Tutorial Calculus2

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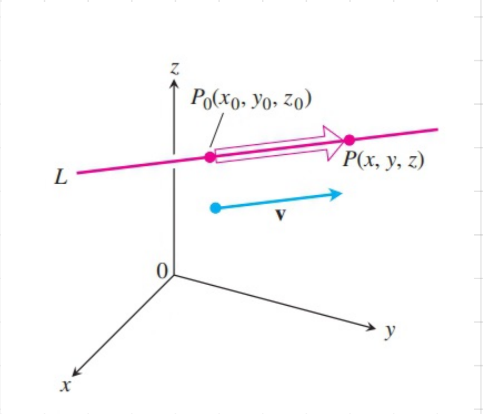
Lines and Plans in space

Line and Segment in space

- ใน plane : เส้น = จุด + ค่าของความชัน
- ใน space : เส้น = จุด + เวกเตอร์ทิศทางของเส้น



ค่า scalar ที่ขึ้นอยู่กับตำแหน่งของจุด P

จากรูป $\overrightarrow{P_0P} = t\underline{v}$

$$(x-x_0)\underline{i} + (y-y_0)\underline{j} + (z-z_0)\underline{k} = t(v_1\underline{i} + v_2\underline{j} + v_3\underline{k})$$

$$\underbrace{x\underline{i} + y\underline{j} + z\underline{k}}_{\underline{r}} = \underbrace{x_0\underline{i} + y_0\underline{j} + z_0\underline{k}}_{\underline{r}_0} + t(v_1\underline{i} + v_2\underline{j} + v_3\underline{k})$$

Vector Equation for a line

สมการของเวกเตอร์เส้น L ที่ผ่านจุด $P_0(x_0, y_0, z_0)$ ขนานกับ $\underline{v}$

$$\underline{r}(t) = \underline{r}_0 + t\underline{v} \qquad -\infty < t < \infty$$

เวกเตอร์ตำแหน่งของ P　　　　เวกเตอร์ตำแหน่งของ P_0

Parametic Equation for a line

$$x = x_0 + tv_1 \quad , \quad y = y_0 + tv_2 \quad , \quad z = z_0 + tv_3$$

ถ้ามองว่าเส้นตรง คือเส้นทางการเคลื่อนที่ของวัตถุที่มีจุดเริ่มต้นคือ $P_0(x_0, y_0, z_0)$ และเคลื่อนที่ในทิศทางของ $\underline{v}$ จะได้

$$\underline{r}(t) = \underline{r}_0 + t\underline{v}$$

$$= \underline{r}_0 + t|\underline{v}| \cdot \frac{\underline{v}}{|\underline{v}|}$$

ทิศทาง

จุดเริ่มต้น　　เวลา　　อัตราเร็ว

Example 1 : parametic equation for the line through $P(\underbrace{-3, 2, -3}_{P_0})$ and $Q(1,-1,4)$

$x_0 \; y_0 \; z_0 \qquad x_1 \; y_1 \; z_1$

Solution

$$\overrightarrow{PQ} = \overset{x_1 - x_0}{(1-(-3))}\underline{i} + \overset{y_1 - y_0}{(-1-2)}\underline{j} + \overset{z_1 - z_0}{(4-(-3))}\underline{k}$$

$$= \overset{v_x}{4\underline{i}} - \overset{v_y}{3\underline{j}} + \overset{v_z}{7\underline{k}} \longrightarrow \underline{v}$$

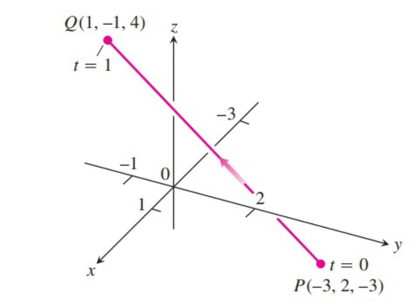
จาก Parametic Equation

$$x = \overset{x_0 + v_x t}{-3+4t}, \quad y = 2-3t, \quad z = -3+7t \quad ※$$

Example 2 : Parameterize the line segment joining the points $P(-3,2,-3)$ and $Q(1,-1,4)$

Solution จากตัวอย่างที่ 1 จะได้ parametic Equation

$$x = -3+4t, \quad y = 2-3t, \quad z = -3+7t$$



จะเห็นว่า

ลองแทนค่า t

$$(x, y, z) = (-3+4t, 2-3t, -3+7t)$$

เส้นตรงนี้จะผ่านจุด $P(-3,2,-3)$ เมื่อ $t = 0$

และผ่านจุด $Q(1,-1,4)$ เมื่อ $t = 1$

ดังนั้นสามารถจำกัดขอบเขตได้เป็น $0 \le t \le 1$ ※

Example 3 : A helicopter is to fly direcly from helipad at the origin in the direction of the point $(1,1,1)$ at a speed of 60 ft/sec. What is the position of the helicopter after 10 sec?

P_0 $(0,0,0)$ P_1 $t = 0$

Solution เฮลิคอปเตอร์บินจาก $\underbrace{(0,0,0)}_{P_0}$ ไปยัง $\underbrace{(1,1,1)}_{P_1}$

$$\underline{u} = \frac{1}{\sqrt{3}}(\underline{i} + \underline{j} + \underline{k})$$

จากสมการ

$$\underline{r}(t) = \underline{r_0} + t|\underline{v}|\left(\frac{\underline{v}}{|\underline{v}|}\right)$$

$$= 0 + 10(60)\left(\frac{1}{\sqrt{3}}(\underline{i} + \underline{j} + \underline{k})\right)$$

$$= 200\sqrt{3}(\underline{i} + \underline{j} + \underline{k}) \quad ※$$

Note : การหา unit vector (บอกทิศ)

$$\underline{u} = \frac{1}{|\overrightarrow{P_0P_1}|} = \frac{1}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}$$

$$\boxed{r(t) = r_0 + t\vec{v}}$$

เส้นตรง // $\vec{v}$ ผ่านจุด r_0

$(x-x_0)\vec{i} + (y-y_0)\vec{j} + (z-z_0)\vec{k} = t$

$$\boxed{x = x_0 + tv_1 \qquad y = y_0 + tv_2 \qquad z = z_0 + tv_3}$$

$t \in (-\infty, \infty)$

t_0 → A —— B ← t

$t \in [t_0, t]$ (0 → t_0, เลข → t)

A $(-3,2,-3)$ B $(1,-1,4)$

P_0

$\vec{AB} = \langle 4, -3, 7 \rangle = \vec{v}$ (เวกเตอร์ที่ขนานกัน)

