x\,dx$

18. $\displaystyle\int x\tan^{-1}x\,dx$

19. $\int x^2 \tan^{-1} x\, dx$

20. $\int \frac{\tan^{-1} x}{x^2}\, dx$

21. $\int \sin 3x \cos 2x\, dx$

22. $\int \sin 2x \cos 3x\, dx$

23. $\int 8 \sin 4t \sin \frac{t}{2}\, dt$

24. $\int \sin \frac{t}{3} \sin \frac{t}{6}\, dt$

25. $\int \cos \frac{\theta}{3} \cos \frac{\theta}{4}\, d\theta$

26. $\int \cos \frac{\theta}{2} \cos 7\theta\, d\theta$

Substitution and Integral Tables

In Exercises 27–40, use a substitution to change the integral into one you can find in the table. Then evaluate the integral.

27. $\int \frac{x^3 + x + 1}{(x^2 + 1)^2}\, dx$

28. $\int \frac{x^2 + 6x}{(x^2 + 3)^2}\, dx$

29. $\int \sin^{-1} \sqrt{x}\, dx$

30. $\int \frac{\cos^{-1} \sqrt{x}}{\sqrt{x}}\, dx$

31. $\int \frac{\sqrt{x}}{\sqrt{1 - x}}\, dx$

32. $\int \frac{\sqrt{2 - x}}{\sqrt{x}}\, dx$

33. $\int \cot t \sqrt{1 - \sin^2 t}\, dt, \quad 0 < t < \pi/2$

34. $\int \frac{dt}{\tan t \sqrt{4 - \sin^2 t}}$

35. $\int \frac{dy}{y\sqrt{3 + (\ln y)^2}}$

36. $\int \tan^{-1} \sqrt{y}\, dy$

37. $\int \frac{1}{\sqrt{x^2 + 2x + 5}}\, dx$

(*Hint*: Complete the square.)

38. $\int \frac{x^2}{\sqrt{x^2 - 4x + 5}}\, dx$

39. $\int \sqrt{5 - 4x - x^2}\, dx$

40. $\int x^2 \sqrt{2x - x^2}\, dx$

Using Reduction Formulas

Use reduction formulas to evaluate the integrals in Exercises 41–50.

41. $\int \sin^5 2x\, dx$

42. $\int 8 \cos^4 2\pi t\, dt$

43. $\int \sin^2 2\theta \cos^3 2\theta\, d\theta$

44. $\int 2 \sin^2 t \sec^4 t\, dt$

45. $\int 4 \tan^3 2x\, dx$

46. $\int 8 \cot^4 t\, dt$

47. $\int 2 \sec^3 \pi x\, dx$

48. $\int 3 \sec^4 3x\, dx$

49. $\int \csc^5 x\, dx$

50. $\int 16x^3(\ln x)^2\, dx$

Evaluate the integrals in Exercises 51–56 by making a substitution (possibly trigonometric) and then applying a reduction formula.

51. $\int e^t \sec^3 (e^t - 1)\, dt$

52. $\int \frac{\csc^3 \sqrt{\theta}}{\sqrt{\theta}}\, d\theta$

53. $\int_0^1 2\sqrt{x^2 + 1}\, dx$

54. $\int_0^{\sqrt{3}/2} \frac{dy}{(1 - y^2)^{5/2}}$

55. $\int_1^2 \frac{(r^2 - 1)^{3/2}}{r}\, dr$

56. $\int_0^{1/\sqrt{3}} \frac{dt}{(t^2 + 1)^{7/2}}$

Applications

57. **Surface area** Find the area of the surface generated by revolving the curve $y = \sqrt{x^2 + 2}$, $0 \le x \le \sqrt{2}$, about the x-axis.

58. **Arc length** Find the length of the curve $y = x^2$, $0 \le x \le \sqrt{3}/2$.

59. **Centroid** Find the centroid of the region cut from the first quadrant by the curve $y = 1/\sqrt{x + 1}$ and the line $x = 3$.

60. **Moment about *y*-axis** A thin plate of constant density $\delta = 1$ occupies the region enclosed by the curve $y = 36/(2x + 3)$ and the line $x = 3$ in the first quadrant. Find the moment of the plate about the y-axis.

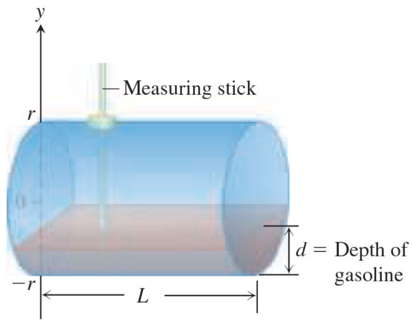
T 61. Use the integral table and a calculator to find to two decimal places the area of the surface generated by revolving the curve $y = x^2$, $-1 \le x \le 1$, about the x-axis.

62. **Volume** The head of your firm's accounting department has asked you to find a formula she can use in a computer program to calculate the year-end inventory of gasoline in the company's tanks. A typical tank is shaped like a right circular cylinder of radius r and length L, mounted horizontally, as shown in the accompanying figure. The data come to the accounting office as depth measurements taken with a vertical measuring stick marked in centimeters.

a. Show, in the notation of the figure, that the volume of gasoline that fills the tank to a depth d is

$$V = 2L \int_{-r}^{-r+d} \sqrt{r^2 - y^2}\, dy.$$

b. Evaluate the integral.



63. What is the largest value

$$\int_a^b \sqrt{x - x^2}\, dx$$

can have for any a and b? Give reasons for your answer.

64. What is the largest value

$$\int_a^b x\sqrt{2x - x^2}\,dx$$

can have for any a and b? Give reasons for your answer.

COMPUTER EXPLORATIONS

In Exercises 65 and 66, use a CAS to perform the integrations.

65. Evaluate the integrals

a. $\int x \ln x\,dx$ **b.** $\int x^2 \ln x\,dx$ **c.** $\int x^3 \ln x\,dx.$

d. What pattern do you see? Predict the formula for $\int x^4 \ln x\,dx$ and then see if you are correct by evaluating it with a CAS.

e. What is the formula for $\int x^n \ln x\,dx$, $n \ge 1$? Check your answer using a CAS.

66. Evaluate the integrals

a. $\int \frac{\ln x}{x^2}\,dx$ **b.** $\int \frac{\ln x}{x^3}\,dx$ **c.** $\int \frac{\ln x}{x^4}\,dx.$

d. What pattern do you see? Predict the formula for

$$\int \frac{\ln x}{x^5}\,dx$$

and then see if you are correct by evaluating it with a CAS.

e. What is the formula for

$$\int \frac{\ln x}{x^n}\,dx, \quad n \ge 2?$$

Check your answer using a CAS.

67. a. Use a CAS to evaluate

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx$$

where n is an arbitrary positive integer. Does your CAS find the result?

b. In succession, find the integral when $n = 1, 2, 3, 5,$ and 7. Comment on the complexity of the results.

c. Now substitute $x = (\pi/2) - u$ and add the new and old integrals. What is the value of

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx?$$

This exercise illustrates how a little mathematical ingenuity solves a problem not immediately amenable to solution by a CAS.

8.7 Numerical Integration

The antiderivatives of some functions, like $\sin(x^2)$, $1/\ln x$, and $\sqrt{1 + x^4}$, have no elementary formulas. When we cannot find a workable antiderivative for a function f that we have to integrate, we can partition the interval of integration, replace f by a closely fitting polynomial on each subinterval, integrate the polynomials, and add the results to approximate the definite integral of f. This procedure is an example of numerical integration. In this section we study two such methods, the *Trapezoidal Rule* and *Simpson's Rule*. A key goal in our analysis is to control the possible error that is introduced when computing an approximation to an integral.

Trapezoidal Approximations

The Trapezoidal Rule for the value of a definite integral is based on approximating the region between a curve and the x-axis with trapezoids instead of rectangles, as in Figure 8.7. It is not necessary for the subdivision points $x_0, x_1, x_2, \ldots, x_n$ in the figure to be evenly spaced, but the resulting formula is simpler if they are. We therefore assume that the length of each subinterval is

$$\Delta x = \frac{b - a}{n}.$$

The length $\Delta x = (b - a)/n$ is called the **step size** or **mesh size**. The area of the trapezoid that lies above the ith subinterval is

$$\Delta x\left(\frac{y_{i-1} + y_i}{2}\right) = \frac{\Delta x}{2}(y_{i-1} + y_i),$$

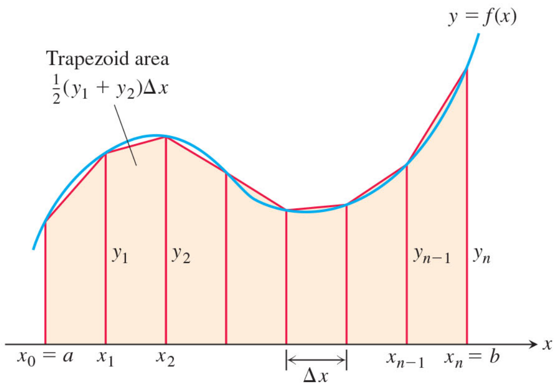


FIGURE 8.7 The Trapezoidal Rule approximates short stretches of the curve $y = f(x)$ with line segments. To approximate the integral of f from a to b, we add the areas of the trapezoids made by joining the ends of the segments to the x-axis.

where $y_{i-1} = f(x_{i-1})$ and $y_i = f(x_i)$. (See Figure 8.7.) The area below the curve $y = f(x)$ and above the x-axis is then approximated by adding the areas of all the trapezoids:

$$\begin{aligned} T &= \frac{1}{2}(y_0 + y_1)\Delta x + \frac{1}{2}(y_1 + y_2)\Delta x + \cdots \\ &\quad + \frac{1}{2}(y_{n-2} + y_{n-1})\Delta x + \frac{1}{2}(y_{n-1} + y_n)\Delta x \\ &= \Delta x\left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right) \\ &= \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n), \end{aligned}$$

where

$$y_0 = f(a), \qquad y_1 = f(x_1), \ldots, \qquad y_{n-1} = f(x_{n-1}), \qquad y_n = f(b).$$

The Trapezoidal Rule says: Use T to estimate the integral of f from a to b.

The Trapezoidal Rule

To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right).$$

The y's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b,$$

where $\Delta x = (b - a)/n$.

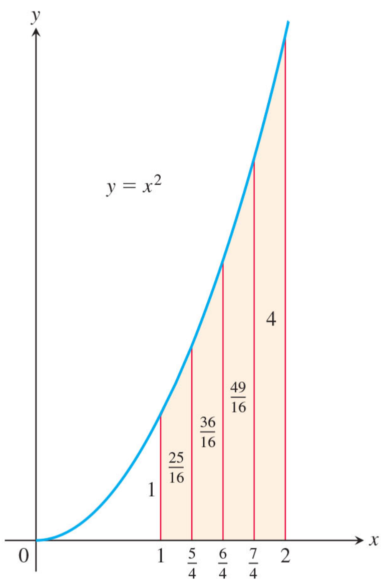


FIGURE 8.8 The trapezoidal approximation of the area under the graph of $y = x^2$ from $x = 1$ to $x = 2$ is a slight overestimate (Example 1).

EXAMPLE 1 Use the Trapezoidal Rule with $n = 4$ to estimate $\int_1^2 x^2\,dx$. Compare the estimate with the exact value.

Solution Partition $[1, 2]$ into four subintervals of equal length (Figure 8.8). Then evaluate $y = x^2$ at each partition point (Table 8.2).

TABLE 8.2

x	$y = x^2$
1	1
$\frac{5}{4}$	$\frac{25}{16}$
$\frac{6}{4}$	$\frac{36}{16}$
$\frac{7}{4}$	$\frac{49}{16}$
2	4

Using these y-values, $n = 4$, and $\Delta x = (2 - 1)/4 = 1/4$ in the Trapezoidal Rule, we have

$$\begin{aligned} T &= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right) \\ &= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right) \\ &= \frac{75}{32} = 2.34375. \end{aligned}$$

Since the parabola is concave *up*, the approximating segments lie above the curve, giving each trapezoid slightly more area than the corresponding strip under the curve. The exact value of the integral is

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\bigg]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

The T approximation overestimates the integral by about half a percent of its true value of $7/3$. The percentage error is $(2.34375 - 7/3)/(7/3) \approx 0.00446$, or 0.446%. ■

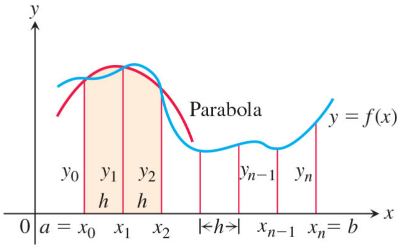


FIGURE 8.9 Simpson's Rule approximates short stretches of the curve with parabolas.

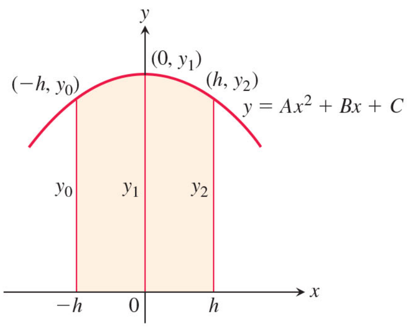


FIGURE 8.10 By integrating from $-h$ to h, we find the shaded area to be

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

Simpson's Rule: Approximations Using Parabolas

Another rule for approximating the definite integral of a continuous function results from using parabolas instead of the straight-line segments that produced trapezoids. As before, we partition the interval $[a, b]$ into n subintervals of equal length $h = \Delta x = (b - a)/n$, but this time we require that n be an *even* number. On each consecutive pair of intervals we approximate the curve $y = f(x) \geq 0$ by a parabola, as shown in Figure 8.9. A typical parabola passes through three consecutive points (x_{i-1}, y_{i-1}), (x_i, y_i), and (x_{i+1}, y_{i+1}) on the curve.

Let's calculate the shaded area beneath a parabola passing through three consecutive points. To simplify our calculations, we first take the case where $x_0 = -h$, $x_1 = 0$, and $x_2 = h$ (Figure 8.10), where $h = \Delta x = (b - a)/n$. The area under the parabola will be the same if we shift the y-axis to the left or right. The parabola has an equation of the form

$$y = Ax^2 + Bx + C,$$

so the area under it from $x = -h$ to $x = h$ is

$$\begin{aligned} A_p &= \int_{-h}^{h} (Ax^2 + Bx + C)\,dx \\ &= \left[\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\right]_{-h}^{h} \\ &= \frac{2Ah^3}{3} + 2Ch = \frac{h}{3}(2Ah^2 + 6C). \end{aligned}$$

Since the curve passes through the three points $(-h, y_0)$, $(0, y_1)$, and (h, y_2), we also have

$$y_0 = Ah^2 - Bh + C, \qquad y_1 = C, \qquad y_2 = Ah^2 + Bh + C,$$

from which we obtain

$$\begin{aligned} C &= y_1, \\ Ah^2 - Bh &= y_0 - y_1, \\ Ah^2 + Bh &= y_2 - y_1, \\ 2Ah^2 &= y_0 + y_2 - 2y_1. \end{aligned}$$

Hence, expressing the area A_p in terms of the ordinates y_0, y_1, and y_2, we have

$$A_p = \frac{h}{3}(2Ah^2 + 6C) = \frac{h}{3}((y_0 + y_2 - 2y_1) + 6y_1) = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Now shifting the parabola horizontally to its shaded position in Figure 8.9 does not change the area under it. Thus the area under the parabola through (x_0, y_0), (x_1, y_1), and (x_2, y_2) in Figure 8.9 is still

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

Similarly, the area under the parabola through the points (x_2, y_2), (x_3, y_3), and (x_4, y_4) is

$$\frac{h}{3}(y_2 + 4y_3 + y_4).$$

Computing the areas under all the parabolas and adding the results gives the approximation

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots \\ &\quad + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n) \\ &= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).\end{aligned}$$

HISTORICAL BIOGRAPHY
Thomas Simpson
(1720–1761)
www.goo.gl/idqvuc

The result is known as Simpson's Rule. The function need not be positive, as in our derivation, but the number n of subintervals must be even to apply the rule because each parabolic arc uses two subintervals.

Simpson's Rule

To approximate $\int_a^b f(x)\,dx$, use

$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right).$$

The y's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b.$$

The number n is even, and $\Delta x = (b - a)/n$.

Note the pattern of the coefficients in the above rule: 1, 4, 2, 4, 2, 4, 2, . . . , 4, 1.

EXAMPLE 2 Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4\,dx$.

Solution Partition $[0, 2]$ into four subintervals and evaluate $y = 5x^4$ at the partition points (Table 8.3). Then apply Simpson's Rule with $n = 4$ and $\Delta x = 1/2$:

$$\begin{aligned}S &= \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right) \\ &= \frac{1}{6}\left(0 + 4\left(\frac{5}{16}\right) + 2(5) + 4\left(\frac{405}{16}\right) + 80\right) \\ &= 32\frac{1}{12}.\end{aligned}$$

This estimate differs from the exact value (32) by only $1/12$, a percentage error of less than three-tenths of one percent, and this was with just four subintervals. ■

TABLE 8.3

x	$y = 5x^4$
0	0
$\frac{1}{2}$	$\frac{5}{16}$
1	5
$\frac{3}{2}$	$\frac{405}{16}$
2	80

Error Analysis

Whenever we use an approximation technique, the issue arises as to how accurate the approximation might be. The following theorem gives formulas for estimating the errors when using the Trapezoidal Rule and Simpson's Rule. The **error** is the difference between the approximation obtained by the rule and the actual value of the definite integral $\int_a^b f(x)\,dx$.

THEOREM 1—Error Estimates in the Trapezoidal and Simpson's Rules
If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then the error E_T in the trapezoidal approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_T| \le \frac{M(b-a)^3}{12n^2}. \qquad \text{Trapezoidal Rule}$$

If $f^{(4)}$ is continuous and M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then the error E_S in the Simpson's Rule approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_S| \le \frac{M(b-a)^5}{180n^4}. \qquad \text{Simpson's Rule}$$

To see why Theorem 1 is true in the case of the Trapezoidal Rule, we begin with a result from advanced calculus, which says that if f'' is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = T - \frac{b-a}{12} \cdot f''(c)(\Delta x)^2$$

for some number c between a and b. Thus, as Δx approaches zero, the error defined by

$$E_T = -\frac{b-a}{12} \cdot f''(c)(\Delta x)^2$$

approaches *zero* as the *square* of Δx.