$x = -3 + 4t$ $\quad y = 2 - 3t$ $\quad z = -3 + 7t$

$t_0 = 0$ $\quad x = -3 + 4t$ $\quad t = 1$

$t \in [0, 1]$

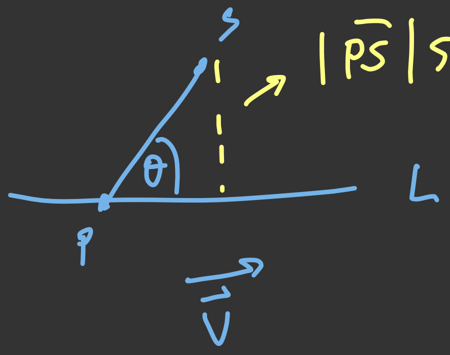
$$r(t) = r_0 + t\vec{v} = r_0 + t\,|v|\left(\frac{v}{|v|}\right)$$

$(0,0,0)$ $\quad (1,1,1)$

H —— 60 kph

$$r(t) = r_0 \times t\,[60]\left[\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle\right]$$

point to line

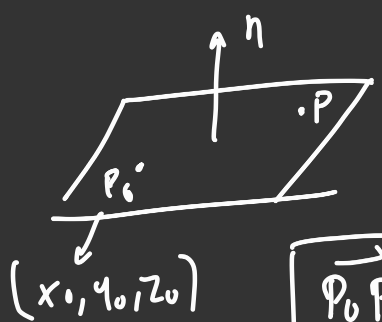


$|\overline{PS}|\sin\theta \rightarrow \dfrac{|\overline{PS}||v|\sin\theta}{|v|}$

$$\frac{|\vec{PS}\times\vec{v}|}{|v|}$$

plane in space

$n = \langle A, B, C \rangle$



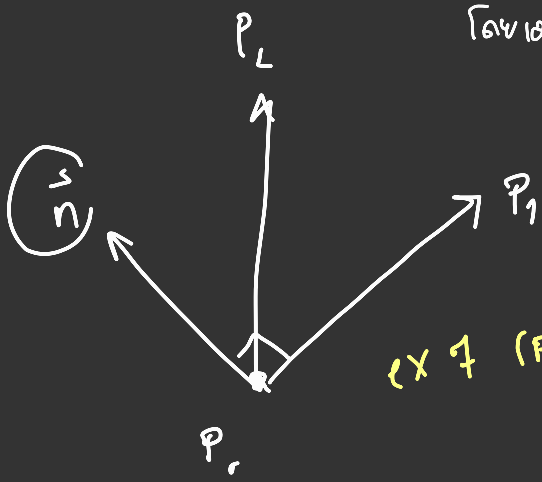
$\boxed{\vec{P_0P} \perp n}$

$\boxed{\vec{P_0P} \cdot n = 0}$

$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$

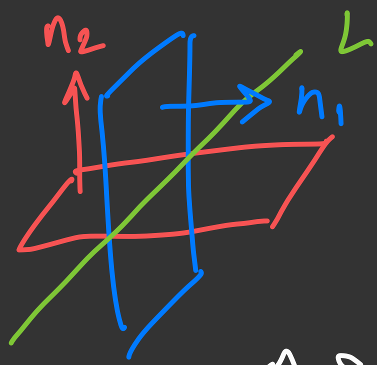
$$\boxed{Ax + By + Cz = \underbrace{x_0A + y_0B + z_0C}_{\text{constant}}}$$

↳ สมการนี้ หาเส้นตั้งฉากระนาบได้
โดยเอา สปส. ข้างหน้ามาแทนค่าได้เลย



ex 7 (p. 711)

Line of intersection



$$\boxed{n_1 \times n_2 = \vec{v}}$$

$$\vec{v} \parallel \vec{L}$$

ตัดจริง

$$\_\_x + \_\_y + \_\_z = \_\_$$

$$\_\_x + \_\_y + \_\_z = \_\_$$

$$(x_{\text{ตัด}}, y_{\text{ตัด}}, z_{\text{ตัด}})$$

$$r(t) = r_0 + tv$$

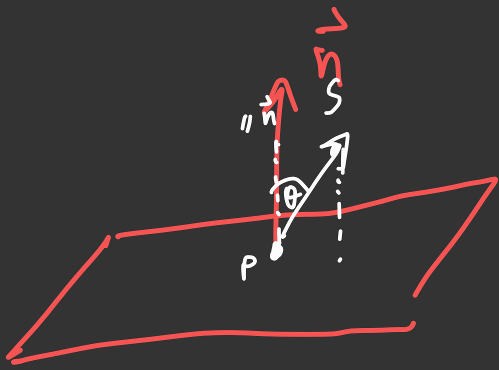
$$x = x_{\text{ตัด}} + tv_1$$

เส้นไปตัดกับระนาบ

$$\_\_x + \_\_y + \_\_z = \_\_$$

$$L: \quad x = x_0 + tv_1 \qquad y = y_0 + tv_2 \qquad z = z_0 + tv_3$$

point to plane

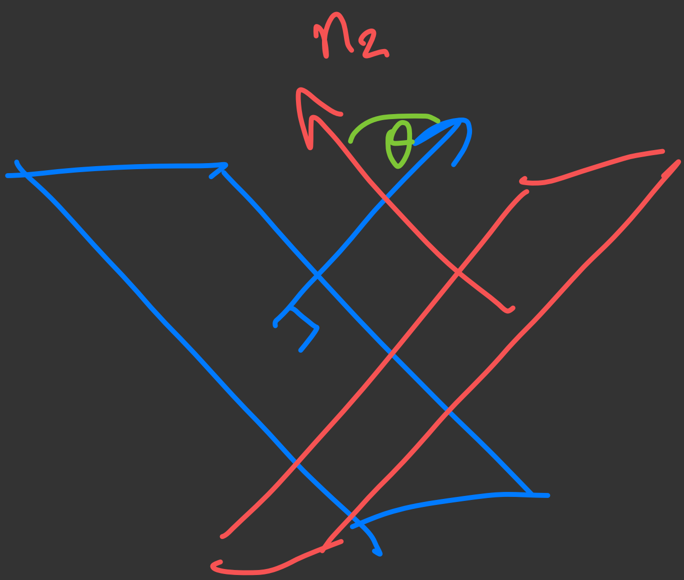


$$\text{สูง} = |\vec{PS}| \cos\theta \left(\frac{|\vec{n}|}{|n|}\right)$$

$$= \frac{|\vec{PS}|\,|n|\cos\theta}{|n|}$$

$$= \frac{|\vec{PS}\cdot n|}{|n|}$$

Angle between plane



$$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1|\,|\vec{n}_2|\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}\right)$$

rad

Distance from a Point to a line is Space

ระยะทางจากจุด S ไปยังเส้นที่ผ่านจุด P และขนานกับ $\underline{v}$

$$d = \frac{|\overrightarrow{PS} \times \underline{v}|}{|\underline{v}|}$$

Example 4 : Find the distance from the point $S(1,1,5)$ to line

$$L : x = 1+t \ , \ y = 3-t \ , \ z = 2t$$

Paramatric Equation

Solution จาก L จะได้ว่า $P(1, 3, 0)$

$$\overrightarrow{PS} = (1-1)\underline{i} + (1-3)\underline{j} + (5-0)\underline{k}$$