The inequality

$$|E_T| \le \frac{b-a}{12} \max|f''(x)|(\Delta x)^2,$$

where max refers to the interval $[a, b]$, gives an upper bound for the magnitude of the error. In practice, we usually cannot find the exact value of $\max|f''(x)|$ and have to estimate an upper bound or "worst case" value for it instead. If M is any upper bound for the values of $|f''(x)|$ on $[a, b]$, so that $|f''(x)| \le M$ on $[a, b]$, then

$$|E_T| \le \frac{b-a}{12} M(\Delta x)^2.$$

If we substitute $(b-a)/n$ for Δx, we get

$$|E_T| \le \frac{M(b-a)^3}{12n^2}.$$

To estimate the error in Simpson's Rule, we start with a result from advanced calculus that says that if the fourth derivative $f^{(4)}$ is continuous, then

$$\int_a^b f(x)\,dx = S - \frac{b-a}{180} \cdot f^{(4)}(c)(\Delta x)^4$$

for some point c between a and b. Thus, as Δx approaches zero, the error,

$$E_S = -\frac{b-a}{180} \cdot f^{(4)}(c)(\Delta x)^4,$$

approaches zero as the *fourth power* of Δx. (This helps to explain why Simpson's Rule is likely to give better results than the Trapezoidal Rule.)

The inequality

$$|E_S| \le \frac{b-a}{180} \max|f^{(4)}(x)| \, (\Delta x)^4,$$

where max refers to the interval $[a, b]$, gives an upper bound for the magnitude of the error. As with $\max|f''|$ in the error formula for the Trapezoidal Rule, we usually cannot find the exact value of $\max|f^{(4)}(x)|$ and have to replace it with an upper bound. If M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then

$$|E_S| \le \frac{b-a}{180} M(\Delta x)^4.$$

Substituting $(b-a)/n$ for Δx in this last expression gives

$$|E_S| \le \frac{M(b-a)^5}{180n^4}.$$

EXAMPLE 3 Find an upper bound for the error in estimating $\int_0^2 5x^4\,dx$ using Simpson's Rule with $n = 4$ (Example 2).

Solution To estimate the error, we first find an upper bound M for the magnitude of the fourth derivative of $f(x) = 5x^4$ on the interval $0 \le x \le 2$. Since the fourth derivative has the constant value $f^{(4)}(x) = 120$, we take $M = 120$. With $b - a = 2$ and $n = 4$, the error estimate for Simpson's Rule gives

$$|E_S| \le \frac{M(b-a)^5}{180n^4} = \frac{120(2)^5}{180 \cdot 4^4} = \frac{1}{12}.$$

This estimate is consistent with the result of Example 2. ■

Theorem 1 can also be used to estimate the number of subintervals required when using the Trapezoidal or Simpson's Rule if we specify a certain tolerance for the error.

EXAMPLE 4 Estimate the minimum number of subintervals needed to approximate the integral in Example 3 using Simpson's Rule with an error of magnitude less than 10^{-4}.

Solution Using the inequality in Theorem 1, if we choose the number of subintervals n to satisfy

$$\frac{M(b-a)^5}{180n^4} < 10^{-4},$$

then the error E_S in Simpson's Rule satisfies $|E_S| < 10^{-4}$ as required.

From the solution in Example 3, we have $M = 120$ and $b - a = 2$, so we want n to satisfy

$$\frac{120(2)^5}{180n^4} < \frac{1}{10^4}$$

or, equivalently,

$$n^4 > \frac{64 \cdot 10^4}{3}.$$

It follows that

$$n > 10\left(\frac{64}{3}\right)^{1/4} \approx 21.5.$$

Since n must be even in Simpson's Rule, we estimate the minimum number of subintervals required for the error tolerance to be $n = 22$. ■

EXAMPLE 5 As we saw in Chapter 7, the value of ln 2 can be calculated from the integral

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx.$$

Table 8.4 shows T and S values for approximations of $\int_1^2 (1/x)\,dx$ using various values of n. Notice how Simpson's Rule dramatically improves over the Trapezoidal Rule.

TABLE 8.4 **Trapezoidal Rule approximations (T_n) and Simpson's Rule approximations (S_n) of ln 2 = $\int_1^2 (1/x)\,dx$**

n	T_n	|Error| less than . . .	S_n	|Error| less than . . .
10	0.6937714032	0.0006242227	0.6931502307	0.0000030502
20	0.6933033818	0.0001562013	0.6931473747	0.0000001942
30	0.6932166154	0.0000694349	0.6931472190	0.0000000385
40	0.6931862400	0.0000390595	0.6931471927	0.0000000122
50	0.6931721793	0.0000249988	0.6931471856	0.0000000050
100	0.6931534305	0.0000062500	0.6931471809	0.0000000004

In particular, notice that when we double the value of n (thereby halving the value of $h = \Delta x$), the T error is divided by 2 *squared*, whereas the S error is divided by 2 *to the fourth*.

This has a dramatic effect as $\Delta x = (2 - 1)/n$ gets very small. The Simpson approximation for $n = 50$ rounds accurately to seven places and for $n = 100$ agrees to nine decimal places (billionths)! ■

If $f(x)$ is a polynomial of degree less than four, then its fourth derivative is zero, and

$$E_S = -\frac{b-a}{180} f^{(4)}(c)(\Delta x)^4 = -\frac{b-a}{180}(0)(\Delta x)^4 = 0.$$

Thus, there will be no error in the Simpson approximation of any integral of f. In other words, if f is a constant, a linear function, or a quadratic or cubic polynomial, Simpson's Rule will give the value of any integral of f exactly, whatever the number of subdivisions. Similarly, if f is a constant or a linear function, then its second derivative is zero, and

$$E_T = -\frac{b-a}{12} f''(c)(\Delta x)^2 = -\frac{b-a}{12}(0)(\Delta x)^2 = 0.$$

The Trapezoidal Rule will therefore give the exact value of any integral of f. This is no surprise, for the trapezoids fit the graph perfectly.

Although decreasing the step size Δx reduces the error in the Simpson and Trapezoidal approximations in theory, it may fail to do so in practice. When Δx is very small, say $\Delta x = 10^{-8}$, computer or calculator round-off errors in the arithmetic required to evaluate S and T may accumulate to such an extent that the error formulas no longer describe what is going on. Shrinking Δx below a certain size can actually make things worse. You should consult a text on numerical analysis for more sophisticated methods if you are having problems with round-off error using the rules discussed in this section.

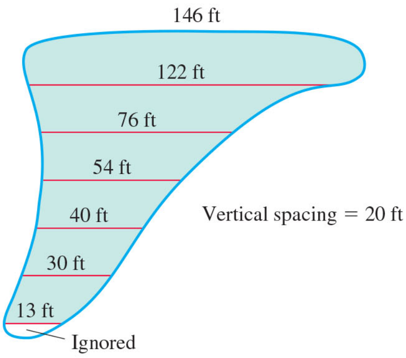


FIGURE 8.11 The dimensions of the swamp in Example 6.

EXAMPLE 6 A town wants to drain and fill a small polluted swamp (Figure 8.11). The swamp averages 5 ft deep. About how many cubic yards of dirt will it take to fill the area after the swamp is drained?

Solution To calculate the volume of the swamp, we estimate the surface area and multiply by 5. To estimate the area, we use Simpson's Rule with $\Delta x = 20$ ft and the y's equal to the distances measured across the swamp, as shown in Figure 8.11.

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6)$$

$$= \frac{20}{3}(146 + 488 + 152 + 216 + 80 + 120 + 13) = 8100$$

The volume is about $(8100)(5) = 40{,}500$ ft^3 or 1500 yd^3. ■

EXERCISES 8.7

Estimating Definite Integrals

The instructions for the integrals in Exercises 1–10 have two parts, one for the Trapezoidal Rule and one for Simpson's Rule.

I. Using the Trapezoidal Rule

a. Estimate the integral with $n = 4$ steps and find an upper bound for $|E_T|$.

b. Evaluate the integral directly and find $|E_T|$.

c. Use the formula $(|E_T|/(\text{true value})) \times 100$ to express $|E_T|$ as a percentage of the integral's true value.

II. Using Simpson's Rule

a. Estimate the integral with $n = 4$ steps and find an upper bound for $|E_S|$.

b. Evaluate the integral directly and find $|E_S|$.

c. Use the formula $(|E_S|/(\text{true value})) \times 100$ to express $|E_S|$ as a percentage of the integral's true value.

1. $\int_1^2 x\,dx$

2. $\int_1^3 (2x - 1)\,dx$

3. $\int_{-1}^1 (x^2 + 1)\,dx$

4. $\int_{-2}^0 (x^2 - 1)\,dx$

5. $\int_0^2 (t^3 + t)\,dt$

6. $\int_{-1}^1 (t^3 + 1)\,dt$

7. $\int_1^2 \frac{1}{s^2}\,ds$

8. $\int_2^4 \frac{1}{(s-1)^2}\,ds$

9. $\int_0^\pi \sin t\,dt$

10. $\int_0^1 \sin \pi t\,dt$

Estimating the Number of Subintervals

In Exercises 11–22, estimate the minimum number of subintervals needed to approximate the integrals with an error of magnitude less than 10^{-4} by **(a)** the Trapezoidal Rule and **(b)** Simpson's Rule. (The integrals in Exercises 11–18 are the integrals from Exercises 1–8.)

11. $\int_1^2 x\,dx$

12. $\int_1^3 (2x - 1)\,dx$

13. $\int_{-1}^1 (x^2 + 1)\,dx$

14. $\int_{-2}^0 (x^2 - 1)\,dx$

15. $\int_0^2 (t^3 + t)\,dt$

16. $\int_{-1}^1 (t^3 + 1)\,dt$

17. $\int_1^2 \frac{1}{s^2}\,ds$

18. $\int_2^4 \frac{1}{(s-1)^2}\,ds$

19. $\int_0^3 \sqrt{x+1}\,dx$

20. $\int_0^3 \frac{1}{\sqrt{x+1}}\,dx$

21. $\int_0^2 \sin(x + 1)\,dx$

22. $\int_{-1}^1 \cos(x + \pi)\,dx$

Estimates with Numerical Data

23. Volume of water in a swimming pool A rectangular swimming pool is 30 ft wide and 50 ft long. The accompanying table shows the depth $h(x)$ of the water at 5-ft intervals from one end of the pool to the other. Estimate the volume of water in the pool using the Trapezoidal Rule with $n = 10$ applied to the integral

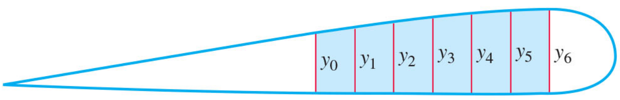
$$V = \int_0^{50} 30 \cdot h(x)\,dx.$$

Position (ft) x	Depth (ft) $h(x)$	Position (ft) x	Depth (ft) $h(x)$
0	6.0	30	11.5
5	8.2	35	11.9
10	9.1	40	12.3
15	9.9	45	12.7
20	10.5	50	13.0
25	11.0		

24. **Distance traveled** The accompanying table shows time-to-speed data for a sports car accelerating from rest to 130 mph. How far had the car traveled by the time it reached this speed? (Use trapezoids to estimate the area under the velocity curve, but be careful: The time intervals vary in length.)

Speed change	**Time (sec)**
Zero to 30 mph	2.2
40 mph	3.2
50 mph	4.5
60 mph	5.9
70 mph	7.8
80 mph	10.2
90 mph	12.7
100 mph	16.0
110 mph	20.6
120 mph	26.2
130 mph	37.1

25. **Wing design** The design of a new airplane requires a gasoline tank of constant cross-sectional area in each wing. A scale drawing of a cross-section is shown here. The tank must hold 5000 lb of gasoline, which has a density of 42 lb/ft^3. Estimate the length of the tank by Simpson's Rule.



$y_0 = 1.5$ ft, $y_1 = 1.6$ ft, $y_2 = 1.8$ ft, $y_3 = 1.9$ ft,
$y_4 = 2.0$ ft, $y_5 = y_6 = 2.1$ ft Horizontal spacing = 1 ft

26. **Oil consumption on Pathfinder Island** A diesel generator runs continuously, consuming oil at a gradually increasing rate until it must be temporarily shut down to have the filters replaced. Use the Trapezoidal Rule to estimate the amount of oil consumed by the generator during that week.

Day	**Oil consumption rate (liters / h)**
Sun	0.019
Mon	0.020
Tue	0.021
Wed	0.023
Thu	0.025
Fri	0.028
Sat	0.031
Sun	0.035

Theory and Examples

27. **Usable values of the sine-integral function** *The sine-integral function,*

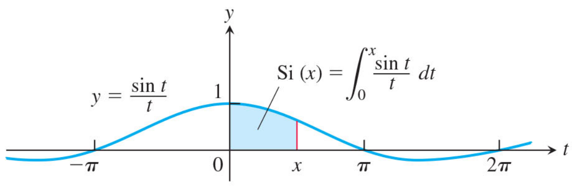
$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt, \qquad \text{"Sine integral of } x\text{"}$$

is one of the many functions in engineering whose formulas cannot be simplified. There is no elementary formula for the antiderivative of $(\sin t)/t$. The values of Si(x), however, are readily estimated by numerical integration.

Although the notation does not show it explicitly, the function being integrated is

$$f(t) = \begin{cases} \dfrac{\sin t}{t}, & t \neq 0 \\ 1, & t = 0, \end{cases}$$

the continuous extension of $(\sin t)/t$ to the interval $[0, x]$. The function has derivatives of all orders at every point of its domain. Its graph is smooth, and you can expect good results from Simpson's Rule.



a. Use the fact that $|f^{(4)}| \leq 1$ on $[0, \pi/2]$ to give an upper bound for the error that will occur if

$$\mathrm{Si}\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{\sin t}{t}\,dt$$

is estimated by Simpson's Rule with $n = 4$.

b. Estimate Si$(\pi/2)$ by Simpson's Rule with $n = 4$.

c. Express the error bound you found in part (a) as a percentage of the value you found in part (b).

28. **The error function** *The error function,*

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt,$$

important in probability and in the theories of heat flow and signal transmission, must be evaluated numerically because there is no elementary expression for the antiderivative of e^{-t^2}.

a. Use Simpson's Rule with $n = 10$ to estimate erf (1).

b. In $[0, 1]$,

$$\left|\frac{d^4}{dt^4}\left(e^{-t^2}\right)\right| \leq 12.$$

Give an upper bound for the magnitude of the error of the estimate in part (a).

29. Prove that the sum T in the Trapezoidal Rule for $\int_a^b f(x)\,dx$ is a Riemann sum for f continuous on $[a, b]$. (*Hint:* Use the Intermediate Value Theorem to show the existence of c_k in the subinterval $[x_{k-1}, x_k]$ satisfying $f(c_k) = (f(x_{k-1}) + f(x_k))/2$.)

30. Prove that the sum S in Simpson's Rule for $\int_a^b f(x)\,dx$ is a Riemann sum for f continuous on $[a, b]$. (See Exercise 29.)

T 31. **Elliptic integrals** The length of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

turns out to be

$$\text{Length} = 4a\int_0^{\pi/2} \sqrt{1 - e^2\cos^2 t}\, dt,$$

where $e = \sqrt{a^2 - b^2}/a$ is the ellipse's eccentricity. The integral in this formula, called an *elliptic integral*, is nonelementary except when $e = 0$ or 1.

a. Use the Trapezoidal Rule with $n = 10$ to estimate the length of the ellipse when $a = 1$ and $e = 1/2$.

b. Use the fact that the absolute value of the second derivative of $f(t) = \sqrt{1 - e^2\cos^2 t}$ is less than 1 to find an upper bound for the error in the estimate you obtained in part (a).

Applications

T **32.** The length of one arch of the curve $y = \sin x$ is given by

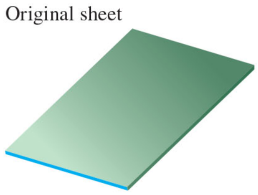
$$L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx.$$

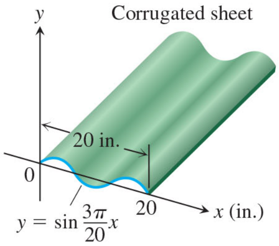
Estimate L by Simpson's Rule with $n = 8$.

T **33.** Your metal fabrication company is bidding for a contract to make sheets of corrugated iron roofing like the one shown here. The cross-sections of the corrugated sheets are to conform to the curve

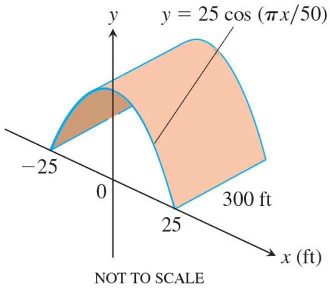
$$y = \sin\frac{3\pi}{20}x, \quad 0 \le x \le 20 \text{ in.}$$

If the roofing is to be stamped from flat sheets by a process that does not stretch the material, how wide should the original material be? To find out, use numerical integration to approximate the length of the sine curve to two decimal places.





T **34.** Your engineering firm is bidding for the contract to construct the tunnel shown here. The tunnel is 300 ft long and 50 ft wide at the base. The cross-section is shaped like one arch of the curve $y = 25\cos(\pi x/50)$. Upon completion, the tunnel's inside surface (excluding the roadway) will be treated with a waterproof sealer that costs \$2.35 per square foot to apply. How much will it cost to apply the sealer? (*Hint:* Use numerical integration to find the length of the cosine curve.)



Find, to two decimal places, the areas of the surfaces generated by revolving the curves in Exercises 35 and 36 about the x-axis.

35. $y = \sin x, \quad 0 \le x \le \pi$ **36.** $y = x^2/4, \quad 0 \le x \le 2$

37. Use numerical integration to estimate the value of

$$\sin^{-1} 0.6 = \int_0^{0.6} \frac{dx}{\sqrt{1 - x^2}}.$$

For reference, $\sin^{-1} 0.6 = 0.64350$ to five decimal places.

38. Use numerical integration to estimate the value of

$$\pi = 4\int_0^1 \frac{1}{1 + x^2}\, dx.$$

39. Drug assimilation An average adult under age 60 years assimilates a 12-hr cold medicine into his or her system at a rate modeled by

$$\frac{dy}{dt} = 6 - \ln(2t^2 - 3t + 3),$$

where y is measured in milligrams and t is the time in hours since the medication was taken. What amount of medicine is absorbed into a person's system over a 12-hr period?