$$= -2\underline{j} + 5\underline{k}$$

$$\overrightarrow{PS} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \underline{i} - 5\underline{j} + 2\underline{k}$$

$$\begin{vmatrix} \text{ก} & \text{ล} \end{vmatrix} i - \begin{vmatrix} \text{ห} & \text{ล} \end{vmatrix} j + \begin{vmatrix} \text{ห} & \text{ก} \end{vmatrix} k$$

$$d = \frac{|\overrightarrow{PS} \times \underline{v}|}{|\underline{v}|}$$

$$= \frac{\sqrt{1+25+4}}{\sqrt{1+1+4}}$$

$$= \sqrt{5}$$

✗✗

An Equation for Plane in Space

Equation for a plane

เมื่อระนาบผ่านจุด $P_0(x_0, y_0, z_0)$ ตั้งฉากกับ

$\underline{n} = A\underline{i} + B\underline{j} + C\underline{k}$

- Vector equation : $\underline{n} \cdot \overrightarrow{P_0P} = 0$
- Component equation : $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$
- Component equation symplified : $Ax + By + Cz = D$ เมื่อ $D = Ax_0 + By_0 + Cz_0$

Example 5 Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\underline{n} = 5\underline{i} + 2\underline{j} - \underline{k}$

Solution Component equation

$$5(x-(-3)) + 2(y-0) - (z-7) = 0$$

$$5x + 15 + 2y - z + 7 = 0$$

$$5x + 2y - z = -22 \quad \#$$

Example 6 Find an equation for the plane through $A(0,0,1)$, $B(2,0,0)$, and $C(0,3,0)$

Solution หา $\underline{n}$ จากระนาบ

$$n = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\underline{i} + 2\underline{j} + 6\underline{k}$$

$$3(x-0) + 2(y-0) + 6(z-1) = 0$$

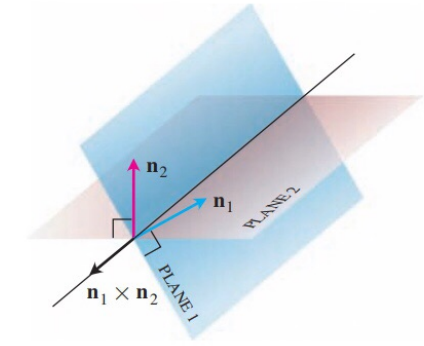
$$3x + 2y + 6 = -6$$

Lines of Intersection

- ระนาบสองระนาบจะขนานกัน ก็ต่อเมื่อ normal vector ขนานกัน

$$\underline{n}_1 = k\,\underline{n}_2$$

Example 7 Find a vector parallel to the line of intersection of the plane $\underbrace{3x - 6y - 2z = 15}_{n_1}$ and $\underbrace{2x + y - 2x = 5}_{n_2}$



Solution เส้นของการอินเตอร์เซคของสองระนาบจะตั้งฉากกับ normal vector $\underline{n}_1$ และ $\underline{n}_2$ ดังนั้นจะขนานกับ $\underline{n}_1 \times \underline{n}_2$

$$\underline{n}_1 \times \underline{n}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\underline{i} + 2\underline{j} + 15\underline{k}$$

Example 8 Find the point where the line

$$x = \frac{8}{3} + 2t, \quad y = -2t, \quad z = 1 + t$$

intersects the plane $3x + 2y + 6z = 6$

Solution จุดบนระนาบจะต้องทำให้สมการของระนาบเป็นจริง

$$3\Big(\underbrace{\tfrac{8}{3} + 2t}_{x}\Big) + 2\big(\underbrace{-2t}_{y}\big) + 6\big(\underbrace{1+t}_{z}\big) = 6$$

$$8 + 6t - 4t + 6 + 6t = 6$$

$$8t = -8$$

$$t = -1$$

จุดของการอินเตอร์เซค คือ (แทนค่า t)

$$(x, y, z)_{t=-1} = \left(\frac{8}{3} - 2,\ 2,\ 1 - 1\right) = \left(\frac{2}{3},\ 2,\ 0\right)$$

Distance from a Point to a Plane

$$\boxed{d = \left| \vec{PS} \cdot \frac{\underline{n}}{|\underline{n}|} \right|}$$

Example 9 Find distance from $S(1,1,3)$ to plane $3x+2y+6z=6$

Solution หาจุด P บนระนาบ และคำนวณความยาวที่ $\vec{PS}$ project บน $\underline{n}$

① จาก ส.ป.ส. ของสมการของระนาบจะได้

$$n = 3\underline{i} + 2\underline{j} + 6\underline{k}$$

② จากสมการของระนาบ สามารถกำหนดจุดได้โดยแทนค่า x, y, z แล้วทำให้สมการเป็นจริง

ในที่นี้เลือก $P(0,3,0)$

③ หา $\vec{PS}$ จะได้

$$\vec{PS} = (1-0)\underline{i} + (1-3)\underline{j} + (3-0)\underline{k} = \underline{i} - 2\underline{j} + 3\underline{k}$$

④ หา $|\underline{n}|$ จะได้

$$|\underline{n}| = \sqrt{3^2 + 2^2 + 6^2} = 7$$

⑤ ระยะทางจาก S ถึงระนาบ คือ

$$d = \left| \vec{PS} \cdot \frac{\underline{n}}{|\underline{n}|} \right|$$

$$= \left| (\underline{i} - 2\underline{j} + 3\underline{k}) \cdot \frac{3\underline{i} + 2\underline{j} + 6\underline{k}}{7} \right|$$

$$= \left| (\underline{i} - 2\underline{j} + 3\underline{k}) \cdot \left(\frac{3}{7}\underline{i} + \frac{2}{7}\underline{j} + \frac{6}{7}\underline{k} \right) \right|$$

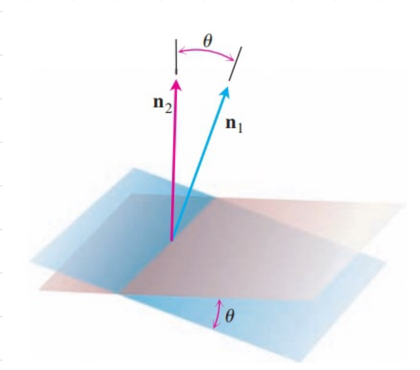
$$= \left| \frac{3}{7} - \frac{4}{7} + \frac{18}{7} \right|$$

$$= \frac{17}{7} \quad \#$$

Angles Between Planes

มุมระหว่าง 2 ระนาบที่ตัดกัน จะเท่ากับมุมระหว่าง normal vector ของทั้งสองระนาบ

$$\boxed{\theta = \cos^{-1}\left(\frac{\underline{n}_1 \cdot \underline{n}_2}{|\underline{n}_1||\underline{n}_2|}\right)}$$



Example 10 Find the angle between the planes $3x-6y-2z=15$ and $2x+y-2z=5$

Solution ① หา normal vector

$$\underline{n}_1 = 3\underline{i} - 6\underline{j} - 2\underline{k}, \quad \underline{n}_2 = 2\underline{i} + \underline{j} - 2\underline{k}$$

② หามุมระหว่าง 2 normal vector

$$\theta = \cos^{-1}\left(\frac{\underline{n}_1 \cdot \underline{n}_2}{|\underline{n}_1||\underline{n}_2|}\right)$$

$$= \cos^{-1}\left(\frac{6-6+4}{\sqrt{3^2+(-6)^2+(-2)^2}\cdot\sqrt{(2)^2+1^2+(-2)^2}}\right)$$

$$= \cos^{-1}\left(\frac{4}{21}\right)$$

$$= 1.38 \text{ radians}$$

จำไว้!! หน่วยของมุมที่ได้เป็น radians นะจ๊ะ

Cylinders and Quadric Surfaces

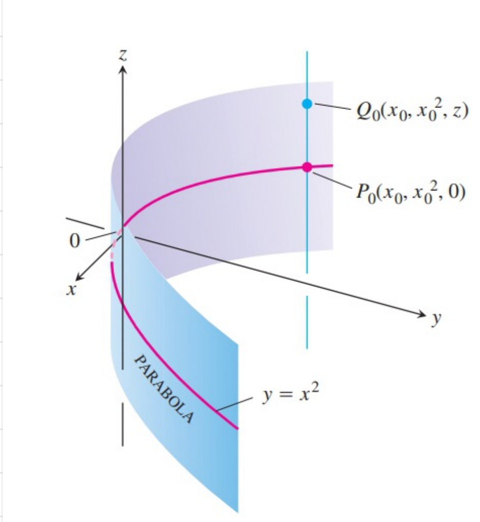
Cylinders

นิยาม พื้นผิวที่สร้างจากการเคลื่อนที่ของเส้นตรงไปตามแนวเส้นโค้ง โดยที่เส้นตรงนั้นต้อง ขนาน กับเส้นตรงที่ fixed ไว้

Example 1 Find equation for the cylinder made by the lines parallel to the z-axis that pass through the parabola $y = x^2$, $z = 0$

Solution จากสมการพาราโบลา จะได้จุด $P_0(x_0, x_0^2, 0)$ เป็นจุดบนระนาบ xy ดังนั้นที่ค่า z ใดๆ จุด $Q(x_0, x_0^2, z)$ จะอยู่บน cylinder เนื่องจากจุด $Q(x_0, x_0^2, z)$ อยู่บนเส้น $x = x_0$ และ $y = x_0^2$ ผ่าน P_0 ขนานกับแกน z

ดังนั้นจุดที่อยู่บนพื้นผิวจะต้องเป็นจุดที่ทำให้ $y = x^2$ เป็นจริง ทำให้เราได้สมการของ cylinder คือ $y = x^2$



Quadric Surface

นิยาม กราฟใน space ของสมการมุม 2 สมการ ใน x, y และ z

Basic quadric surface : ellipsoids , paraboloids, elliptical cones, hyperboloids

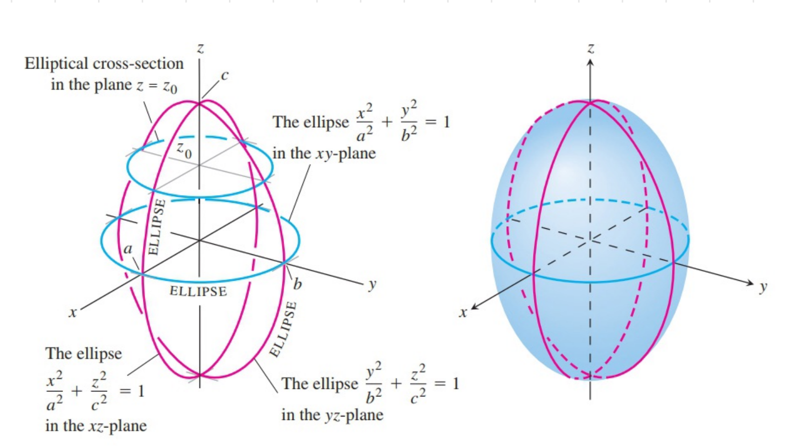
$$Ax^2 + By^2 + Cz^2 + Dz + E$$

General Equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gz + Hy + Iz + J = 0$$

Example 2 ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

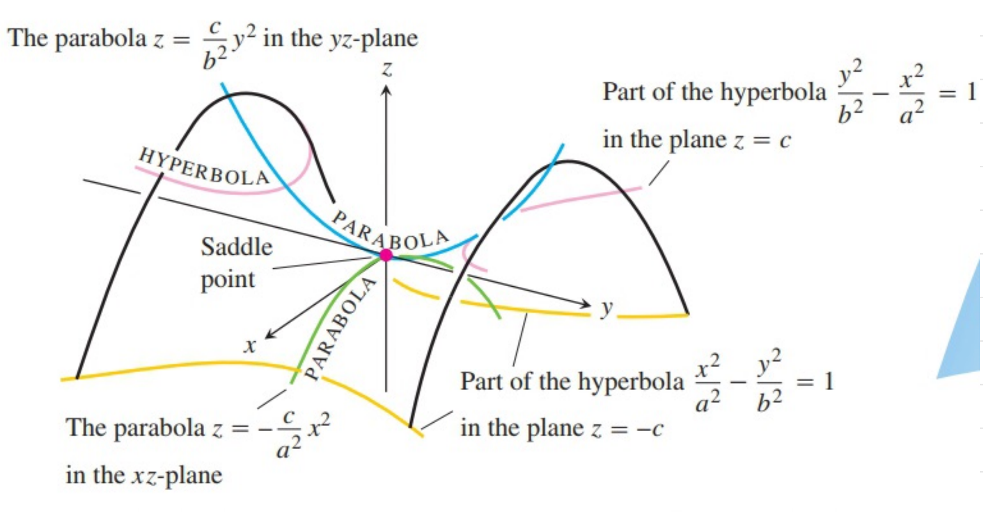


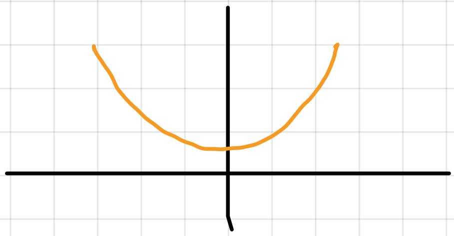
พิจารณา 3 ระนาบ

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ เมื่อ $z = 0$
2. $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$ เมื่อ $y = 0$
3. $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ เมื่อ $x = 0$