40. Effects of an antihistamine The concentration of an antihistamine in the bloodstream of a healthy adult is modeled by

$$C = 12.5 - 4\ln(t^2 - 3t + 4),$$

where C is measured in grams per liter and t is the time in hours since the medication was taken. What is the average level of concentration in the bloodstream over a 6-hr period?

8.8 Improper Integrals

Up to now, we have required definite integrals to satisfy two properties. First, the domain of integration $[a, b]$ must be finite. Second, the range of the integrand must be finite on this domain. In practice, we may encounter problems that fail to meet one or both of these conditions. The integral for the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ is an example for which the domain is infinite (Figure 8.12a). The integral for the area under the curve of $y = 1/\sqrt{x}$ between $x = 0$ and $x = 1$ is an example for which the range of the integrand is infinite (Figure 8.12b). In either case, the integrals are said to be *improper* and are calculated as limits. We will see in Section 8.9 that improper integrals play an important role in probability. They are also useful when investigating the convergence of certain infinite series in Chapter 10.

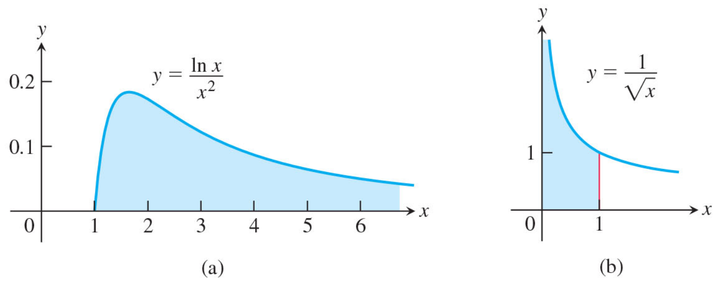


FIGURE 8.12 Are the areas under these infinite curves finite? We will see that the answer is yes for both curves.

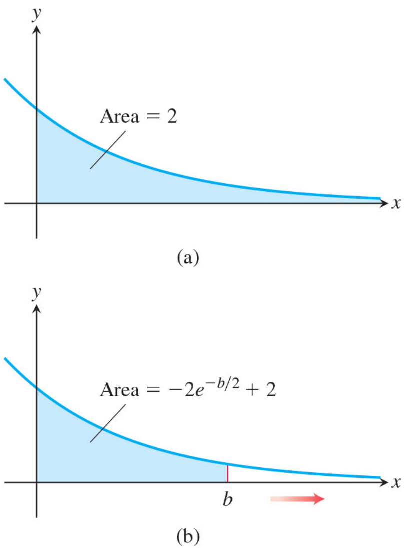


FIGURE 8.13 (a) The area in the first quadrant under the curve $y = e^{-x/2}$. (b) The area is an improper integral of the first type.

Infinite Limits of Integration

Consider the infinite region (unbounded on the right) that lies under the curve $y = e^{-x/2}$ in the first quadrant (Figure 8.13a). You might think this region has infinite area, but we will see that the value is finite. We assign a value to the area in the following way. First find the area $A(b)$ of the portion of the region that is bounded on the right by $x = b$ (Figure 8.13b).

$$A(b) = \int_0^b e^{-x/2}\,dx = -2e^{-x/2}\Big]_0^b = -2e^{-b/2} + 2$$

Then find the limit of $A(b)$ as $b \to \infty$

$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty} \left(-2e^{-b/2} + 2\right) = 2.$$

The value we assign to the area under the curve from 0 to ∞ is

$$\int_0^\infty e^{-x/2}\,dx = \lim_{b\to\infty} \int_0^b e^{-x/2}\,dx = 2.$$

DEFINITION Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

* where c is any real number.

In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

หาค่า limit

หาค่าไม่ได้

The choice of c in Part 3 of the definition is unimportant. We can evaluate or determine the convergence or divergence of $\int_{-\infty}^{\infty} f(x)\,dx$ with any convenient choice.

Any of the integrals in the above definition can be interpreted as an area if $f \geq 0$ on the interval of integration. For instance, we interpreted the improper integral in Figure 8.13 as an area. In that case, the area has the finite value 2. If $f \geq 0$ and the improper integral diverges, we say the area under the curve is **infinite**.

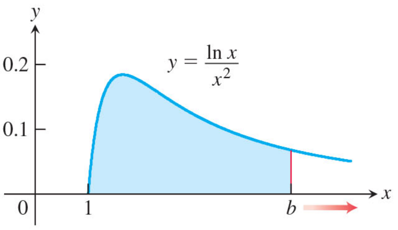


FIGURE 8.14 The area under this curve is an improper integral (Example 1).

EXAMPLE 1 Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is its value?

Solution We find the area under the curve from $x = 1$ to $x = b$ and examine the limit as $b \to \infty$. If the limit is finite, we take it to be the area under the curve (Figure 8.14). The area from 1 to b is

$$\int_1^b \frac{\ln x}{x^2}\,dx = \underbrace{\left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b}_{\text{แทนค่า}} - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$$

* Integration by parts with $u = \ln x$, $dv = dx/x^2$, $du = dx/x$, $v = -1/x$

อินทิเกรต

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b$$

$$= -\frac{\ln b}{b} - \frac{1}{b} + 1.$$

The limit of the area as $b \to \infty$ is

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx$$

take limit

$$= \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right]$$

b ค่าเยอะกว่า ∴ ln b → 0

$$= -\left[\lim_{b\to\infty} \frac{\ln b}{b}\right] - 0 + 1$$

$$= -\left[\lim_{b\to\infty} \frac{1/b}{1}\right] + 1 = 0 + 1 = 1. \qquad \text{l'Hôpital's Rule}$$

Thus, the improper integral converges and the area has finite value 1. ■

EXAMPLE 2 Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}.$$

จะไม่ Error
เพื่อง่ายต่อการคำนวณ

Solution According to the definition (Part 3), we can choose $c = 0$ and write

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \int_{-\infty}^{0} \frac{dx}{1 + x^2} + \int_{0}^{\infty} \frac{dx}{1 + x^2}.$$

take limit, แทนค่า

Next we evaluate each improper integral on the right side of the equation above.

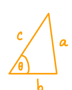
$$\int_{-\infty}^{0} \frac{dx}{1 + x^2} = \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1 + x^2}$$

$$= \lim_{a\to-\infty} \tan^{-1} x\Big]_a^0$$

$$= \lim_{a\to-\infty} (\overbrace{\tan^{-1} 0}^{\text{องศา } 0} - \underbrace{\tan^{-1} a}_{-90\text{ องศา}}) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

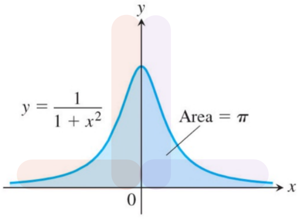
$-\infty$

* จะมีช่วง -90 ถึง 90 องศา



∴ $\tan\theta = \frac{a}{b}$

HISTORICAL BIOGRAPHY
Lejeune Dirichlet
(1805–1859)
www.goo.gl/QGwXLL



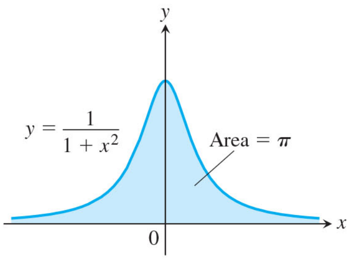


FIGURE 8.15 The area under this curve is finite (Example 2).

$$\int_0^{\infty} \frac{dx}{1+x^2} = \lim_{b \to \infty} \int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{b \to \infty} \tan^{-1} x \Big]_0^b$$

$$= \lim_{b \to \infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Since $1/(1+x^2) > 0$, the improper integral can be interpreted as the (finite) area beneath the curve and above the x-axis (Figure 8.15). ■

The Integral $\int_1^{\infty} \frac{dx}{x^p}$

The function $y = 1/x$ is the boundary between the convergent and divergent improper integrals with integrands of the form $y = 1/x^p$. As the next example shows, the improper integral converges if $p > 1$ and diverges if $p \leq 1$.

EXAMPLE 3 For what values of p does the integral $\int_1^{\infty} dx/x^p$ converge? When the integral does converge, what is its value?

Solution If $p \neq 1$,

$$\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\Big]_1^b = \frac{1}{1-p}(b^{-p+1} - 1) = \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right).$$

Thus,

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{b \to \infty} \int_1^b \frac{dx}{x^p}$$

$$= \lim_{b \to \infty} \left[\boxed{\frac{1}{1-p}}\boxed{\left(\frac{1}{b^{p-1}} - 1\right)}\right] = \begin{cases} \dfrac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}$$

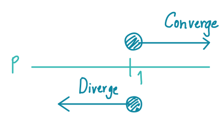
because

$$\lim_{b \to \infty} \frac{1}{b^{\boxed{p-1}}} = \begin{cases} 0, & p > 1 \\ \infty, & p < 1. \end{cases}$$

รู้ได้โดยการแทนค่า *ค่าติดลบ

Therefore, the integral converges to the value $1/(p-1)$ if $p > 1$ and it diverges if $p < 1$.

If $p = 1$, the integral also diverges:



$$\int_1^{\infty} \frac{dx}{x^p} = \int_1^{\infty} \frac{dx}{x}$$

$$= \lim_{b \to \infty} \int_1^b \frac{dx}{x}$$

$$= \lim_{b \to \infty} \ln x \Big]_1^b$$

$$= \lim_{b \to \infty} (\ln b - \ln 1) = \infty.$$ ■

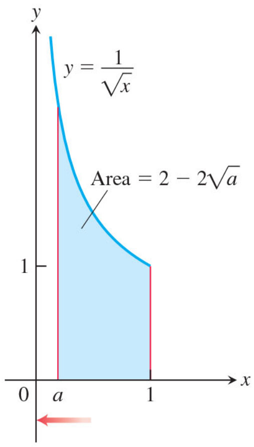


FIGURE 8.16 The area under this curve is an example of an improper integral of the second kind.

Integrands with Vertical Asymptotes

Another type of improper integral arises when the integrand has a vertical asymptote—an infinite discontinuity—at a limit of integration or at some point between the limits of integration. If the integrand f is positive over the interval of integration, we can again interpret the improper integral as the area under the graph of f and above the x-axis between the limits of integration.

Consider the region in the first quadrant that lies under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$ (Figure 8.12b). First we find the area of the portion from a to 1 (Figure 8.16):

$$\int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_a^1 = 2 - 2\sqrt{a}.$$

Then we find the limit of this area as $a \to 0^+$:

$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0}\left(2 - 2\sqrt{a}\right) = 2.$$

Therefore the area under the curve from 0 to 1 is finite and is defined to be

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = 2.$$

DEFINITION Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit exists and is finite, we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

In Part 3 of the definition, the integral on the left side of the equation converges if *both* integrals on the right side converge; otherwise it diverges.

EXAMPLE 4 Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx.$$

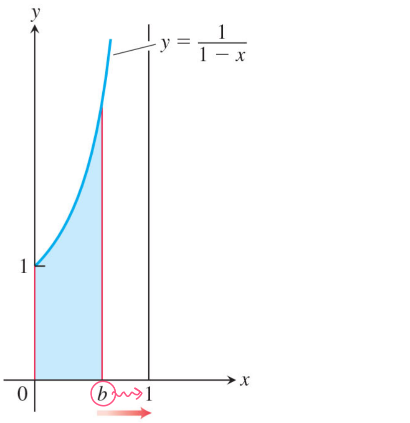


FIGURE 8.17 The area beneath the curve and above the x-axis for $[0, 1)$ is not a real number (Example 4).

Solution The integrand $f(x) = 1/(1 - x)$ is continuous on $[0, 1)$ but is discontinuous at $x = 1$ and becomes infinite as $x \to 1^-$ (Figure 8.17). We evaluate the integral as

ไม่ต่อเนื่องที่ x = 1

$$\lim_{b \to 1^-} \int_0^b \frac{1}{1 - x}\, dx = \lim_{b \to 1^-} \Big[-\ln |1 - x| \Big]_0^b$$

$$= \lim_{b \to 1^-} \Big[\boxed{-\ln (1 - b)} + 0 \Big] = \infty.$$

ได้ ∞

มีค่าเกือบๆ 1

-ln $\frac{1}{b}$

1 - 0.9999 = 0.0001

จะได้ -∞

The limit is infinite, so the integral diverges. ■

EXAMPLE 5 Evaluate

$$\int_0^3 \frac{dx}{(x - 1)^{2/3}}.$$

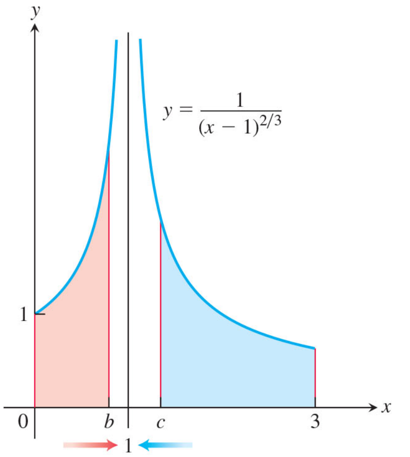


FIGURE 8.18 Example 5 shows that the area under the curve exists (so it is a real number).

Solution The integrand has a vertical asymptote at $x = 1$ and is continuous on $[0, 1)$ and $(1, 3]$ (Figure 8.18). Thus, by Part 3 of the definition above,

$$\int_0^3 \frac{dx}{(x - 1)^{2/3}} = \int_0^1 \frac{dx}{(x - 1)^{2/3}} + \int_1^3 \frac{dx}{(x - 1)^{2/3}}.$$

* ถ้า ∫ ไม่ได้ ต้องใช้วิธี take limit

Next, we evaluate each improper integral on the right-hand side of this equation.

$$\int_0^1 \frac{dx}{(x - 1)^{2/3}} = \lim_{b \to 1^-} \int_0^b \frac{dx}{(x - 1)^{2/3}}$$

$$= \lim_{b \to 1^-} 3(x - 1)^{1/3} \Big]_0^b$$

$$= \lim_{b \to 1^-} \left[3(b - 1)^{1/3} + 3 \right] = 3$$

ค่าใกล้ 1 ทางซ้าย

$$\int_1^3 \frac{dx}{(x - 1)^{2/3}} = \lim_{c \to 1^+} \int_c^3 \frac{dx}{(x - 1)^{2/3}}$$

$$= \lim_{c \to 1^+} 3(x - 1)^{1/3} \Big]_c^3$$

$$= \lim_{c \to 1^+} \left[3(3 - 1)^{1/3} - 3(c - 1)^{1/3} \right] = 3\sqrt[3]{2}$$

→ 0.0001 → $\frac{1}{\infty}$ = 0

We conclude that

$$\int_0^3 \frac{dx}{(x - 1)^{2/3}} = 3 + 3\sqrt[3]{2}.$$ ■

Improper Integrals with a CAS

Computer algebra systems can evaluate many convergent improper integrals. To evaluate the integral

$$\int_2^\infty \frac{x + 3}{(x - 1)(x^2 + 1)}\, dx$$

(which converges) using Maple, enter

$$> f := (x + 3)/((x - 1) * (x\text{\textasciicircum}2 + 1));$$

Then use the integration command

$$> \text{int}(f, x = 2..\text{infinity});$$

Maple returns the answer

$$-\frac{1}{2}\pi + \ln (5) + \arctan (2).$$

To obtain a numerical result, use the evaluation command **evalf** and specify the number of digits as follows:

$$> \text{evalf}(\%, 6);$$

The symbol % instructs the computer to evaluate the last expression on the screen, in this case $(-1/2)\pi + \ln(5) + \arctan(2)$. Maple returns 1.14579.

Using Mathematica, entering

$$\textit{In[1]:=} \text{ Integrate}[(x + 3)/((x - 1)(x\text{^}2 + 1)),\ \{x, 2, \text{Infinity}\}]$$

returns

$$\textit{Out[1]=} -\frac{\pi}{2} + \text{ArcTan}[2] + \text{Log}[5].$$

To obtain a numerical result with six digits, use the command "N[%, 6]"; it also yields 1.14579.

Tests for Convergence and Divergence

When we cannot evaluate an improper integral directly, we try to determine whether it converges or diverges. If the integral diverges, that's the end of the story. If it converges, we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

EXAMPLE 6 Does the integral $\int_1^\infty e^{-x^2}\,dx$ converge?

Solution By definition,

$$\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx.$$

We cannot evaluate this integral directly because it is nonelementary. But we *can* show that its limit as $b\to\infty$ is finite. We know that $\int_1^b e^{-x^2}\,dx$ is an increasing function of b because the area under the curve increases as b increases. Therefore either it becomes infinite as $b\to\infty$ or it has a finite limit as $b\to\infty$. For our function it does not become infinite: For every value of $x \ge 1$, we have $e^{-x^2} \le e^{-x}$ (Figure 8.19) so that

$$\int_1^b e^{-x^2}\,dx \le \int_1^b e^{-x}\,dx = -e^{-b} + e^{-1} < e^{-1} \approx 0.36788.$$

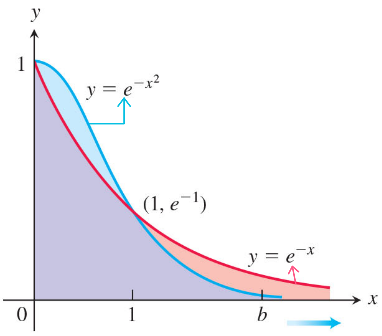


FIGURE 8.19 The graph of e^{-x^2} lies below the graph of e^{-x} for $x > 1$ (Example 6).

Hence,

$$\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx$$ Converges

converges to some finite value. We do not know exactly what the value is except that it is something positive and less than 0.37. Here we are relying on the completeness property of the real numbers, discussed in Appendix 6. ■

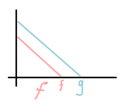
The comparison of e^{-x^2} and e^{-x} in Example 6 is a special case of the following test.

THEOREM 2—Direct Comparison Test

Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

ฟังก์ชันต่อเนื่อง

1. If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ also converges.



2. If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ also diverges.

Proof The reasoning behind the argument establishing Theorem 2 is similar to that in Example 6. If $0 \le f(x) \le g(x)$ for $x \ge a$, then from Rule 7 in Theorem 2 of Section 5.3 we have

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx, \qquad b > a.$$

From this it can be argued, as in Example 6, that

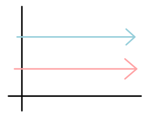
$$\int_a^\infty f(x)\,dx \quad \text{converges if} \quad \int_a^\infty g(x)\,dx \quad \text{converges.}$$

Turning this around to its contrapositive form, this says that

$$\int_a^\infty g(x)\,dx \quad \text{diverges if} \quad \int_a^\infty f(x)\,dx \quad \text{diverges.} \qquad \blacksquare$$

Although the theorem is stated for Type I improper integrals, a similar result is true for integrals of Type II as well.

EXAMPLE 7 These examples illustrate how we use Theorem 2.



HISTORICAL BIOGRAPHY
Karl Weierstrass
(1815–1897)
www.goo.gl/3RH2r0

(a) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges because

$$0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x^2}\,dx \quad \text{converges.}$$ Example 3

(b) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}}\,dx$ diverges because

ค่ามากกว่า , ต้องลองแทนค่า

$$\frac{1}{\sqrt{x^2 - 0.1}} \ge \frac{1}{x} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x}\,dx \quad \text{diverges.}$$ Example 3

(c) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\,dx$ converges because

$$0 \le \frac{\cos x}{\sqrt{x}} \le \frac{1}{\sqrt{x}} \quad \text{on} \quad \left[0, \frac{\pi}{2}\right],$$ $0 \le \cos x \le 1$ on $\left[0, \frac{\pi}{2}\right]$

and

$$\begin{aligned}
\int_0^{\pi/2} \frac{dx}{\sqrt{x}} &= \lim_{a \to 0^+} \int_a^{\pi/2} \frac{dx}{\sqrt{x}} \\
&= \lim_{a \to 0^+} \sqrt{4x}\,\Big]_a^{\pi/2} \qquad 2\sqrt{x} = \sqrt{4x} \\
&= \lim_{a \to 0^+} \left(\sqrt{2\pi} - \sqrt{4a}\right) = \sqrt{2\pi} \qquad \text{converges.} \qquad \blacksquare
\end{aligned}$$

THEOREM 3—Limit Comparison Test
If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

* ตัวเปรียบเทียบหารกัน ไปหา $\lim_{x\to\infty}$ ได้ค่าคงที่ เป็น +

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

either *both converge* or *both diverge*.

We omit the proof of Theorem 3, which is similar to that of Theorem 2.

Although the improper integrals of two functions from a to ∞ may both converge, this does not mean that their integrals necessarily have the same value, as the next example shows.

EXAMPLE 8 Show that

$$\int_1^\infty \frac{dx}{1+x^2}$$

converges by comparison with $\int_1^\infty (1/x^2)\,dx$. Find and compare the two integral values.

Solution The functions $f(x) = 1/x^2$ and $g(x) = 1/(1+x^2)$ are positive and continuous on $[1, \infty)$. Also,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1/x^2}{1/(1+x^2)} = \lim_{x\to\infty} \frac{1+x^2}{x^2}$$

$$= \lim_{x\to\infty} \left(\frac{1}{x^2} + 1\right) = 0 + 1 = 1,$$

which is a positive finite limit (Figure 8.20). Therefore, $\int_1^\infty \frac{dx}{1+x^2}$ converges because $\int_1^\infty \frac{dx}{x^2}$ converges.

The integrals converge to different values, however:

$$\int_1^\infty \frac{dx}{x^2} = \frac{1}{2-1} = 1 \qquad \text{Example 3}$$

and

$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{1+x^2} = \lim_{b\to\infty} \left[\tan^{-1} b - \tan^{-1} 1\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$ ■

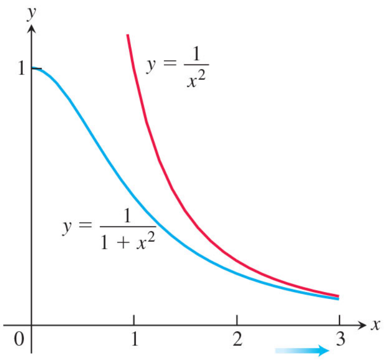


FIGURE 8.20 The functions in Example 8.

TABLE 8.5

b	$\int_1^b \frac{1-e^{-x}}{x}\,dx$
2	0.5226637569
5	1.3912002736
10	2.0832053156
100	4.3857862516
1000	6.6883713446
10000	8.9909564376
100000	11.2935415306

EXAMPLE 9 Investigate the convergence of $\int_1^\infty \overbrace{\boxed{\frac{1-e^{-x}}{x}}}^{f(x)}\,dx.$

$\frac{1}{x}$ ...... Diverges

$\int_1^\infty \frac{1}{x}\,dx = \ln x \Big|_1^\infty$

Solution The integrand suggests a comparison of $f(x) = (1 - e^{-x})/x$ with $g(x) = 1/x$. However, we cannot use the Direct Comparison Test because $f(x) \le g(x)$ and the integral of $g(x)$ *diverges*. On the other hand, using the Limit Comparison Test we find that

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$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(\frac{1-e^{-x}}{\cancel{x}}\right)\left(\frac{\cancel{x}}{1}\right) = \lim_{x\to\infty} (1 - \underbrace{e^{-x}}_{\to\infty}) = 1,$$

which is a positive finite limit. Therefore, $\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ diverges because $\int_1^\infty \frac{dx}{x}$ diverges. Approximations to the improper integral are given in Table 8.5. Note that the values do not appear to approach any fixed limiting value as $b \to \infty$. ■

EXERCISES 8.8

Evaluating Improper Integrals

The integrals in Exercises 1–34 converge. Evaluate the integrals without using tables.

1. $\int_0^\infty \frac{dx}{x^2+1}$

2. $\int_1^\infty \frac{dx}{x^{1.001}}$

3. $\int_0^1 \frac{dx}{\sqrt{x}}$

4. $\int_0^4 \frac{dx}{\sqrt{4-x}}$

5. $\int_{-1}^1 \frac{dx}{x^{2/3}}$

6. $\int_{-8}^1 \frac{dx}{x^{1/3}}$

7. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

8. $\int_0^1 \frac{dr}{r^{0.999}}$

9. $\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1}$

10. $\int_{-\infty}^{2} \frac{2\,dx}{x^2+4}$

11. $\int_2^\infty \frac{2}{v^2-v}\,dv$

12. $\int_2^\infty \frac{2\,dt}{t^2-1}$

13. $\int_{-\infty}^\infty \frac{2x\,dx}{(x^2+1)^2}$

14. $\int_{-\infty}^\infty \frac{x\,dx}{(x^2+4)^{3/2}}$

15. $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$

16. $\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds$

17. $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$

18. $\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\,dx$

19. $\int_0^\infty \frac{dv}{(1+v^2)(1+\tan^{-1} v)}$

20. $\int_0^\infty \frac{16\tan^{-1} x}{1+x^2}\,dx$

21. $\int_{-\infty}^0 \theta e^\theta\,d\theta$

22. $\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta$

23. $\int_{-\infty}^0 e^{-|x|}\,dx$

24. $\int_{-\infty}^\infty 2xe^{-x^2}\,dx$

25. $\int_0^1 x\ln x\,dx$

26. $\int_0^1 (-\ln x)\,dx$

27. $\int_0^2 \frac{ds}{\sqrt{4-s^2}}$

28. $\int_0^1 \frac{4r\,dr}{\sqrt{1-r^4}}$

29. $\int_1^2 \frac{ds}{s\sqrt{s^2-1}}$

30. $\int_2^4 \frac{dt}{t\sqrt{t^2-4}}$

31. $\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$

32. $\int_0^2 \frac{dx}{\sqrt{|x-1|}}$

33. $\int_{-1}^\infty \frac{d\theta}{\theta^2+5\theta+6}$

34. $\int_0^\infty \frac{dx}{(x+1)(x^2+1)}$

Testing for Convergence

In Exercises 35–68, use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence. If more than one method applies, use whatever method you prefer.

35. $\int_{1/2}^2 \frac{dx}{x\ln x}$

36. $\int_{-1}^1 \frac{d\theta}{\theta^2-2\theta}$

37. $\int_{1/2}^\infty \frac{dx}{x(\ln x)^3}$

38. $\int_0^\infty \frac{d\theta}{\theta^2-1}$

39. $\int_0^{\pi/2} \tan\theta\,d\theta$

40. $\int_0^{\pi/2} \cot\theta\,d\theta$

41. $\int_0^1 \frac{\ln x}{x^2}\,dx$

42. $\int_1^2 \frac{dx}{x\ln x}$

43. $\int_0^{\ln 2} x^{-2}e^{-1/x}\,dx$

44. $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

45. $\int_0^\pi \frac{dt}{\sqrt{t}+\sin t}$

46. $\int_0^1 \frac{dt}{t-\sin t}$ (*Hint:* $t \ge \sin t$ for $t \ge 0$)

47. $\int_0^2 \frac{dx}{1-x^2}$

48. $\int_0^2 \frac{dx}{1-x}$

49. $\int_{-1}^1 \ln|x|\,dx$

50. $\int_{-1}^1 -x\ln|x|\,dx$

51. $\int_1^\infty \frac{dx}{x^3+1}$

52. $\int_4^\infty \frac{dx}{\sqrt{x}-1}$

53. $\int_2^\infty \frac{dv}{\sqrt{v-1}}$

54. $\int_0^\infty \frac{d\theta}{1+e^\theta}$

55. $\int_0^\infty \frac{dx}{\sqrt{x^6+1}}$

56. $\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$

57. $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$

58. $\int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$

59. $\int_\pi^\infty \frac{2+\cos x}{x}\,dx$

60. $\int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$

61. $\int_4^\infty \frac{2\,dt}{t^{3/2}-1}$

62. $\int_2^\infty \frac{1}{\ln x}\,dx$

63. $\int_1^\infty \frac{e^x}{x}\,dx$

64. $\int_{e^e}^\infty \ln(\ln x)\,dx$

65. $\int_1^\infty \frac{1}{\sqrt{e^x-x}}\,dx$

66. $\int_1^\infty \frac{1}{e^x-2^x}\,dx$

67. $\int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}}$

68. $\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$

Theory and Examples

69. Find the values of p for which each integral converges.

a. $\int_1^2 \frac{dx}{x(\ln x)^p}$

b. $\int_2^\infty \frac{dx}{x(\ln x)^p}$

70. $\int_{-\infty}^{\infty} f(x)\,dx$ **may not equal** $\lim_{b\to\infty} \int_{-b}^{b} f(x)\,dx$ Show that

$$\int_0^\infty \frac{2x\,dx}{x^2+1}$$

diverges and hence that

$$\int_{-\infty}^\infty \frac{2x\,dx}{x^2+1}$$

diverges. Then show that

$$\lim_{b\to\infty}\int_{-b}^{b} \frac{2x\,dx}{x^2+1} = 0.$$

Exercises 71–74 are about the infinite region in the first quadrant between the curve $y = e^{-x}$ and the x-axis.

71. Find the area of the region.

72. Find the centroid of the region.

73. Find the volume of the solid generated by revolving the region about the y-axis.

74. Find the volume of the solid generated by revolving the region about the x-axis.

75. Find the area of the region that lies between the curves $y = \sec x$ and $y = \tan x$ from $x = 0$ to $x = \pi/2$.

76. The region in Exercise 75 is revolved about the x-axis to generate a solid.

a. Find the volume of the solid.

b. Show that the inner and outer surfaces of the solid have infinite area.

77. Consider the infinite region in the first quadrant bounded by the graphs of $y = \dfrac{1}{x^2}$, $y = 0$, and $x = 1$.

a. Find the area of the region.

b. Find the volume of the solid formed by revolving the region (i) about the x-axis; (ii) about the y-axis.

78. Consider the infinite region in the first quadrant bounded by the graphs of $y = \dfrac{1}{\sqrt{x}}$, $y = 0$, $x = 0$, and $x = 1$.

a. Find the area of the region.

b. Find the volume of the solid formed by revolving the region (i) about the x-axis; (ii) about the y-axis.

79. Evaluate the integrals.

a. $\displaystyle\int_0^1 \frac{dt}{\sqrt{t}(1+t)}$ **b.** $\displaystyle\int_0^\infty \frac{dt}{\sqrt{t}(1+t)}$

80. Evaluate $\displaystyle\int_3^\infty \frac{dx}{x\sqrt{x^2-9}}$.

81. Estimating the value of a convergent improper integral whose domain is infinite

a. Show that

$$\int_3^\infty e^{-3x}\,dx = \frac{1}{3}e^{-9} < 0.000042,$$

and hence that $\int_3^\infty e^{-x^2}\,dx < 0.000042$. Explain why this means that $\int_0^\infty e^{-x^2}\,dx$ can be replaced by $\int_0^3 e^{-x^2}\,dx$ without introducing an error of magnitude greater than 0.000042.

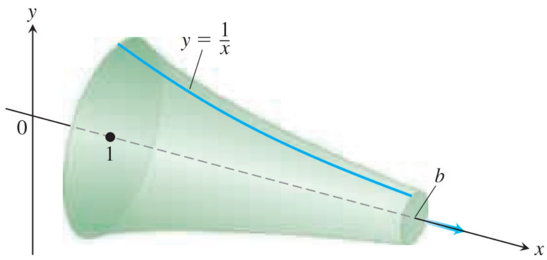
T **b.** Evaluate $\int_0^3 e^{-x^2}\,dx$ numerically.

82. The infinite paint can or Gabriel's horn As Example 3 shows, the integral $\int_1^\infty (dx/x)$ diverges. This means that the integral

$$\int_1^\infty 2\pi \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx,$$

which measures the *surface area* of the solid of revolution traced out by revolving the curve $y = 1/x$, $1 \le x$, about the x-axis, diverges also. By comparing the two integrals, we see that, for every finite value $b > 1$,

$$\int_1^b 2\pi \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx > 2\pi\int_1^b \frac{1}{x}\,dx.$$



However, the integral

$$\int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx$$

for the *volume* of the solid converges.

a. Calculate it.

b. This solid of revolution is sometimes described as a can that does not hold enough paint to cover its own interior. Think about that for a moment. It is common sense that a finite amount of paint cannot cover an infinite surface. But if we fill the horn with paint (a finite amount), then we *will* have covered an infinite surface. Explain the apparent contradiction.

83. Sine-integral function The integral

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt,$$

called the *sine-integral function*, has important applications in optics.

T **a.** Plot the integrand $(\sin t)/t$ for $t > 0$. Is the sine-integral function everywhere increasing or decreasing? Do you think $\mathrm{Si}(x) = 0$ for $x \ge 0$? Check your answers by graphing the function $\mathrm{Si}(x)$ for $0 \le x \le 25$.

b. Explore the convergence of

$$\int_0^\infty \frac{\sin t}{t}\,dt.$$

If it converges, what is its value?

84. Error function The function

$$\mathrm{erf}(x) = \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}}\,dt,$$

called the *error function*, has important applications in probability and statistics.

T **a.** Plot the error function for $0 \le x \le 25$.

b. Explore the convergence of

$$\int_0^{\infty} \frac{2e^{-t^2}}{\sqrt{\pi}}\, dt.$$

If it converges, what appears to be its value? You will see how to confirm your estimate in Section 15.4, Exercise 41.

85. Normal probability distribution The function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

is called the *normal probability density function* with mean μ and standard deviation σ. The number μ tells where the distribution is centered, and σ measures the "scatter" around the mean. (See Section 8.9.)

From the theory of probability, it is known that

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

In what follows, let $\mu = 0$ and $\sigma = 1$.

T **a.** Draw the graph of f. Find the intervals on which f is increasing, the intervals on which f is decreasing, and any local extreme values and where they occur.

b. Evaluate

$$\int_{-n}^{n} f(x)\, dx$$

for $n = 1, 2,$ and 3.

c. Give a convincing argument that

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

(*Hint:* Show that $0 < f(x) < e^{-x/2}$ for $x > 1$, and for $b > 1$,

$$\int_b^{\infty} e^{-x/2}\, dx \to 0 \quad \text{as} \quad b \to \infty.)$$

86. Show that if $f(x)$ is integrable on every interval of real numbers and a and b are real numbers with $a < b$, then

a. $\int_{-\infty}^{a} f(x)\, dx$ and $\int_{a}^{\infty} f(x)\, dx$ both converge if and only if $\int_{-\infty}^{b} f(x)\, dx$ and $\int_{b}^{\infty} f(x)\, dx$ both converge.

b. $\int_{-\infty}^{a} f(x)\, dx + \int_{a}^{\infty} f(x)\, dx = \int_{-\infty}^{b} f(x)\, dx + \int_{b}^{\infty} f(x)\, dx$ when the integrals involved converge.

COMPUTER EXPLORATIONS

In Exercises 87–90, use a CAS to explore the integrals for various values of p (include noninteger values). For what values of p does the integral converge? What is the value of the integral when it does converge? Plot the integrand for various values of p.

87. $\displaystyle\int_0^{e} x^p \ln x\, dx$

88. $\displaystyle\int_e^{\infty} x^p \ln x\, dx$

89. $\displaystyle\int_0^{\infty} x^p \ln x\, dx$

90. $\displaystyle\int_{-\infty}^{\infty} x^p \ln |x|\, dx$

Use a CAS to evaluate the integrals.

91. $\displaystyle\int_0^{2/\pi} \sin\frac{1}{x}\, dx$

92. $\displaystyle\int_0^{2/\pi} x \sin\frac{1}{x}\, dx$

8.9 Probability

The outcome of some events, such as a heavy rock falling from a great height, can be modeled so that we can predict with high accuracy what will happen. On the other hand, many events have more than one possible outcome and which one of them will occur is uncertain. If we toss a coin, a head or a tail will result with each outcome being equally likely, but we do not know in advance which one it will be. If we randomly select and then weigh a person from a large population, there are many possible weights the person might have, and it is not certain whether the weight will be between 180 and 190 lb. We are told it is highly likely, but not known for sure, that an earthquake of magnitude 6.0 or greater on the Richter scale will occur near a major population area in California within the next one hundred years. Events having more than one possible outcome are *probabilistic* in nature, and when modeling them we assign a *probability* to the likelihood that a particular outcome may occur. In this section we show how calculus plays a central role in making predictions with probabilistic models.

Random Variables

We begin our discussion with some familiar examples of uncertain events for which the collection of all possible outcomes is finite.

EXAMPLE 1

(a) If we toss a coin once, there are two possible outcomes $\{H, T\}$, where H represents the coin landing head face up and T a tail landing face up. If we toss a coin three times, there are eight possible outcomes, taking into account the order in which a head or tail occurs. The set of outcomes is $\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

(b) If we roll a six-sided die once, the set of possible outcomes is $\{1, 2, 3, 4, 5, 6\}$ representing the six faces of the die.