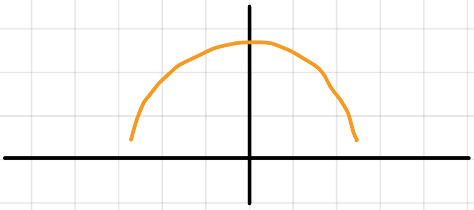
Example 3 Hyperbolic paraboloid

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}$$

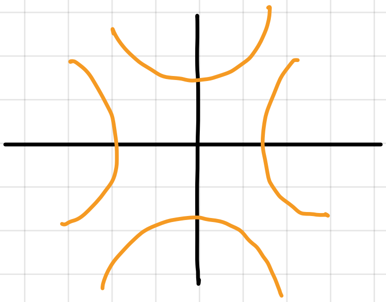




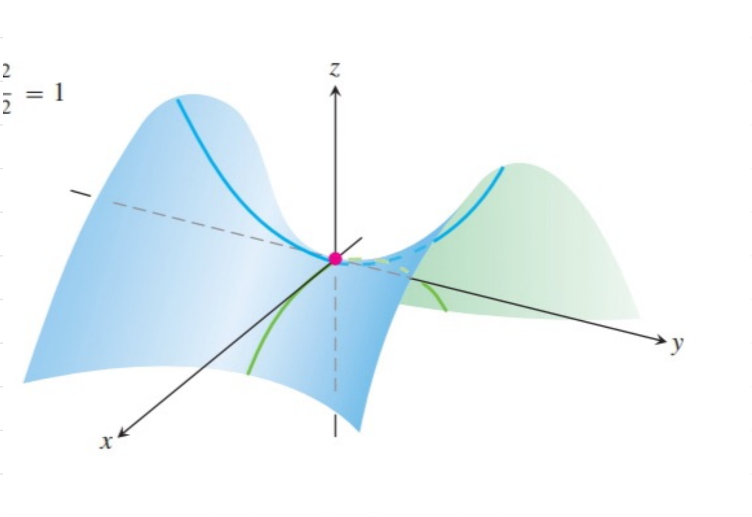
$x = 0$



$y = 0$

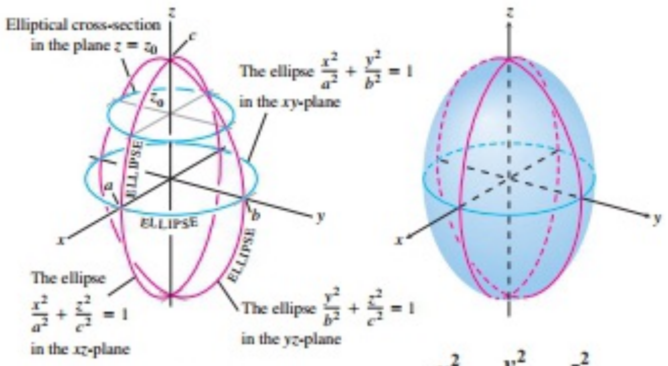


$z = z_0$

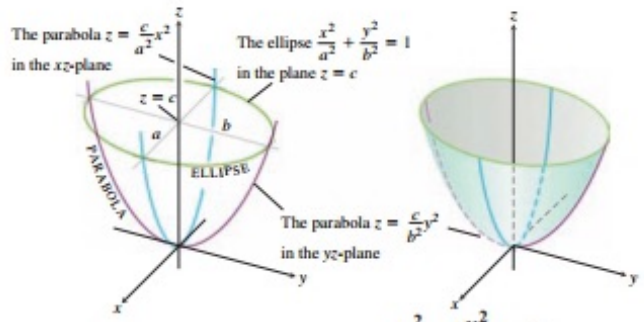


Hint!

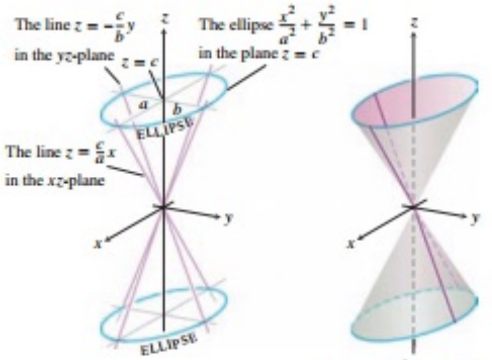
จำสมการของแต่ละรูป แล้วมองหน้าตัดให้เป็น



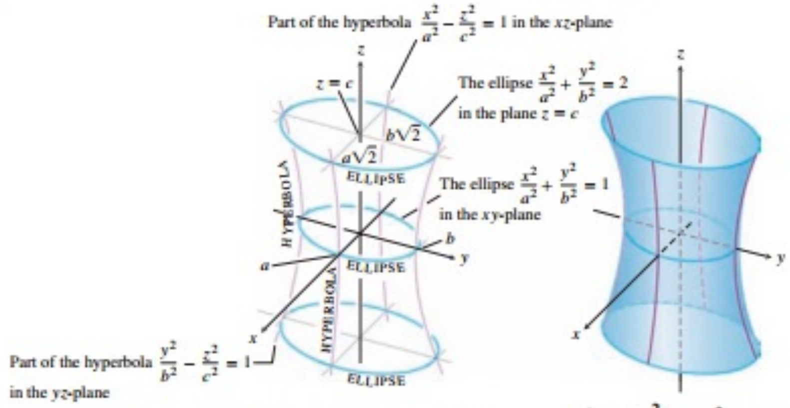
ELLIPSOID $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$



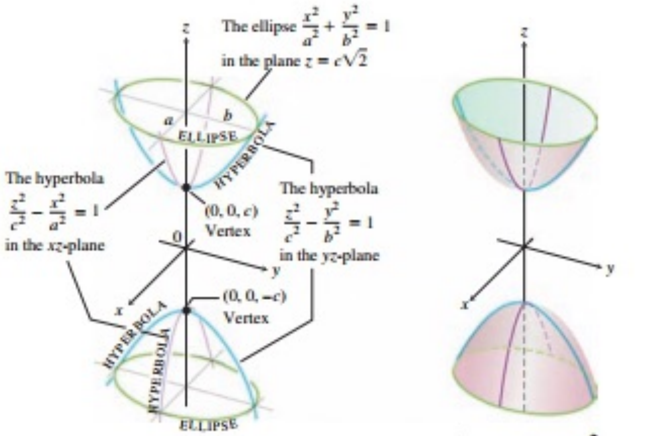
ELLIPTICAL PARABOLOID $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z}{c}$



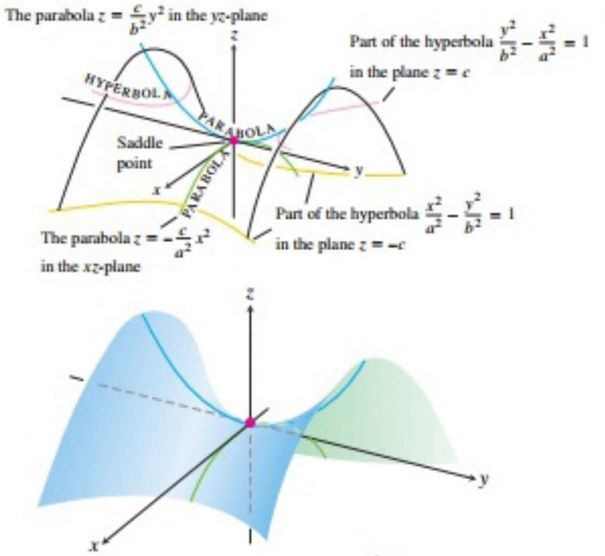
ELLIPTICAL CONE $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$