(c) If we select at random two cards from a 52-card deck, there are 52 possible outcomes for the first card drawn and then 51 possibilities for the second card. Since the order of the cards does not matter, there are $(52 \cdot 51)/2 = 1{,}326$ possible outcomes altogether. ■

It is customary to refer to the set of all possible outcomes as the *sample space* for an event. With an uncertain event we are usually interested in which outcomes, if any, are more likely to occur than others, and to how large an extent. In tossing a coin three times, is it more likely that two heads or that one head will result? To answer such questions, we need a way to quantify the outcomes.

> DEFINITION A **random variable** is a function X that assigns a numerical value to each outcome in a sample space.

Random variables that have only finitely many values are called **discrete** random variables. A **continuous random variable** can take on values in an entire interval, and it is associated with a *distribution function*, which we explain later.

EXAMPLE 2

(a) Suppose we toss a coin three times giving the possible outcomes $\{$HHH, HHT, HTH, THH, HTT, THT, TTH, TTT$\}$. Define the random variable X to be the number of heads that appear. So $X(\text{HHT}) = 2$, $X(\text{THT}) = 1$, and so forth. Since X can only assume the values 0, 1, 2, or 3, it is a discrete random variable.

(b) We spin an arrow anchored by a pin located at the origin. The arrow can wind up pointing in any possible direction and we define the random variable X as the radian angle the arrow makes with the positive x-axis, measured counterclockwise. In this case, X is a continuous random variable that can take on any value in the interval $[0, 2\pi)$.

(c) The weight of a randomly selected person in a given population is a continuous random variable W. The cholesterol level of a randomly chosen person, and the waiting time for service of a person in a queue at a bank, are also continuous random variables.

(d) The scores on the national ACT Examination for college admissions in a particular year are described by a discrete random variable S taking on integer values between 1 and 36. If the number of outcomes is large, or for reasons involving statistical analysis, discrete random variables such as test scores are often modeled as continuous random variables (Example 13).

(e) We roll a pair of dice and define the random variable X to be the sum of the numbers on the top faces. This sum can only assume the integer values from 2 through 12, so X is a discrete random variable.

(f) A tire company produces tires for mid-sized sedans. The tires are guaranteed to last for 30,000 miles, but some will fail sooner and some will last many more miles beyond 30,000. The lifetime in miles of a tire is described by a continuous random variable L. ■

Probability Distributions

A *probability distribution* describes the probabilistic behavior of a random variable. Our chief interest is in probability distributions associated with continuous random variables, but to gain some perspective we first consider a distribution for a discrete random variable.

Suppose we toss a coin three times, with each side H or T equally likely to occur on a given toss. We define the discrete random variable X that assigns the number of heads appearing in each outcome, giving

$$\begin{array}{cccccccccc} & \{\text{HHH}, & \text{HHT}, & \text{HTH}, & \text{HTT}, & \text{THH}, & \text{THT}, & \text{TTH}, & \text{TTT}\} \\ X & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 3 & 2 & 2 & 1 & 2 & 1 & 1 & 0 \end{array}$$

Next we count the *frequency* or number of times a specific value of X occurs. Because each of the eight outcomes is equally likely to occur, we can calculate the probability of the random variable X by dividing the frequency of each value by the total number of outcomes. We summarize our results as follows:

Value of X	0	1	2	3
Frequency	1	3	3	1
***P*(*X*)**	1/8	3/8	3/8	1/8

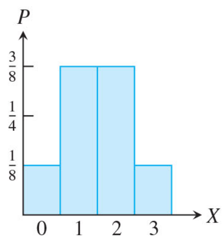


FIGURE 8.21 Probability bar graph for the random variable X when tossing a fair coin three times.

We display this information in a probability bar graph of the discrete random variable X, as shown in Figure 8.21. The values of X are portrayed by intervals of length 1 on the x-axis so the area of each bar in the graph is the probability of the corresponding outcome. For instance, the probability that exactly two heads occurs in the three tosses of the coin is the area of the bar associated with the value $X = 2$, which is $3/8$. Similarly, the probability that two or more heads occurs is the sum of areas of the bars associated with the values $X = 2$ and $X = 3$, or $4/8$. The probability that either zero or three heads occurs is $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$, and so forth. Note that the total area of all the bars in the graph is 1, which is the sum of all the probabilities for X.

With a continuous random variable, even when the outcomes are equally likely, we cannot simply count the number of outcomes in the sample space or the frequencies of outcomes that lead to a specific value of X. In fact, the probability that X takes on any particular one of its values is zero. What *is* meaningful to ask is how probable it is that the random variable takes on a value within some specified *interval* of values.

We capture the information we need about the probabilities of X in a function whose graph behaves much like the bar graph in Figure 8.21. That is, we take a nonnegative function f defined over the range of the random variable with the property that the total area beneath the graph of f is 1. The probability that a value of the random variable X lies within some specified interval $[c, d]$ is then the area under the graph of f over that interval. The following definition assumes the range of the continuous random variable X is any real value, but the definition is general enough to account for random variables having a range of finite length.

DEFINITIONS A **probability density function** for a continuous random variable is a function f defined over $(-\infty, \infty)$ and having the following properties:

1. f is continuous, except possibly at a finite number of points.
2. f is nonnegative, so $f \geq 0$.
3. $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1.$

If X is a continuous random variable with probability density function f, the **probability** that X assumes a value in the interval between $X = c$ and $X = d$ is given by the integral

$$P(c \leq X \leq d) = \int_c^d f(x)\,dx.$$

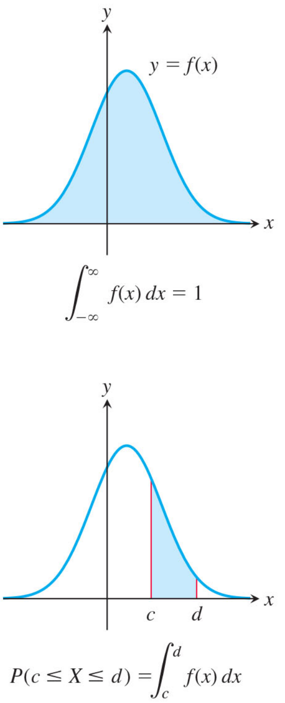


FIGURE 8.22 A probability density function for the continuous random variable X.

The probability that a continuous random variable X assumes a particular real value c is $P(X = c) = \int_c^c f(x)\, dx = 0$, consistent with our previous assertion. Since the area under the graph of f over the interval $[c, d]$ is only a portion of the total area beneath the graph, the probability $P(c \leq X \leq d)$ is always a number between zero and one. Figure 8.22 illustrates a probability density function.

A probability density function for a random variable X resembles the density function for a wire of varying density. To obtain the mass of a segment of the wire, we integrate the density of the wire over an interval. To obtain the probability that a random variable has values in a particular interval, we integrate the probability density function over that interval.

EXAMPLE 3 Let $f(x) = 2e^{-2x}$ if $0 \leq x < \infty$ and $f(x) = 0$ for all negative values of x.

(a) Verify that f is a probability density function.

(b) The time T in hours until a car passes a spot on a remote road is described by the probability density function f. Find the probability $P(T \leq 1)$ that a hitchhiker at that spot will see a car within one hour.

(c) Find the probability $P(T = 1)$ that a car passes by the spot after precisely one hour.

Solution

(a) The function f is continuous except at $x = 0$, and is everywhere nonnegative. Moreover,

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^{\infty} 2e^{-2x}\, dx = \lim_{b \to \infty} \int_0^b 2e^{-2x}\, dx = \lim_{b \to \infty} \left(1 - e^{-2b}\right) = 1.$$

So all of the conditions are satisfied and we have shown that f is a probability density function.

(b) The probability that a car comes after a time lapse between zero and one hour is given by integrating the probability density function over the interval $[0, 1]$. So

$$P(T \leq 1) = \int_0^1 2e^{-2t}\, dt = -e^{-2t}\Big]_0^1 = 1 - e^{-2} \approx 0.865.$$

This result can be interpreted to mean that if 100 people were to hitchhike at that spot, about 87 of them can expect to see a car within one hour.

(c) This probability is the integral $\int_1^1 f(t)\, dt$, which equals zero. We interpret this to mean that a sufficiently accurate measurement of the time until a car comes by the spot would have no possibility of being precisely equal to one hour. It might be very close, perhaps, but it would not be exactly one hour. ■

We can extend the definition to finite intervals. If f is a nonnegative function with at most finitely many discontinuities over the interval $[a, b]$, and its extension F to $(-\infty, \infty)$, obtained by defining F to be 0 outside of $[a, b]$, satisfies the definition for a probability density function, then f is a **probability density function for $[a, b]$**. This means that $\int_a^b f(x)\, dx = 1$. Similar definitions can be made for the intervals (a, b), $(a, b]$, and $[a, b)$.

EXAMPLE 4 Show that $f(x) = \frac{4}{27}x^2(3 - x)$ is a probability density function over the interval $[0, 3]$.

Solution The function f is continuous and nonnegative over $[0, 3]$. Also,

$$\int_0^3 \frac{4}{27}x^2(3 - x)\, dx = \frac{4}{27}\left[x^3 - \frac{1}{4}x^4\right]_0^3 = \frac{4}{27}\left(27 - \frac{81}{4}\right) = 1.$$

We conclude that f is a probability density function over $[0, 3]$. ■

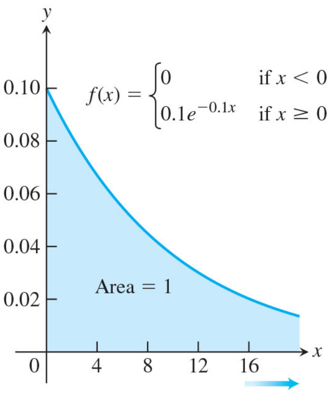


FIGURE 8.23 An exponentially decreasing probability density function.

Exponentially Decreasing Distributions

The distribution in Example 3 is called an *exponentially decreasing probability density function*. These probability density functions always take on the form

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ ce^{-cx} & \text{if } x \geq 0 \end{cases}$$

(see Exercise 23). Exponential density functions can provide models for describing random variables such as the lifetimes of light bulbs, radioactive particles, tooth crowns, and many kinds of electronic components. They also model the amount of time until some specific event occurs, such as the time until a pollinator arrives at a flower, the arrival times of a bus at a stop, the time between individuals joining a queue, the waiting time between phone calls at a help desk, and even the lengths of the phone calls themselves. A graph of an exponential density function is shown in Figure 8.23.

Random variables with exponential distributions are *memoryless*. If we think of X as describing the lifetime of some object, then the probability that the object survives for at least $s + t$ hours, given that it has survived t hours, is the same as the initial probability that it survives for at least s hours. For instance, the current age t of a radioactive particle does not change the probability that it will survive for at least another time period of length s. Sometimes the exponential distribution is used as a model when the memoryless principle is violated, because it provides reasonable approximations that are good enough for their intended use. For instance, this might be the case when predicting the lifetime of an artificial hip replacement or heart valve for a particular patient. Here is an application illustrating the exponential distribution.

EXAMPLE 5 An electronics company models the lifetime T in years of a chip they manufacture with the exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ 0.1e^{-0.1t} & \text{if } t \geq 0. \end{cases}$$

Using this model,

(a) Find the probability $P(T > 2)$ that a chip will last for more than two years.

(b) Find the probability $P(4 \leq T \leq 5)$ that a chip will fail in the fifth year.

(c) If 1000 chips are shipped to a customer, how many can be expected to fail within three years?

Solution

(a) The probability that a chip lasts at least two years is

$$\begin{aligned} P(T > 2) &= \int_2^\infty 0.1e^{-0.1t}\,dt = \lim_{b\to\infty} \int_2^b 0.1e^{-0.1t}\,dt \\ &= \lim_{b\to\infty} \left[e^{-0.2} - e^{-0.1b}\right] = e^{-0.2} \approx 0.819. \end{aligned}$$

That is, about 82% of the chips last more than two years.

(b) The probability is

$$P(4 \leq T \leq 5) = \int_4^5 0.1e^{-0.1t}\,dt = -e^{-0.1t}\Big]_4^5 = e^{-0.4} - e^{-0.5} \approx 0.064$$

which means that about 6% of the chips fail during the fifth year.

(c) We want the probability

$$P(0 \leq T \leq 3) = \int_0^3 0.1e^{-0.1t}\,dt = -e^{-0.1t}\Big]_0^3 = 1 - e^{-0.3} \approx 0.259.$$

We can expect that about 259 of the 1000 chips will fail within three years. ■

Expected Values, Means, and Medians

Suppose the weight in lbs of a steer raised on a cattle ranch is described by a continuous random variable W with probability density function $f(w)$ and that the rancher can sell a steer of weight w for $g(w)$ dollars. How much can the rancher expect to earn for a randomly chosen steer on the ranch?

To answer this question, we consider a small interval $[w_i, w_{i+1}]$ of width Δw_i and note that the probability a steer has weight in this interval is

$$\int_{w_i}^{wi+1} f(w)\,dw \approx f(w_i)\,\Delta w_i.$$

The earning on a steer in this interval is approximately $g(w_i)$. The Riemann sum

$$\sum g(w_i)\,f(w_i)\,\Delta w_i$$

then approximates the amount the rancher would receive for a steer. We assume that steers have a maximum weight, so f is zero outside some finite interval $[0, b]$. Then taking the limit of the Riemann sum as the width of each interval approaches zero gives the integral

$$\int_{-\infty}^{\infty} g(w)\,f(w)\,dw.$$

This integral estimates how much the rancher can expect to earn for a typical steer on the ranch and is the *expected value of the function g*.

The expected values of certain functions of a random variable X have particular importance in probability and statistics. One of the most important of these functions is the expected value of the function $g(x) = x$.

DEFINITION The **expected value** or **mean** of a continuous random variable X with probability density function f is the number

$$\mu = \mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx.$$

The expected value E(X) can be thought of as a weighted average of the random variable X, where each value of X is weighted by $f(X)$. The mean can also be interpreted as the long-run average value of the random variable X, and it is one measure of the centrality of the random variable X.

EXAMPLE 6 Find the mean of the random variable X with exponential probability density function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ ce^{-cx} & \text{if } x \ge 0. \end{cases}$$

Solution From the definition we have

$$\begin{aligned}
\mu &= \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} xce^{-cx}\,dx \\
&= \lim_{b\to\infty} \int_0^b xce^{-cx}\,dx = \lim_{b\to\infty}\left(-xe^{-cx}\Big]_0^b + \int_0^b e^{-cx}\,dx\right) \\
&= \lim_{b\to\infty}\left(-be^{-cb} - \frac{1}{c}e^{-cb} + \frac{1}{c}\right) = \frac{1}{c}. \qquad \text{l'Hôpital's Rule on first term}
\end{aligned}$$

Therefore, the mean is $\mu = 1/c$. ■

From the result in Example 6, knowing the mean or expected value μ of a random variable X having an exponential density function allows us to write its entire formula.

Exponential Density Function for a Random Variable X with Mean μ

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \mu^{-1}e^{-x/\mu} & \text{if } x \geq 0 \end{cases}$$

EXAMPLE 7 Suppose the time T before a chip fails in Example 5 is modeled instead by the exponential density function with a mean of eight years. Find the probability that a chip will fail within five years.

Solution The exponential density function with mean $\mu = 8$ is

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \dfrac{1}{8}e^{-t/8} & \text{if } t \geq 0 \end{cases}$$

Then the probability a chip will fail within five years is the definite integral

$$P(0 \leq T \leq 5) = \int_0^5 0.125e^{-0.125t}\, dt = -e^{-0.125t}\Big]_0^5 = 1 - e^{-0.625} \approx 0.465$$

so about 47% of the chips can be expected to fail within five years. ■

EXAMPLE 8 Find the expected value for the random variable X with probability density function given by Example 4.

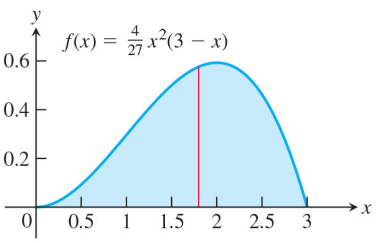


FIGURE 8.24 The expected value of a random variable with this probability density function is $\mu = 1.8$ (Example 8).

Solution The expected value is

$$\mu = E(X) = \int_0^3 \frac{4}{27}x^3(3 - x)\, dx = \frac{4}{27}\left[\frac{3}{4}x^4 - \frac{1}{5}x^5\right]_0^3$$

$$= \frac{4}{27}\left(\frac{243}{4} - \frac{243}{5}\right) = 1.8$$

From Figure 8.24, you can see that this expected value is reasonable because the region beneath the probability density function appears to be balanced about the vertical line $x = 1.8$. That is, the horizontal coordinate of the centroid of a plate described by the region is $\bar{x} = 1.8$. ■

There are other ways to measure the centrality of a random variable with a given probability density function.

DEFINITION The **median** of a continuous random variable X with probability density function f is the number m for which

$$\int_{-\infty}^{m} f(x)\, dx = \frac{1}{2} \quad \text{and} \quad \int_{m}^{\infty} f(x)\, dx = \frac{1}{2}.$$

The definition of the median means that there is an equal likelihood that the random variable X will be smaller than m or larger than m.

EXAMPLE 9 Find the median of a random variable X with exponential probability density function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ ce^{-cx} & \text{if } x \ge 0. \end{cases}$$

Solution The median m must satisfy

$$\frac{1}{2} = \int_0^m ce^{-cx}\,dx = -e^{-cx}\Big]_0^m = 1 - e^{-cm}.$$

It follows that

$$e^{-cm} = \frac{1}{2} \qquad \text{or} \qquad m = \frac{1}{c}\ln 2.$$

Also,

$$\frac{1}{2} = \lim_{b\to\infty}\int_m^b ce^{-cx}\,dx = \lim_{b\to\infty}\Big[-e^{-cx}\Big]_m^b = \lim_{b\to\infty}\left(e^{-cm} - e^{-cb}\right) = e^{-cm}$$

giving the same value for m. Since $1/c$ is the mean μ of X with an exponential distribution, we conclude that the median is $m = \mu \ln 2$. The mean and median differ because the probability density function is skewed and spreads toward the right. ■

Variance and Standard Deviation

Random variables with exactly the same mean μ but different distributions can behave very differently (see Figure 8.25). The *variance* of a random variable X measures how spread out the values of X are in relation to the mean, and we measure this dispersion by the expected value of $(X - \mu)^2$. Since the variance measures the expected square of the difference from the mean, we often work instead with its square root.

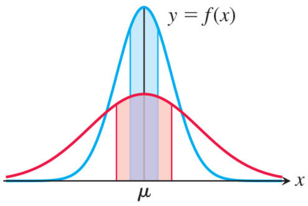


FIGURE 8.25 Probability density functions with the same mean can have different spreads in relation to the mean. The blue and red regions under the curves have equal area.

DEFINITIONS The **variance** of a random variable X with probability density function f is the expected value of $(X - \mu)^2$:

$$\text{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\,dx$$

The **standard deviation** of X is

$$\sigma_X = \sqrt{\text{Var}(X)} = \sqrt{\int_{-\infty}^{\infty} (x - \mu)^2 f(x)\,dx}.$$

EXAMPLE 10 Find the standard deviation of the random variable T in Example 5, and find the probability that T lies within one standard deviation of the mean.

Solution The probability density function is the exponential density function with mean $\mu = 10$ by Example 6. To find the standard deviation we first calculate the variance integral:

$$\begin{aligned}\int_{-\infty}^{\infty} (t - \mu)^2 f(t)\,dt &= \int_0^{\infty} (t - 10)^2 \left(0.1e^{-0.1t}\right) dt \\ &= \lim_{b\to\infty}\int_0^b (t - 10)^2 \left(0.1e^{-0.1t}\right) dt\end{aligned}$$

$$= \lim_{b\to\infty} \left[\left(-(t-10)^2 - 20(t-10) \right) e^{-0.1t} \right]_0^b$$

$$+ \lim_{b\to\infty} \int_0^b 20e^{-0.1t}\,dt \qquad \text{Integrating by parts}$$

$$= [0 + (-10)^2 + 20(-10)] - 20 \lim_{b\to\infty} \left(10e^{-0.1t}\right)\Big]_0^b$$

$$= -100 - 200 \lim_{b\to\infty} \left(e^{-0.1b} - 1\right) = 100.$$

The standard deviation is the square root of the variance, so $\sigma = 10.0$.

To find the probability that T lies within one standard deviation of the mean, we find the probability $P(\mu - \sigma \le T \le \mu + \sigma)$. For this example, we have

$$P(10 - 10 \le T \le 10 + 10) = \int_0^{20} 0.1e^{-0.1t}\,dt = -e^{-0.1t}\Big]_0^{20} = 1 - e^{-2} \approx 0.865$$

This means that about 87% of the chips will fail within twenty years. ■

Uniform Distributions

The **uniform distribution** is very simple, but it occurs commonly in applications. The probability density function for this distribution on the interval $[a, b]$ is

$$f(x) = \frac{1}{b-a}, \qquad a \le x \le b.$$

If each outcome in the sample space is equally likely to occur, then the random variable X has a uniform distribution. Since f is constant on $[a, b]$, a random variable with a uniform distribution is just as likely to be in one subinterval of a fixed length as in any other of the same length. The probability that X assumes a value in a subinterval of $[a, b]$ is the length of that subinterval divided by $(b - a)$.

EXAMPLE 11 An anchored arrow is spun around the origin, and the random variable X is the radian angle the arrow makes with the positive x-axis, measured within the interval $[0, 2\pi)$. Assuming there is equal probability for the arrow pointing in any direction, find the probability density function and the probability that the arrow ends up pointing between North and East.

Solution We model the probability density function with the uniform distribution $f(x) = 1/2\pi$, $0 \le x < 2\pi$, and $f(x) = 0$ elsewhere.

The probability that the arrow ends up pointing between North and East is given by

$$P\left(0 \le X \le \frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{1}{2\pi}\,dx = \frac{1}{4}.$$ ■

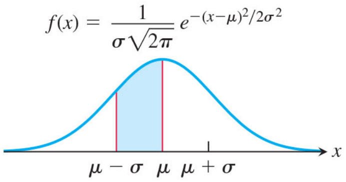


FIGURE 8.26 The normal probability density function with mean μ and standard deviation σ.

Normal Distributions

Numerous applications use the **normal distribution,** which is defined by the probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}.$$

It can be shown that the mean of a random variable X with this probability density function is μ and its standard deviation is σ. In applications the values of μ and σ are often estimated using large sets of data. The function is graphed in Figure 8.26, and the graph is sometimes called a *bell curve* because of its shape. Since the curve is symmetric about the mean, the median for X is the same as its mean. It is often observed in practice that many random variables have approximately a normal distribution. Some examples illustrating this phenomenon are the height of a man, the annual rainfall in a certain region, an individual's blood pressure, the serum cholesterol level in the blood, the brain weights in a certain population of adults, and the amount of growth in a given period for a population of sunflower seeds.

The normal probability density function does not have an antiderivative expressible in terms of familiar functions. Once μ and σ are fixed, however, an integral involving the normal probability density function can be computed using numerical integration methods. Usually we use the numerical integration capability of a computer or calculator to estimate the values of these integrals. Such computations show that for any normal distribution, we get the following values for the probability that the random variable X lies within $k = 1, 2, 3$, or 4 standard deviations of the mean:

$$\begin{aligned}
P(\mu - \sigma < X < \mu + \sigma) &\approx 0.68269\\
P(\mu - 2\sigma < X < \mu + 2\sigma) &\approx 0.95450\\
P(\mu - 3\sigma < X < \mu + 3\sigma) &\approx 0.99730\\
P(\mu - 4\sigma < X < \mu + 4\sigma) &\approx 0.99994
\end{aligned}$$

This means, for instance, that the random variable X will take on a value within two standard deviations of the mean about 95% of the time. About 68% of the time, X will lie within one standard deviation of the mean (see Figure 8.27).

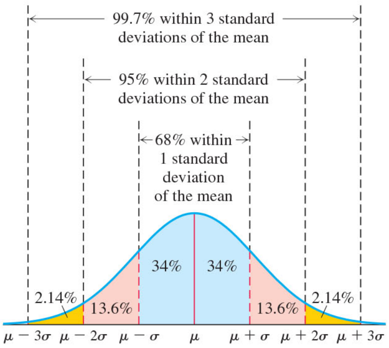


FIGURE 8.27 Probabilities of the normal distribution within its standard deviation bands.

EXAMPLE 12 An individual's blood pressure is an important indicator of overall health. A medical study of healthy individuals between 14 and 70 years of age modeled their systolic blood pressure using a normal distribution with mean 119.7 mm Hg and standard deviation 10.9 mm Hg.

(a) Using this model, what percentage of the population has a systolic blood pressure between 140 and 160 mm Hg, the levels set by the American Heart Association for Stage 1 hypertension?

(b) What percentage has a blood pressure between 160 and 180 mm Hg, the levels set by the American Heart Association for Stage 2 hypertension?

(c) What percentage has a blood pressure in the normal range of 90–120, as set by the American Heart Association?

Solution

(a) Since we cannot find an antiderivative, we use a computer to evaluate the probability integral of the normal probability density function with $\mu = 119.7$ and $\sigma = 10.9$:

$$P(140 \le X \le 160) = \int_{140}^{160} \frac{1}{10.9\sqrt{2\pi}} e^{-(x-119.7)^2/2(10.9)^2}\,dx \approx 0.03117.$$

This means that about 3% of the population in the studied age range have Stage 1 hypertension.

(b) Again we use a computer to calculate the probability that the blood pressure is between 160 and 180 mm Hg:

$$P(160 \le X \le 180) = \int_{160}^{180} \frac{1}{10.9\sqrt{2\pi}} e^{-(x-119.7)^2/2(10.9)^2}\,dx \approx 0.00011.$$

We conclude that about 0.011% of the population has Stage 2 hypertension.

(c) The probability that the blood pressure falls in the normal range is

$$P(90 \le X \le 120) = \int_{90}^{120} \frac{1}{10.9\sqrt{2\pi}} e^{-(x-119.7)^2/2(10.9)^2}\,dx \approx 0.50776.$$

That is, about 51% of the population has a normal systolic blood pressure. ■

Many national tests are standardized using the normal distribution. The following example illustrates modeling the discrete random variable for scores on a test using the normal distribution function for a continuous random variable.

EXAMPLE 13 The ACT is a standardized test taken by high school students seeking admission to many colleges and universities. The test measures knowledge skills and proficiency in the areas of English, math, and science, with scores ranging over the interval $[1, 36]$. Nearly 1.5 million high school students took the test in 2009, and the composite mean score across the academic areas was $\mu = 21.1$ with standard deviation $\sigma = 5.1$.

(a) What percentage of the population had an ACT score between 18 and 24?

(b) What is the ranking of a student who scored 27 on the test?

(c) What is the minimal integer score a student needed to get in order to be in the top 8% of the scoring population?

Solution

(a) We use a computer to evaluate the probability integral of the normal probability density function with $\mu = 21.1$ and $\sigma = 5.1$:

$$P(18 \le X \le 24) = \int_{18}^{24} \frac{1}{5.1\sqrt{2\pi}} e^{-(x-21.1)^2/2(5.1)^2}\, dx \approx 0.44355.$$

This means that about 44% of the students had an ACT score between 18 and 24.

(b) Again we use a computer to calculate the probability of a student getting a score lower than 27 on the test:

$$P(1 \le X < 27) = \int_{1}^{27} \frac{1}{5.1\sqrt{2\pi}} e^{-(x-21.1)^2/2(5.1)^2}\, dx \approx 0.87630.$$

We conclude that about 88% of the students scored below a score of 27, so the student ranked in the top 12% of the population.

(c) We look at how many students had a mark higher than 28:

$$P(28 < X \le 36) = \int_{28}^{36} \frac{1}{5.1\sqrt{2\pi}} e^{-(x-21.1)^2/2(5.1)^2}\, dx \approx 0.0863.$$

Since this number gives more than 8% of the students, we look at the next higher integer score:

$$P(29 < X \le 36) = \int_{29}^{36} \frac{1}{5.1\sqrt{2\pi}} e^{-(x-21.1)^2/2(5.1)^2}\, dx \approx 0.0595.$$

Therefore, 29 is the lowest integer score a student could get in order to score in the top 8% of the population (and actually scoring here in the top 6%).

The simplest form for a normal distribution of X occurs when its mean is zero and its standard deviation is one. The *standard normal probability density function* f giving mean $\mu = 0$ and standard deviation $\sigma = 1$ is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Note that the substitution $z = (x - \mu)/\sigma$ gives the equivalent integrals

$$\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-((x-\mu)/\sigma)^2/2}\, dx = \int_\alpha^\beta \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz,$$

where $\alpha = (a - \mu)/\sigma$ and $\beta = (b - \mu)/\sigma$. So we can convert random variable values to the "z-values" to standardize a normal distribution, and then use the integral on the right-hand side of the last equation to calculate probabilities for the original random variable normal distribution with mean μ and standard deviation σ. In a normal distribution, we know that 95.5% of the population lies within two standard deviations of the mean, so a random variable X converted to a z-value has more than a 95% chance of occurring in the interval $[-2, 2]$.

EXERCISES 8.9

Probability Density Functions

In Exercises 1–8, determine which are probability density functions and justify your answer.

1. $f(x) = \frac{1}{18}x$ over $[4, 8]$

2. $f(x) = \frac{1}{2}(2 - x)$ over $[0, 2]$

3. $f(x) = 2^x$ over $\left[0, \frac{\ln(1 + \ln 2)}{\ln 2}\right]$

4. $f(x) = x - 1$ over $[0, 1 + \sqrt{3}]$

5. $f(x) = \begin{cases} \frac{1}{x^2} & x \ge 1 \\ 0 & x < 1 \end{cases}$

6. $f(x) = \begin{cases} \frac{8}{\pi(4 + x^2)} & x \ge 0 \\ 0 & x < 0 \end{cases}$

7. $f(x) = 2\cos 2x$ over $\left[0, \frac{\pi}{4}\right]$

8. $f(x) = \frac{1}{x}$ over $(0, e]$

9. Let f be the probability density function for the random variable L in Example 2f. Explain the meaning of each integral.

a. $\int_{25{,}000}^{32{,}000} f(l)\,dl$ **b.** $\int_{30{,}000}^{\infty} f(l)\,dl$

c. $\int_{0}^{20{,}000} f(l)\,dl$ **d.** $\int_{-\infty}^{15{,}000} f(l)\,dl$

10. Let $f(x)$ be the uniform distribution for the random variable X in Example 11. Express the following probabilities as integrals.

a. The probability that the arrow points either between South and West or between North and West.

b. The probability that the arrow makes an angle of at least 2 radians.

Verify that the functions in Exercises 11–16 are probability density functions for a continuous random variable X over the given interval. Determine the specified probability.

11. $f(x) = xe^{-x}$ over $[0, \infty)$, $P(1 \le X \le 3)$

T **12.** $f(x) = \frac{\ln x}{x^2}$ over $[1, \infty)$, $P(2 < X < 15)$

13. $f(x) = \frac{3}{2}x(2 - x)$ over $[0, 1]$, $P(0.5 > X)$

T **14.** $f(x) = \frac{\sin^2 \pi x}{\pi x^2}$ over $\left[\frac{200}{1059}, \infty\right)$, $P(X < \pi/6)$

15. $f(x) = \begin{cases} \frac{2}{x^3} & x > 1 \\ 0 & x \le 1 \end{cases}$ over $(-\infty, \infty)$, $P(4 \le X < 9)$

16. $f(x) = \sin x$ over $[0, \pi/2]$, $P\left(\frac{\pi}{6} < X \le \frac{\pi}{4}\right)$

In Exercises 17–20, find the value of the constant c so that the given function is a probability density function for a random variable over the specified interval.

17. $f(x) = \frac{1}{6}x$ over $[3, c]$ **18.** $f(x) = \frac{1}{x}$ over $[c, c + 1]$

19. $f(x) = 4e^{-2x}$ over $[0, c]$

20. $f(x) = cx\sqrt{25 - x^2}$ over $[0, 5]$

21. Let $f(x) = \frac{c}{1 + x^2}$. Find the value of c so that f is a probability density function. If f is a probability density function for the random variable X, find the probability $P(1 \le X < 2)$.

22. Find the value of c so that $f(x) = c\sqrt{x}(1 - x)$ is a probability density function for the random variable X over $[0, 1]$, and find the probability $P(0.25 \le X \le 0.5)$.

23. Show that if the exponentially decreasing function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ Ae^{-cx} & \text{if } x \ge 0 \end{cases}$$

is a probability density function, then $A = c$.

24. Suppose f is a probability density function for the random variable X with mean μ. Show that its variance satisfies

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2.$$

Compute the mean and median for a random variable with the probability density functions in Exercises 25–28.

25. $f(x) = \frac{1}{8}x$ over $[0, 4]$ **26.** $f(x) = \frac{1}{9}x^2$ over $[0, 3]$

27. $f(x) = \begin{cases} \frac{2}{x^3} & x \ge 1 \\ 0 & x < 1 \end{cases}$ **28.** $f(x) = \begin{cases} \frac{1}{x} & 1 \le x \le e \\ 0 & \text{Otherwise} \end{cases}$

Exponential Distributions

29. Digestion time The digestion time in hours of a fixed amount of food is exponentially distributed with a mean of 1 hour. What is the probability that the food is digested in less than 30 minutes?

30. Pollinating flowers A biologist models the time in minutes until a bee arrives at a flowering plant with an exponential distribution having a mean of 4 minutes. If 1000 flowers are in a field, how many can be expected to be pollinated within 5 minutes?

31. Lifetime of light bulbs A manufacturer of light bulbs finds that the mean lifetime of a bulb is 1200 hours. Assume the life of a bulb is exponentially distributed.

a. Find the probability that a bulb will last less than its guaranteed lifetime of 1000 hours.

b. In a batch of light bulbs, what is the expected time until half the light bulbs in the batch fail?

32. Lifetime of an electronic component The life expectancy in years of a component in a microcomputer is exponentially distributed, and $1/3$ of the components fail in the first 3 years. The company that manufactures the component offers a 1 year warranty. What is the probability that a component will fail during the warranty period?

33. Lifetime of an organism A *hydra* is a small fresh-water animal, and studies have shown that its probability of dying does not increase with the passage of time. The lack of influence of age on mortality rates for this species indicates that an exponential distribution is an appropriate model for the mortality of hydra. A biologist studies a population of 500 hydra and observes that 200 of them die within the first 2 years. How many of the hydra would you expect to die within the first six months?

34. Car accidents The number of days that elapse between the beginning of a calendar year and the moment a high-risk driver is involved in an accident is exponentially distributed. Based on historical data, an insurance company expects that 30% of high-risk drivers will be involved in an accident during the first 50 days of the calendar year. In a group of 100 high-risk drivers, how many do you expect to be involved in an accident during the first 80 days of the calendar year?

35. Customer service time The mean waiting time to get served after walking into a bakery is 30 seconds. Assume that an exponential density function describes the waiting times.

a. What is the probability a customer waits 15 seconds or less?

b. What is the probability a customer waits longer than one minute?

c. What is the probability a customer waits exactly 5 minutes?

d. If 200 customers come to the bakery in a day, how many are likely to be served within three minutes?

36. Airport waiting time According to the U.S. Customs and Border Protection Agency, the average airport wait time at Chicago's O'Hare International airport is 16 minutes for a traveler arriving during the hours 7–8 A.M., and 32 minutes for arrival during the hours 4–5 P.M. The wait time is defined as the total processing time from arrival at the airport until the completion of a passenger's security screening. Assume the wait time is exponentially distributed.

a. What is the probability of waiting between 10 and 30 minutes for a traveler arriving during the 7–8 A.M. hour?

b. What is the probability of waiting more than 25 minutes for a traveler arriving during the 7–8 P.M. hour?

c. What is the probability of waiting between 35 and 50 minutes for a traveler arriving during the 4–5 P.M. hour?

d. What is the probability of waiting less than 20 minutes for a traveler arriving during the 4–5 P.M. hour?

37. Printer lifetime The lifetime of a $200 printer is exponentially distributed with a mean of 2 years. The manufacturer agrees to pay a full refund to a buyer if the printer fails during the first year following its purchase, and a one-half refund if it fails during the second year. If the manufacturer sells 100 printers, how much should it expect to pay in refunds?

38. Failure time The time between failures of a photocopier is exponentially distributed. Half of the copiers at a university require service during the first 2 years of operations. If the university purchased 150 copiers, how many do you expect to require service during the first year of their operation?