HYPERBOLOID OF ONE SHEET $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

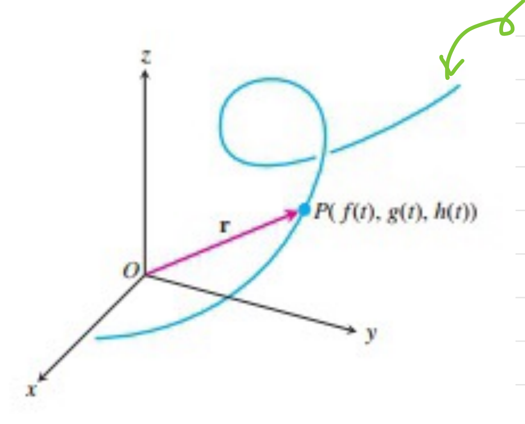


HYPERBOLOID OF TWO SHEETS $\quad \dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\quad \dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = \dfrac{z}{c}, \quad c > 0$

Curves in Space and Their Tangents



curves หรือ เส้นทางการเคลื่อน

ตำแหน่งของวัตถุ ณ เวลา t คือ $P(x, y, z)$

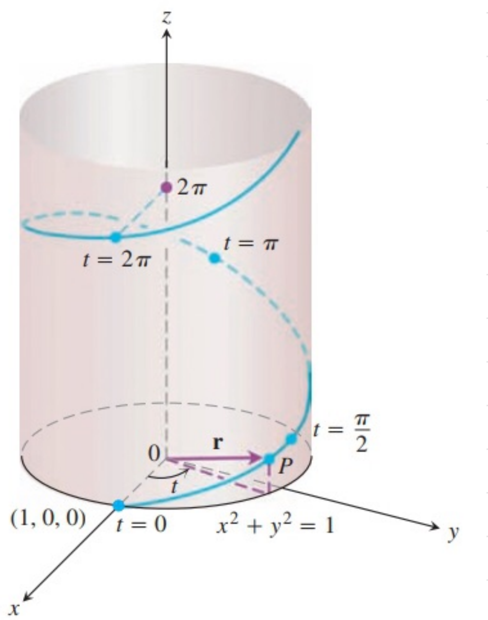
$$x = f(t)\ ,\quad y = g(t)\ ,\quad z = h(t)\ ;\ t \in I$$

สามารถเขียน curve ในรูปเวกเตอร์ได้

$$\boxed{\underline{r}(t) = f(t)\underline{i} + g(t)\underline{j} + h(t)\underline{k}}$$

position vector (บอกตำแหน่งวัตถุ) ... component function

Example 1 Graph the vector function $\underline{r}(t) = (\cos t)\underline{i} + (\sin t)\underline{j} + t\underline{k}$



solution จากรูปจะเห็นว่า $\underline{r}$ จะหมุนรอบทรงกระบอก $x^2+y^2=1$

ดังนั้น $\underline{i}$ และ $\underline{j}$ จะขึ้นอยู่กับ x และ y ของทรงกระบอก

$$x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1$$

$\underline{k}$-component จะเพิ่มขึ้นตาม z

$$z = t$$

$$\therefore \quad x = \cos t\ ,\quad y = \sin t\ ,\quad z = t$$

Limit and Continuity

นิยาม กำหนดให้ $\underline{r}(t) = f(t)\underline{i} + g(t)\underline{j} + h(t)\underline{k}$ เป็น vector function ในโดเมน D

และ $\underline{L}$ เป็นเวกเตอร์

$$\lim_{t \to t_0} \underline{r}(t) = \underline{L}$$

ถ้าทุกๆ ค่า $\varepsilon > 0$ จะมี $\delta > 0$ โดยที่ $t \in D$

$$|\underline{r}(t) - \underline{L}| < \varepsilon \qquad \text{เมื่อ } 0 < |t - t_0| < \delta$$

Example 2 ถ้า $\underline{r}(t) = (\cos t)\underline{i} + (\sin t)\underline{j} + t\underline{k}$; $t_0 = \pi/4$

Solution $\lim\limits_{t \to \pi/4} \underline{r}(t) = \left(\lim\limits_{t \to \pi/4} \cos t\right)\underline{i} + \left(\lim\limits_{t \to \pi/4} \sin t\right)\underline{j} + \left(\lim\limits_{t \to \pi/4} t\right)\underline{k}$

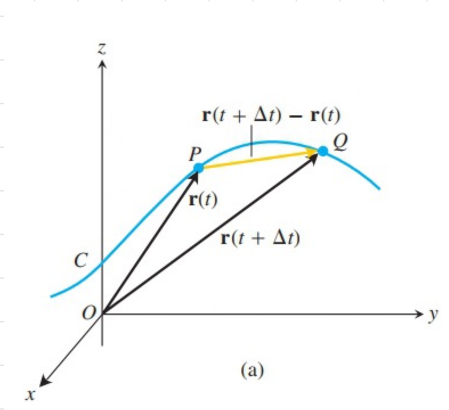
$$\underline{L} = \frac{\sqrt{2}}{2}\underline{i} + \frac{\sqrt{2}}{2}\underline{j} + \frac{\pi}{4}\underline{k} \quad \#$$

นิยาม vector function $\underline{r}(t)$ จะต่อเนื่องที่จุด $t = t_0$ ในโดเมน ถ้า $\lim\limits_{t \to t_0} \underline{r}(t) = \underline{r}(t_0)$
function นี้จะเป็นแบบต่อเนื่องก็ต่อเมื่อมีความต่อเนื่องทุกจุด

Derivative and Motion

position vector $\underline{r}(t) = f(t)\underline{i} + g(t)\underline{j} + h(t)\underline{k}$ โดยที่ f, g, h are differentiable

$$\Delta\underline{r}(t) = \underline{r}(t + \Delta t) - \underline{r}(t)$$
$$= f(t+\Delta t)\underline{i} + g(t+\Delta t)\underline{j} + h(t+\Delta t)\underline{k} - [f(t)\underline{i} + g(t)\underline{j} + h(t)\underline{k}]$$
$$= \Delta f\,\underline{i} + \Delta g\,\underline{j} + \Delta h\,\underline{k}$$



(a)

เมื่อ $t \to 0$

1. $Q \to P$ บนเส้นโค้ง
2. $\overrightarrow{PQ} \to$ เส้นสัมผัสที่จุด P
3. $\lim\limits_{\Delta t \to 0} \frac{\Delta \underline{r}}{\Delta t} = \left(\lim\limits_{\Delta t \to 0} \frac{\Delta f}{\Delta t}\right)\underline{i} + \left(\lim\limits_{\Delta t \to 0} \frac{\Delta g}{\Delta t}\right)\underline{j} + \left(\lim\limits_{\Delta t \to 0} \frac{\Delta h}{\Delta t}\right)\underline{k}$

$$\boxed{\underline{r}'(t) = \frac{d\underline{r}}{dt} = \frac{df}{dt}\underline{i} + \frac{dg}{dt}\underline{j} + \frac{dh}{dt}\underline{k}}$$