T Normal Distributions

39. Cholesterol levels The serum cholesterol levels of children aged 12 to 14 years follows a normal distribution with mean $\mu = 162$ mg/dl and standard deviation $\sigma = 28$ mg/dl. In a population of 1000 of these children, how many would you expect to have serum cholesterol levels between 165 and 193? between 148 and 167?

40. Annual rainfall The annual rainfall in inches for San Francisco, California, is approximately a normal random variable with mean 20.11 in. and standard deviation 4.7 in. What is the probability that next year's rainfall will exceed 17 in.?

41. Manufacturing time The assembly time in minutes for a component at an electronic manufacturing plant is normally distributed with a mean of $\mu = 55$ and standard deviation $\sigma = 4$. What is the probability that a component will be made in less than one hour?

42. Lifetime of a tire Assume the random variable L in Example 2f is normally distributed with mean $\mu = 22{,}000$ miles and $\sigma = 4{,}000$ miles.

a. In a batch of 4000 tires, how many can be expected to last for at least 18,000 miles?

b. What is the minimum number of miles you would expect to find as the lifetime for 90% of the tires?

43. Height The average height of American females aged 18–24 is normally distributed with mean $\mu = 65.5$ inches and $\sigma = 2.5$ inches.

a. What percentage of females are taller than 68 inches?

b. What is the probability a female is between $5'1''$ and $5'4''$ tall?

44. Life expectancy At birth, a French citizen has an average life expectancy of 82 years with a standard deviation of 7 years. If 100 newly born French babies are selected at random, how many would you expect to live between 75 and 85 years? Assume life expectancy is normally distributed.

45. Length of pregnancy A team of medical practitioners determines that in a population of 1000 females with ages ranging from 20 to 35 years, the length of pregnancy from conception to birth is approximately normally distributed with a mean of 266 days and a standard deviation of 16 days. How many of these females would you expect to have a pregnancy lasting from 36 weeks to 40 weeks?

46. Brain weights In a population of 500 adult Swedish males, medical researchers find their brain weights to be approximately normally distributed with mean $\mu = 1400$ gm and standard deviation $\sigma = 100$ gm.

a. What percentage of brain weights are between 1325 and 1450 gm?

b. How many males in the population would you expect to have a brain weight exceeding 1480 gm?

47. Blood pressure Diastolic blood pressure in adults is normally distributed with $\mu = 80$ mm Hg and $\sigma = 12$ mm Hg. In a random sample of 300 adults, how many would be expected to have a diastolic blood pressure below 70 mm Hg?

48. Albumin levels Serum albumin in healthy 20-year-old males is normally distributed with $\mu = 4.4$ and $\sigma = 0.2$. How likely is it for a healthy 20-year-old male to have a level in the range 4.3 to 4.45?

49. Quality control A manufacturer of generator shafts finds that it needs to add additional weight to its shafts in order to achieve proper static and dynamic balance. Based on experimental tests, the average weight it needs to add is $\mu = 35$ gm with $\sigma = 9$ gm. Assuming a normal distribution, from 1000 randomly selected shafts, how many would be expected to need an added weight in excess of 40 gm?

50. Miles driven A taxicab company in New York City analyzed the daily number of miles driven by each of its drivers. It found the average distance was 200 mi with a standard deviation of 30 mi. Assuming a normal distribution, what prediction can we make about the percentage of drivers who will log in either more than 260 mi or less than 170 mi?

51. Germination of sunflower seeds The germination rate of a particular seed is the percentage of seeds in the batch which successfully emerge as plants. Assume that the germination rate for a batch of sunflower seeds is 80%, and that among a large population of n seeds the number of successful germinations is normally distributed with mean $\mu = 0.8n$ and $\sigma = 0.4\sqrt{n}$.

a. In a batch of $n = 2500$ seeds, what is the probability that at least 1960 will successfully germinate?

b. In a batch of $n = 2500$ seeds, what is the probability that at most 1980 will successfully germinate?

c. In a batch of $n = 2500$ seeds, what is the probability that between 1940 and 2020 will successfully germinate?

52. Suppose you toss a fair coin n times and record the number of heads that land. Assume that n is large and approximate the discrete random variable X with a continuous random variable that is normally distributed with $\mu = n/2$ and $\sigma = \sqrt{n}/2$. If $n = 400$, find the given probabilities.

a. $P(190 \le X < 210)$ **b.** $P(X < 170)$

c. $P(X > 220)$ **d.** $P(X = 300)$

Discrete Random Variables

53. A fair coin is tossed four times and the random variable X assigns the number of tails that appear in each outcome.

a. Determine the set of possible outcomes.

b. Find the value of X for each outcome.

c. Create a probability bar graph for X, as in Figure 8.21. What is the probability that at least two heads appear in the four tosses of the coin?

54. You roll a pair of six-sided dice, and the random variable X assigns to each outcome the sum of the number of dots showing on each face, as in Example 2e.

a. Find the set of possible outcomes.

b. Create a probability bar graph for X.

c. What is the probability that $X = 8$?

d. What is the probability that $X \le 5$? $X > 9$?

55. Three people are asked their opinion in a poll about a particular brand of a common product found in grocery stores. They can answer in one of three ways: "Like the product brand" (L), "Dislike the product brand" (D), or "Undecided" (U). For each outcome, the random variable X assigns the number of L's that appear.

a. Find the set of possible outcomes and the range of X.

b. Create a probability bar graph for X.

c. What is the probability that at least two people like the product brand?

d. What is the probability that no more than one person dislikes the product brand?

56. Spacecraft components A component of a spacecraft has both a main system and a backup system operating throughout a flight. The probability that both systems fail sometime during the flight is 0.0148. Assuming that each system separately has the same failure rate, what is the probability that the main system fails during the flight?

CHAPTER 8 Questions to Guide Your Review

1. What is the formula for integration by parts? Where does it come from? Why might you want to use it?
2. When applying the formula for integration by parts, how do you choose the u and dv? How can you apply integration by parts to an integral of the form $\int f(x)\,dx$?
3. If an integrand is a product of the form $\sin^n x \cos^m x$, where m and n are nonnegative integers, how do you evaluate the integral? Give a specific example of each case.
4. What substitutions are made to evaluate integrals of $\sin mx \sin nx$, $\sin mx \cos nx$, and $\cos mx \cos nx$? Give an example of each case.
5. What substitutions are sometimes used to transform integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$ into integrals that can be evaluated directly? Give an example of each case.
6. What restrictions can you place on the variables involved in the three basic trigonometric substitutions to make sure the substitutions are reversible (have inverses)?

7. What is the goal of the method of partial fractions?
8. When the degree of a polynomial $f(x)$ is less than the degree of a polynomial $g(x)$, how do you write $f(x)/g(x)$ as a sum of partial fractions if $g(x)$
   a. is a product of distinct linear factors?
   b. consists of a repeated linear factor?
   a. contains an irreducible quadratic factor?
   What do you do if the degree of f is *not* less than the degree of g?
9. How are integral tables typically used? What do you do if a particular integral you want to evaluate is not listed in the table?
10. What is a reduction formula? How are reduction formulas used? Give an example.
11. How would you compare the relative merits of Simpson's Rule and the Trapezoidal Rule?
12. What is an improper integral of Type I? Type II? How are the values of various types of improper integrals defined? Give examples.
13. What tests are available for determining the convergence and divergence of improper integrals that cannot be evaluated directly? Give examples of their use.
14. What is a random variable? What is a continuous random variable? Give some specific examples.
15. What is a probability density function? What is the probability that a continuous random variable has a value in the interval $[c, d]$?
16. What is an exponentially decreasing probability density function? What are some typical events that might be modeled by this distribution? What do we mean when we say such distributions are *memoryless*?
17. What is the expected value of a continuous random variable? What is the expected value of an exponentially distributed random variable?
18. What is the median of a continuous random variable? What is the median of an exponential distribution?
19. What does the variance of a random variable measure? What is the standard deviation of a continuous random variable X?
20. What probability density function describes the normal distribution? What are some examples typically modeled by a normal distribution? How do we usually calculate probabilities for a normal distribution?
21. In a normal distribution, what percentage of the population lies within 1 standard deviation of the mean? Within 2 standard deviations?

CHAPTER 8 Practice Exercises

Integration by Parts

Evaluate the integrals in Exercises 1–8 using integration by parts.

1. $\displaystyle\int \ln (x+1)\, dx$ **2.** $\displaystyle\int x^2 \ln x\, dx$

3. $\displaystyle\int \tan^{-1} 3x\, dx$ **4.** $\displaystyle\int \cos^{-1}\left(\frac{x}{2}\right) dx$

5. $\displaystyle\int (x+1)^2 e^x\, dx$ **6.** $\displaystyle\int x^2 \sin (1-x)\, dx$

7. $\displaystyle\int e^x \cos 2x\, dx$ **8.** $\displaystyle\int x \sin x \cos x\, dx$

Partial Fractions

Evaluate the integrals in Exercises 9–28. It may be necessary to use a substitution first.

9. $\displaystyle\int \frac{x\, dx}{x^2-3x+2}$ **10.** $\displaystyle\int \frac{x\, dx}{x^2+4x+3}$

11. $\displaystyle\int \frac{dx}{x(x+1)^2}$ **12.** $\displaystyle\int \frac{x+1}{x^2(x-1)}\, dx$

13. $\displaystyle\int \frac{\sin\theta\, d\theta}{\cos^2\theta+\cos\theta-2}$ **14.** $\displaystyle\int \frac{\cos\theta\, d\theta}{\sin^2\theta+\sin\theta-6}$

15. $\displaystyle\int \frac{3x^2+4x+4}{x^3+x}\, dx$ **16.** $\displaystyle\int \frac{4x\, dx}{x^3+4x}$

17. $\displaystyle\int \frac{v+3}{2v^3-8v}\, dv$ **18.** $\displaystyle\int \frac{(3v-7)\, dv}{(v-1)(v-2)(v-3)}$

19. $\displaystyle\int \frac{dt}{t^4+4t^2+3}$ **20.** $\displaystyle\int \frac{t\, dt}{t^4-t^2-2}$

21. $\displaystyle\int \frac{x^3+x^2}{x^2+x-2}\, dx$ **22.** $\displaystyle\int \frac{x^3+1}{x^3-x}\, dx$

23. $\displaystyle\int \frac{x^3+4x^2}{x^2+4x+3}\, dx$ **24.** $\displaystyle\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\, dx$

25. $\displaystyle\int \frac{dx}{x(3\sqrt{x+1})}$ **26.** $\displaystyle\int \frac{dx}{x\left(1+\sqrt[3]{x}\right)}$

27. $\displaystyle\int \frac{ds}{e^s-1}$ **28.** $\displaystyle\int \frac{ds}{\sqrt{e^s+1}}$

Trigonometric Substitutions

Evaluate the integrals in Exercises 29–32 **(a)** without using a trigonometric substitution, **(b)** using a trigonometric substitution.

29. $\displaystyle\int \frac{y\, dy}{\sqrt{16-y^2}}$ **30.** $\displaystyle\int \frac{x\, dx}{\sqrt{4+x^2}}$

31. $\displaystyle\int \frac{x\, dx}{4-x^2}$ **32.** $\displaystyle\int \frac{t\, dt}{\sqrt{4t^2-1}}$

Evaluate the integrals in Exercises 33–36.

33. $\displaystyle\int \frac{x\, dx}{9-x^2}$ **34.** $\displaystyle\int \frac{dx}{x(9-x^2)}$

35. $\displaystyle\int \frac{dx}{9-x^2}$ **36.** $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$

Trigonometric Integrals

Evaluate the integrals in Exercises 37–44.

37. $\int \sin^3 x \cos^4 x \, dx$

38. $\int \cos^5 x \sin^5 x \, dx$

39. $\int \tan^4 x \sec^2 x \, dx$

40. $\int \tan^3 x \sec^3 x \, dx$

41. $\int \sin 5\theta \cos 6\theta \, d\theta$

42. $\int \sec^2 \theta \sin^3 \theta \, d\theta$

43. $\int \sqrt{1 + \cos(t/2)} \, dt$

44. $\int e^t \sqrt{\tan^2 e^t + 1} \, dt$

Numerical Integration

45. According to the error-bound formula for Simpson's Rule, how many subintervals should you use to be sure of estimating the value of

$$\ln 3 = \int_1^3 \frac{1}{x} \, dx$$

by Simpson's Rule with an error of no more than 10^{-4} in absolute value? (Remember that for Simpson's Rule, the number of subintervals has to be even.)

46. A brief calculation shows that if $0 \le x \le 1$, then the second derivative of $f(x) = \sqrt{1 + x^4}$ lies between 0 and 8. Based on this, about how many subdivisions would you need to estimate the integral of f from 0 to 1 with an error no greater than 10^{-3} in absolute value using the Trapezoidal Rule?

47. A direct calculation shows that

$$\int_0^{\pi} 2 \sin^2 x \, dx = \pi.$$

How close do you come to this value by using the Trapezoidal Rule with $n = 6$? Simpson's Rule with $n = 6$? Try them and find out.

48. You are planning to use Simpson's Rule to estimate the value of the integral

$$\int_1^2 f(x) \, dx$$

with an error magnitude less than 10^{-5}. You have determined that $|f^{(4)}(x)| \le 3$ throughout the interval of integration. How many subintervals should you use to ensure the required accuracy? (Remember that for Simpson's Rule the number has to be even.)

T **49. Mean temperature** Use Simpson's Rule to approximate the average value of the temperature function

$$f(x) = 37 \sin\left(\frac{2\pi}{365}(x - 101)\right) + 25$$

for a 365-day year. This is one way to estimate the annual mean air temperature in Fairbanks, Alaska. The National Weather Service's official figure, a numerical average of the daily normal mean air temperatures for the year, is 25.7°F, which is slightly higher than the average value of $f(x)$.

50. Heat capacity of a gas Heat capacity C_v is the amount of heat required to raise the temperature of a given mass of gas with constant volume by 1°C, measured in units of cal/deg-mol (calories per degree gram molecular weight). The heat capacity of oxygen depends on its temperature T and satisfies the formula

$$C_v = 8.27 + 10^{-5}(26T - 1.87T^2).$$

Use Simpson's Rule to find the average value of C_v and the temperature at which it is attained for $20° \le T \le 675°$C.

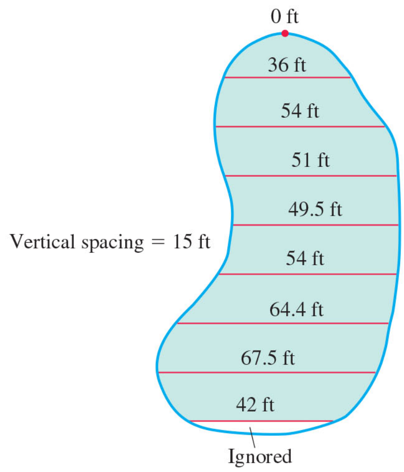
51. Fuel efficiency An automobile computer gives a digital readout of fuel consumption in gallons per hour. During a trip, a passenger recorded the fuel consumption every 5 min for a full hour of travel.

Time	Gal/h	Time	Gal/h
0	2.5	35	2.5
5	2.4	40	2.4
10	2.3	45	2.3
15	2.4	50	2.4
20	2.4	55	2.4
25	2.5	60	2.3
30	2.6		

a. Use the Trapezoidal Rule to approximate the total fuel consumption during the hour.

b. If the automobile covered 60 mi in the hour, what was its fuel efficiency (in miles per gallon) for that portion of the trip?

52. A new parking lot To meet the demand for parking, your town has allocated the area shown here. As the town engineer, you have been asked by the town council to find out if the lot can be built for \$11,000. The cost to clear the land will be \$0.10 a square foot, and the lot will cost \$2.00 a square foot to pave. Use Simpson's Rule to find out if the job can be done for \$11,000.



Improper Integrals

Evaluate the improper integrals in Exercises 53–62.

53. $\int_0^3 \frac{dx}{\sqrt{9 - x^2}}$

54. $\int_0^1 \ln x \, dx$

55. $\int_0^2 \frac{dy}{(y - 1)^{2/3}}$

56. $\int_{-2}^0 \frac{d\theta}{(\theta + 1)^{3/5}}$

57. $\int_3^{\infty} \frac{2\,du}{u^2 - 2u}$

58. $\int_1^{\infty} \frac{3v - 1}{4v^3 - v^2}\,dv$

59. $\int_0^{\infty} x^2 e^{-x}\,dx$

60. $\int_{-\infty}^{0} x e^{3x}\,dx$

61. $\int_{-\infty}^{\infty} \frac{dx}{4x^2 + 9}$

62. $\int_{-\infty}^{\infty} \frac{4\,dx}{x^2 + 16}$

Which of the improper integrals in Exercises 63–68 converge and which diverge?

63. $\int_6^{\infty} \frac{d\theta}{\sqrt{\theta^2 + 1}}$

64. $\int_0^{\infty} e^{-u} \cos u\,du$

65. $\int_1^{\infty} \frac{\ln z}{z}\,dz$

66. $\int_1^{\infty} \frac{e^{-t}}{\sqrt{t}}\,dt$

67. $\int_{-\infty}^{\infty} \frac{2\,dx}{e^x + e^{-x}}$

68. $\int_{-\infty}^{\infty} \frac{dx}{x^2(1 + e^x)}$

Assorted Integrations

Evaluate the integrals in Exercises 69–136. The integrals are listed in random order so you need to decide which integration technique to use.

69. $\int x e^{2x}\,dx$

70. $\int_0^1 x^2 e^{x^3}\,dx$

71. $\int (\tan^2 x + \sec^2 x)\,dx$

72. $\int_0^{\pi/4} \cos^2 2x\,dx$

73. $\int x \sec^2 x\,dx$

74. $\int x \sec^2 (x^2)\,dx$

75. $\int \sin x \cos^2 x\,dx$

76. $\int \sin 2x \sin (\cos 2x)\,dx$

77. $\int_{-1}^{0} \frac{e^x}{e^x + e^{-x}}\,dx$

78. $\int (e^{2x} + e^{-x})^2\,dx$

79. $\int \frac{x + 1}{x^4 - x^3}\,dx$

80. $\int \frac{e^x + 1}{e^x(e^{2x} - 4)}\,dx$

81. $\int \frac{e^x + e^{3x}}{e^{2x}}\,dx$

82. $\int (e^x - e^{-x})(e^x + e^{-x})^3\,dx$

83. $\int_0^{\pi/3} \tan^3 x \sec^2 x\,dx$

84. $\int \tan^4 x \sec^4 x\,dx$

85. $\int_0^3 (x + 2)\sqrt{x + 1}\,dx$

86. $\int (x + 1)\sqrt{x^2 + 2x}\,dx$

87. $\int \cot x \csc^3 x\,dx$

88. $\int \sin x (\tan x - \cot x)^2\,dx$

89. $\int \frac{x\,dx}{1 + \sqrt{x}}$

90. $\int \frac{x^3 + 2}{4 - x^2}\,dx$

91. $\int \sqrt{2x - x^2}\,dx$

92. $\int \frac{dx}{\sqrt{-2x - x^2}}$

93. $\int \frac{2 - \cos x + \sin x}{\sin^2 x}\,dx$

94. $\int \sin^2 \theta \cos^5 \theta\,d\theta$

95. $\int \frac{9\,dv}{81 - v^4}$

96. $\int_2^{\infty} \frac{dx}{(x - 1)^2}$

97. $\int \theta \cos (2\theta + 1)\,d\theta$

98. $\int \frac{x^3\,dx}{x^2 - 2x + 1}$

99. $\int \frac{\sin 2\theta\,d\theta}{(1 + \cos 2\theta)^2}$

100. $\int_{\pi/4}^{\pi/2} \sqrt{1 + \cos 4x}\,dx$

101. $\int \frac{x\,dx}{\sqrt{2 - x}}$

102. $\int \frac{\sqrt{1 - v^2}}{v^2}\,dv$

103. $\int \frac{dy}{y^2 - 2y + 2}$

104. $\int \frac{x\,dx}{\sqrt{8 - 2x^2 - x^4}}$

105. $\int \frac{z + 1}{z^2(z^2 + 4)}\,dz$

106. $\int x^2(x - 1)^{1/3}\,dx$

107. $\int \frac{t\,dt}{\sqrt{9 - 4t^2}}$

108. $\int \frac{\tan^{-1} x}{x^2}\,dx$

109. $\int \frac{e^t\,dt}{e^{2t} + 3e^t + 2}$

110. $\int \tan^3 t\,dt$

111. $\int_1^{\infty} \frac{\ln y}{y^3}\,dy$

112. $\int y^{3/2}(\ln y)^2\,dy$

113. $\int e^{\ln \sqrt{x}}\,dx$

114. $\int e^{\theta}\sqrt{3 + 4e^{\theta}}\,d\theta$

115. $\int \frac{\sin 5t\,dt}{1 + (\cos 5t)^2}$

116. $\int \frac{dv}{\sqrt{e^{2v} - 1}}$

117. $\int \frac{dr}{1 + \sqrt{r}}$

118. $\int \frac{4x^3 - 20x}{x^4 - 10x^2 + 9}\,dx$

119. $\int \frac{x^3}{1 + x^2}\,dx$

120. $\int \frac{x^2}{1 + x^3}\,dx$

121. $\int \frac{1 + x^2}{1 + x^3}\,dx$

122. $\int \frac{1 + x^2}{(1 + x)^3}\,dx$

123. $\int \sqrt{x} \cdot \sqrt{1 + \sqrt{x}}\,dx$

124. $\int \sqrt{1 + \sqrt{1 + x}}\,dx$

125. $\int \frac{1}{\sqrt{x} \cdot \sqrt{1 + x}}\,dx$

126. $\int_0^{1/2} \sqrt{1 + \sqrt{1 - x^2}}\,dx$

127. $\int \frac{\ln x}{x + x \ln x}\,dx$

128. $\int \frac{1}{x \cdot \ln x \cdot \ln (\ln x)}\,dx$

129. $\int \frac{x^{\ln x} \ln x}{x}\,dx$

130. $\int (\ln x)^{\ln x} \left[\frac{1}{x} + \frac{\ln (\ln x)}{x}\right] dx$

131. $\int \frac{1}{x\sqrt{1 - x^4}}\,dx$

132. $\int \frac{\sqrt{1 - x}}{x}\,dx$

133. a. Show that $\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$.

b. Use part (a) to evaluate

$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx.$$

134. $\int \frac{\sin x}{\sin x + \cos x}\,dx$

135. $\int \frac{\sin^2 x}{1 + \sin^2 x}\,dx$

136. $\int \frac{1 - \cos x}{1 + \cos x}\,dx$

CHAPTER 8 Additional and Advanced Exercises

Evaluating Integrals

Evaluate the integrals in Exercises 1–6.

1. $\int (\sin^{-1} x)^2\, dx$

2. $\int \frac{dx}{x(x+1)(x+2)\cdots(x+m)}$

3. $\int x \sin^{-1} x\, dx$

4. $\int \sin^{-1} \sqrt{y}\, dy$

5. $\int \frac{dt}{t - \sqrt{1 - t^2}}$

6. $\int \frac{dx}{x^4 + 4}$

Evaluate the limits in Exercise 7 and 8.

7. $\lim_{x\to\infty} \int_{-x}^{x} \sin t\, dt$

8. $\lim_{x\to 0^+} x \int_{x}^{1} \frac{\cos t}{t^2}\, dt$

Evaluate the limits in Exercise 9 and 10 by identifying them with definite integrals and evaluating the integrals.

9. $\lim_{n\to\infty} \sum_{k=1}^{n} \ln \sqrt[n]{1 + \frac{k}{n}}$

10. $\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{\sqrt{n^2 - k^2}}$

Applications

11. Finding arc length Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t}\, dt, \quad 0 \le x \le \pi/4.$$

12. Finding arc length Find the length of the graph of the function $y = \ln(1 - x^2)$, $0 \le x \le 1/2$.

13. Finding volume The region in the first quadrant that is enclosed by the x-axis and the curve $y = 3x\sqrt{1 - x}$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

14. Finding volume The region in the first quadrant that is enclosed by the x-axis, the curve $y = 5/\left(x\sqrt{5 - x}\right)$, and the lines $x = 1$ and $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

15. Finding volume The region in the first quadrant enclosed by the coordinate axes, the curve $y = e^x$, and the line $x = 1$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

16. Finding volume The region in the first quadrant that is bounded above by the curve $y = e^x - 1$, below by the x-axis, and on the right by the line $x = \ln 2$ is revolved about the line $x = \ln 2$ to generate a solid. Find the volume of the solid.

17. Finding volume Let R be the "triangular" region in the first quadrant that is bounded above by the line $y = 1$, below by the curve $y = \ln x$, and on the left by the line $x = 1$. Find the volume of the solid generated by revolving R about

a. the x-axis. **b.** the line $y = 1$.

18. Finding volume (*Continuation of Exercise 17.*) Find the volume of the solid generated by revolving the region R about

a. the y-axis. **b.** the line $x = 1$.

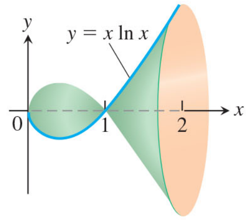
19. Finding volume The region between the x-axis and the curve

$$y = f(x) = \begin{cases} 0, & x = 0 \\ x \ln x, & 0 < x \le 2 \end{cases}$$

is revolved about the x-axis to generate the solid shown here.

a. Show that f is continuous at $x = 0$.

b. Find the volume of the solid.



20. Finding volume The infinite region bounded by the coordinate axes and the curve $y = -\ln x$ in the first quadrant is revolved about the x-axis to generate a solid. Find the volume of the solid.

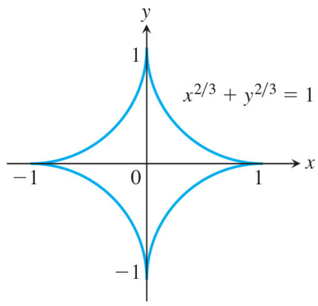
21. Centroid of a region Find the centroid of the region in the first quadrant that is bounded below by the x-axis, above by the curve $y = \ln x$, and on the right by the line $x = e$.

22. Centroid of a region Find the centroid of the region in the plane enclosed by the curves $y = \pm(1 - x^2)^{-1/2}$ and the lines $x = 0$ and $x = 1$.

23. Length of a curve Find the length of the curve $y = \ln x$ from $x = 1$ to $x = e$.

24. Finding surface area Find the area of the surface generated by revolving the curve in Exercise 23 about the y-axis.

25. The surface generated by an astroid The graph of the equation $x^{2/3} + y^{2/3} = 1$ is an *astroid* (see accompanying figure). Find the area of the surface generated by revolving the curve about the x-axis.



26. Length of a curve Find the length of the curve

$$y = \int_1^x \sqrt{\sqrt{t} - 1}\, dt, \qquad 1 \le x \le 16.$$

27. For what value or values of a does

$$\int_1^\infty \left(\frac{ax}{x^2 + 1} - \frac{1}{2x}\right) dx$$

converge? Evaluate the corresponding integral(s).

28. For each $x > 0$, let $G(x) = \int_0^\infty e^{-xt}\,dt$. Prove that $xG(x) = 1$ for each $x > 0$.

29. Infinite area and finite volume What values of p have the following property: The area of the region between the curve $y = x^{-p}$, $1 \le x < \infty$, and the x-axis is infinite but the volume of the solid generated by revolving the region about the x-axis is finite.

30. Infinite area and finite volume What values of p have the following property: The area of the region in the first quadrant enclosed by the curve $y = x^{-p}$, the y-axis, the line $x = 1$, and the interval $[0, 1]$ on the x-axis is infinite but the volume of the solid generated by revolving the region about one of the coordinate axes is finite.

31. Integrating the square of the derivative If f is continuously differentiable on $[0, 1]$ and $f(1) = f(0) = -1/6$, prove that

$$\int_0^1 (f'(x))^2\,dx \ge 2\int_0^1 f(x)\,dx + \frac{1}{4}.$$

Hint: Consider the inequality $0 \le \int_0^1 \left(f'(x) + x - \frac{1}{2}\right)^2 dx.$

Source: Mathematics Magazine, vol. 84, no. 4, Oct. 2011.

32. (*Continuation of Exercise 31.*) If f is continuously differentiable on $[0, a]$ for $a > 0$, and $f(a) = f(0) = b$, prove that

$$\int_0^a (f'(x))^2\,dx \ge 2\int_0^a f(x)\,dx - \left(2ab + \frac{a^3}{12}\right).$$

Hint: Consider the inequality $0 \le \int_0^a \left(f'(x) + x - \frac{a}{2}\right)^2 dx.$

Source: Mathematics Magazine, vol. 84, no. 4, Oct. 2011.

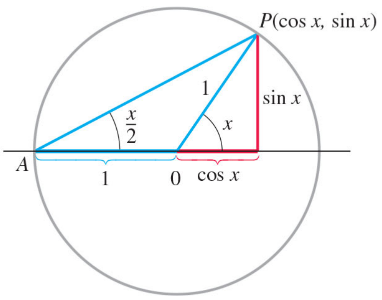
The Substitution $z = \tan(x/2)$

The substitution

$$z = \tan\frac{x}{2} \tag{1}$$

reduces the problem of integrating a rational expression in $\sin x$ and $\cos x$ to a problem of integrating a rational function of z. This in turn can be integrated by partial fractions.

From the accompanying figure



we can read the relation

$$\tan\frac{x}{2} = \frac{\sin x}{1 + \cos x}.$$

To see the effect of the substitution, we calculate

$$\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1 = \frac{2}{\sec^2(x/2)} - 1$$

$$= \frac{2}{1 + \tan^2(x/2)} - 1 = \frac{2}{1 + z^2} - 1$$

$$\cos x = \frac{1 - z^2}{1 + z^2}, \tag{2}$$

and

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\,\frac{\sin(x/2)}{\cos(x/2)}\cdot\cos^2\left(\frac{x}{2}\right)$$

$$= 2\tan\frac{x}{2}\cdot\frac{1}{\sec^2(x/2)} = \frac{2\tan(x/2)}{1 + \tan^2(x/2)}$$

$$\sin x = \frac{2z}{1 + z^2}. \tag{3}$$

Finally, $x = 2\tan^{-1} z$, so

$$dx = \frac{2\,dz}{1 + z^2}. \tag{4}$$

Examples

a.
$$\int \frac{1}{1 + \cos x}\,dx = \int \frac{1 + z^2}{2}\,\frac{2\,dz}{1 + z^2}$$
$$= \int dz = z + C$$
$$= \tan\left(\frac{x}{2}\right) + C$$

b.
$$\int \frac{1}{2 + \sin x}\,dx = \int \frac{1 + z^2}{2 + 2z + 2z^2}\,\frac{2\,dz}{1 + z^2}$$
$$= \int \frac{dz}{z^2 + z + 1} = \int \frac{dz}{(z + (1/2))^2 + 3/4}$$
$$= \int \frac{du}{u^2 + a^2}$$
$$= \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$
$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2z + 1}{\sqrt{3}} + C$$
$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{1 + 2\tan(x/2)}{\sqrt{3}} + C$$

Use the substitutions in Equations (1)–(4) to evaluate the integrals in Exercises 33–40. Integrals like these arise in calculating the average angular velocity of the output shaft of a universal joint when the input and output shafts are not aligned.

33. $\displaystyle\int \frac{dx}{1 - \sin x}$

34. $\displaystyle\int \frac{dx}{1 + \sin x + \cos x}$

35. $\displaystyle\int_0^{\pi/2} \frac{dx}{1 + \sin x}$

36. $\displaystyle\int_{\pi/3}^{\pi/2} \frac{dx}{1 - \cos x}$

37. $\displaystyle\int_0^{\pi/2} \frac{d\theta}{2 + \cos\theta}$

38. $\displaystyle\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\, d\theta}{\sin\theta\cos\theta + \sin\theta}$

39. $\displaystyle\int \frac{dt}{\sin t - \cos t}$

40. $\displaystyle\int \frac{\cos t\, dt}{1 - \cos t}$

Use the substitution $z = \tan(\theta/2)$ to evaluate the integrals in Exercises 41 and 42.

41. $\displaystyle\int \sec\theta\, d\theta$

42. $\displaystyle\int \csc\theta\, d\theta$

The Gamma Function and Stirling's Formula

Euler's gamma function $\Gamma(x)$ ("gamma of x"; Γ is a Greek capital g) uses an integral to extend the factorial function from the nonnegative integers to other real values. The formula is

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt, \quad x > 0.$$

For each positive x, the number $\Gamma(x)$ is the integral of $t^{x-1}e^{-t}$ with respect to t from 0 to ∞. Figure 8.28 shows the graph of Γ near the origin. You will see how to calculate $\Gamma(1/2)$ if you do Additional Exercise 23 in Chapter 15.

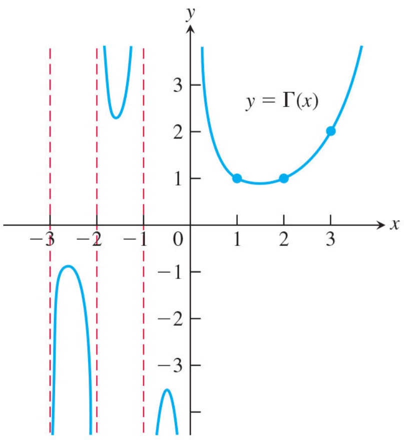


FIGURE 8.28 Euler's gamma function $\Gamma(x)$ is a continuous function of x whose value at each positive integer $n + 1$ is $n!$. The defining integral formula for Γ is valid only for $x > 0$, but we can extend Γ to negative noninteger values of x with the formula $\Gamma(x) = (\Gamma(x + 1))/x$, which is the subject of Exercise 43.

43. If n is a nonnegative integer, $\Gamma(n + 1) = n!$

a. Show that $\Gamma(1) = 1$.

b. Then apply integration by parts to the integral for $\Gamma(x + 1)$ to show that $\Gamma(x + 1) = x\Gamma(x)$. This gives

$$\begin{aligned}\Gamma(2) &= 1\Gamma(1) = 1\\ \Gamma(3) &= 2\Gamma(2) = 2\\ \Gamma(4) &= 3\Gamma(3) = 6\\ &\vdots\\ \Gamma(n + 1) &= n\,\Gamma(n) = n!\end{aligned} \tag{5}$$

c. Use mathematical induction to verify Equation (5) for every nonnegative integer n.

44. Stirling's formula Scottish mathematician James Stirling (1692–1770) showed that

$$\lim_{x\to\infty}\left(\frac{e}{x}\right)^x \sqrt{\frac{x}{2\pi}}\,\Gamma(x) = 1,$$

so, for large x,

$$\Gamma(x) = \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}\,(1 + \varepsilon(x)), \qquad \varepsilon(x) \to 0 \text{ as } x \to \infty. \tag{6}$$

Dropping $\varepsilon(x)$ leads to the approximation

$$\Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \qquad \textbf{(Stirling's formula).} \tag{7}$$

a. Stirling's approximation for $n!$ Use Equation (7) and the fact that $n! = n\Gamma(n)$ to show that

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2n\pi} \qquad \textbf{(Stirling's approximation).} \tag{8}$$

As you will see if you do Exercise 114 in Section 10.1, Equation (8) leads to the approximation

$$\sqrt[n]{n!} \approx \frac{n}{e}. \tag{9}$$

T **b.** Compare your calculator's value for $n!$ with the value given by Stirling's approximation for $n = 10, 20, 30, \ldots,$ as far as your calculator can go.

T **c.** A refinement of Equation (6) gives

$$\Gamma(x) = \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}\, e^{1/(12x)}(1 + \varepsilon(x))$$

or

$$\Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}\, e^{1/(12x)},$$

which tells us that

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2n\pi}\, e^{1/(12n)}. \tag{10}$$

Compare the values given for 10! by your calculator, Stirling's approximation, and Equation (10).

CHAPTER 8 Technology Application Projects

Mathematica/Maple Projects

Projects can be found within MyMathLab.

- ***Riemann, Trapezoidal, and Simpson Approximations***
  Part I: Visualize the error involved in using Riemann sums to approximate the area under a curve.
  Part II: Build a table of values and compute the relative magnitude of the error as a function of the step size Δx.
  Part III: Investigate the effect of the derivative function on the error.
  Parts IV and V: Trapezoidal Rule approximations.
  Part VI: Simpson's Rule approximations.
- ***Games of Chance: Exploring the Monte Carlo Probabilistic Technique for Numerical Integration***
  Graphically explore the Monte Carlo method for approximating definite integrals.
- ***Computing Probabilities with Improper Integrals***
  More explorations of the Monte Carlo method for approximating definite integrals.