- $\underline{r}$ is differentiable if it is differentiable at ALL point on D.
- $\underline{r}$ is SMOOTH if $\frac{d\underline{r}}{dt}$ is continuous and never 0

$\frac{df}{dt}, \frac{dg}{dt}, \frac{dh}{dt}$ ไม่เป็น 0 พร้อมกัน

ความเร็ว และ ความเร่ง

① ความเร็ว $\quad \underline{v} = \dfrac{d\underline{r}}{dt}$

② อัตราเร็ว $\quad \text{speed} = |\underline{v}|$

③ ความเร่ง $\quad a = \dfrac{d\underline{v}}{dt} = \dfrac{d^2\underline{r}}{dt^2}$

④ ทิศทางการเคลื่อนที่ $\quad \text{unit vector} = \dfrac{\underline{v}}{|\underline{v}|}$

Example 3 Find the velocity, speed and acceleration of particle when whose motion in Space is given by the position vector $\underline{r}(t) = 2\cos t\,\underline{i} + 2\sin t\,\underline{j} + 5\cos^2 t\,\underline{k}$

Solution

$$\underline{v}(t) = \underline{r}'(t) = -2\sin t\,\underline{i} + 2\cos t\,\underline{j} - 10\cos t\sin t\,\underline{k}$$

$$= -2\sin t\,\underline{i} + 2\cos t\,\underline{j} - 5\sin 2t\,\underline{k}$$

$$\underline{a}(t) = \underline{v}'(t) = -2\cos t\,\underline{i} - 2\sin t\,\underline{j} - 10\cos 2t\,\underline{k}$$

$$|\underline{v}(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-5\sin 2t)^2}$$

$$= \sqrt{4 + 25\sin^2 2t}$$

เมื่อ $t = 7\pi/4$ จะได้

$$\underline{v}\left(\frac{7\pi}{4}\right) = \sqrt{2}\,\underline{i} + \sqrt{2}\,\underline{j} + 5\,\underline{k}$$

$$a\left(\frac{7\pi}{4}\right) = -\sqrt{2}\,\underline{i} + \sqrt{2}\,\underline{j}$$

$$\left|\underline{v}\left(\frac{7\pi}{4}\right)\right| = \sqrt{29}$$

Differentiation Rules

Differentiation Rules for Vector Functions

Let **u** and **v** be differentiable vector functions of t, **C** a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\frac{d}{dt}\mathbf{C} = \mathbf{0}$
2. *Scalar Multiple Rules:* $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

   $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
3. *Sum Rule:* $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
4. *Difference Rule:* $\frac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$
5. *Dot Product Rule:* $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
6. *Cross Product Rule:* $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$
7. *Chain Rule:* $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Vector function of Constant length

if $\underline{r}$ is a differentiable vector function of t of constant length, then

$$\underline{r} \cdot \frac{d\underline{r}}{dt} = 0$$

Integrals of Vector Functions ; Project Motion

$$\boxed{\int \underline{r}(t)\,dt = \underline{R}(t) + \underline{C}} \quad \underline{C} = C_1\underline{i} + C_2\underline{j} + C_3\underline{k}$$

$$\boxed{\int_a^b \underline{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\underline{i} + \left(\int_a^b g(t)\,dt\right)\underline{j} + \left(\int_a^b h(t)\,dt\right)\underline{k}}$$

Example 1 Suppose we do not know the path of a hang glider, but only its acceleration vector $\underline{a}(t) = -(3\cos t)\underline{i} - (3\sin t)\underline{j} + 2\underline{k}$. We also know that initially (at time $t=0$) the glider departed from the point $(3,0,0)$ with velocity $\underline{v}(0) = 3\underline{j}$. Find the glider's position as function of t.

Solution จากโจทย์ จะได้ว่า

$$\underline{a}(t) = \frac{d^2\underline{r}}{dt^2} = -(3\cos t)\underline{i} - (3\sin t)\underline{j} + 2\underline{k}$$

$$\underline{v}(0) = 3\underline{j}$$

$$\underline{r}(0) = 3\underline{i}$$

ดังนั้น เราสามารถหา $\underline{v}(t)$ ได้ดังนี้

$$\underline{v}(t) = \int -(3\cos t)\underline{i} - (3\sin t)\underline{j} + 2\underline{k}$$

$$= -3\sin t\,\underline{i} + 3\cos t\,\underline{j} + 2t\,\underline{k} + C_1$$

จาก $\underline{v}(0) = 3\underline{j}$ จะหา C_1 ได้ดังนี้

$$\underline{v}(0) = -3\sin(0)\,\underline{i} + 3\cos(0)\,\underline{j} + 2(0)\,\underline{k} + C_1 = 3\underline{j}$$

$$C_1 = 0$$

$$\therefore \underline{v}(t) = -3\sin t\,\underline{i} + 3\cos t\,\underline{j} + 2t\,\underline{k}$$

หา position vector

$$r(t) = \int v(t)\,dt = \int -3\sin t\,\underline{i} + 3\cos t\,\underline{j} + 2t\underline{k}$$

$$= 3\cos t\,\underline{i} - 3\sin t\,\underline{j} + t^2\,\underline{k} + C_2$$

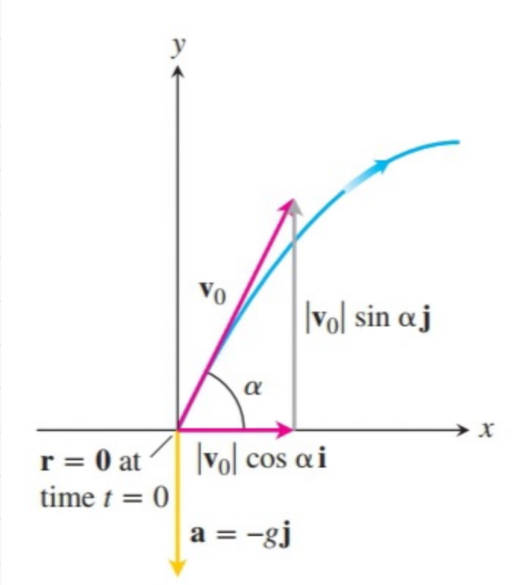
จาก $\underline{r}(0) = 3\underline{i}$ จะหา C_2 ได้ดังนี้

$$r(0) = 3\cos(0)\underline{i} - 3\sin(0)\underline{j} + (0)^2\underline{k} + C_2 = 3\underline{i}$$

$$C_2 = 0$$

$$\therefore r(t) = 3\cos t\,\underline{i} - 3\sin t\,\underline{j} + t^2\underline{k}$$

The vector and Parametic Equation for Ideal Projectile Motion



initial velocity

$$\underline{v}_0 = (|\underline{v}_0|\cos\alpha)\underline{i} + (|\underline{v}_0|\sin\alpha)\underline{j}$$

$$= v_0\cos\alpha\,\underline{i} + v_0\sin\alpha\,\underline{j}$$

initial Position

$$r_0 = 0\underline{i} + 0\underline{j} = \underline{0}$$

Newton's second law of motion

$$\cancel{m}\frac{d^2\underline{r}}{dt^2} = -\cancel{m}g\underline{j}$$

Ideal Projectile Motion Equation

$$\underline{r} = (v_0\cos\alpha)t\,\underline{i} + \left((v_0\sin\alpha)t - \frac{1}{2}gt^2\right)\underline{j}$$

จากสมการข้างต้นจะได้ parametric equation คือ

$$x = (v_0\cos\alpha)t \quad \text{และ} \quad y = (v_0\sin\alpha)t - \frac{1}{2}gt^2$$

Height, Flight Time, and Range for Ideal Projectile Motion

จุดสูงสุด : $\quad y_{max} = \dfrac{(v_0\sin\alpha)^2}{2g}$

เวลา : $\quad t = \dfrac{2v_0\sin\alpha}{g}$

ระยะทาง : $\quad R = \dfrac{v_0^2}{g}\sin 2\alpha$

Example 2 A projectile is fired from the origin over horizontal ground at an initial speed (v_0) of 500 m/sec and a lunch angle (α) of 60°. Where will the projectile be 10 sec later?

Solution

$$\underline{r} = (v_0\cos\alpha)t\,\underline{i} + \left((v_0\sin\alpha)t - \tfrac{1}{2}gt^2\right)\underline{j}$$

$$= 500\cos(60)(10)\,\underline{i} + \left((500\sin(60)(10) - \tfrac{1}{2}(9.8)(10)^2\right)\underline{j}$$

$$= 500\left(\tfrac{1}{2}\right)(10)\underline{i} + \left((500\left(\tfrac{\sqrt{3}}{2}\right)(10) - \tfrac{1}{2}(9.8)(10)^2\right)\underline{j}$$

$$\approx 2500\,\underline{i} + 3840\,\underline{j}$$

Projectile Motion with Wind Gusts

$\underline{r}_0 = 0\underline{i} + 3\underline{j}$

Example 3 A baseball is hit when it is 3 ft above the ground. It leaves the bat with initial speed (v_0) of 152 ft/sec, making an angle of 20° with the horizontal. At the instant the ball is hit, an instantaneous gust of wind blows in the horizontal direction directly opposite the direction the ball is taking the outfield, adding a component of $-8.8\underline{i}$ (ft/sec) to the ball's initial velocity (8.8 ft/sec = 6 mph)

(a) Find a vector equation (position vector) for the path of the baseball.

(b) How high does the baseball go, and when does it reach maximum height?

(c) Assuming that the ball is not caught, find its range and flight time.

Solution (a) หา position vector

ดูทิศทาง เอาไปรวมใน v_0

① หา v_0 ก่อน

$$\underline{v}_0 = (v_0\cos\alpha)\underline{i} + (v_0\sin\alpha)\underline{j} - 8.8\underline{i}$$

$$= (152\cos 20^\circ)\underline{i} + (152\sin 20^\circ)\underline{j} - 8.8\underline{i}$$

$$= (152\cos 20^\circ - 8.8)\underline{i} + (152\sin 20^\circ)\underline{j}$$

9.65

② หา $\underline{r}$ (แกนนอน (j))

$$\frac{d^2\underline{r}}{dt^2} = -g\underline{j}$$

$$\frac{d\underline{r}}{dt} = -g(t)\underline{j} + \underline{v}_0$$

$$\underline{r} = -\frac{1}{2}g(t)^2\underline{j} + \underline{v_0}t + \underline{r_0}$$

③ แทนค่า $\underline{v_0}$, $\underline{r_0}$

$$\underline{r} = -\frac{1}{2}gt^2\underline{j} + (152\cos 20^\circ - 8.8)ti + (152\sin 20^\circ)tj + 3j$$

$$= \underbrace{(152\cos 20^\circ - 8.8)t\underline{i}}_{x} + \underbrace{(3 + (152\sin 20^\circ)t - 16t^2)}_{y}\underline{j} \quad ※$$

(b) baseball จะถึงจุดสูงสุดเมื่อ ความเร็วในแกน y = 0

$$\frac{dy}{dt} = 152\sin 20^\circ - 32t = 0$$

$$t = \frac{152\sin 20^\circ}{32} \approx 1.62 \text{ sec.}$$

แทนค่า เวลาในแกน y ที่เป็นองค์ประกอบของ $\underline{r}$

$$y_{max} = 3 + (152\sin 20^\circ)(1.62) - 16(1.62)^2 \approx 45.2 \text{ ft}$$

(c) เมื่อลูก baseball ถึงพื้น แกน y ของ $\underline{r}$ = 0

$$3 + (152\sin 20)t - 16t^2 = 0$$

$$3 + (51.99)t - 16t^2 = 0$$

แก้สมการจะได้ t = 3.3 sec และ t = -0.06 sec

แทนค่า เวลาที่เป็นบวกในแกน x ของ $\underline{r}$ จะได้

$$R = (152\cos 20^\circ - 8.8)(3.3)$$

$$\approx 442 \text{ ft}$$

Arc Length is Space

Arc Length Along a Space Curve

ความยาวของ smooth curve

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Arc Length Formula

$$L = \int_a^b |\mathbf{v}|\, dt$$

Example 1 A glider is soaring upward along the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$

How long is the glider's path from $t = 0$ to $t = 2\pi$

Solution หา $\mathbf{v}$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$$

$$L = \int_a^b |\mathbf{v}|\, dt$$

$$= \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\, dt \qquad \sin^2 A + \cos^2 A = 1$$

$$= \int_0^{2\pi} \sqrt{2}\, dt$$

$$= 2\pi\sqrt{2}$$ ✱

Arc Length Parameter with Base Point $P(t_0)$

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\, d\tau$$

$$= \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau$$

Speed on a Smooth Curves

$$\boxed{\frac{ds}{dt} = |\mathbf{v}(t)|}$$

Unit tangent Vector

$$\boxed{\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}}$$

Example 2 Find the unit tangent vector of the curve $\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$

Solution

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + 2t\,\mathbf{k}$$

$$|\mathbf{v}| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (2t)^2}$$

$$= \sqrt{9(\sin^2 t + \cos^2 t) + 4t^2}$$

$$= \sqrt{9 + 4t^2}$$

ดังนั้น

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{3\sin t}{\sqrt{9+4t^2}}\mathbf{i} + \frac{3\cos t}{\sqrt{9+4t^2}}\mathbf{j} + \frac{2t}{\sqrt{9+4t^2}}\mathbf{k} \quad \#$$

Curvature and Normal Vectors of a Curve

- ถามไร
- ให้ไร
- สูตรไร
- หาไร

Curvature function

$$\boxed{\kappa = \left|\frac{d\mathbf{T}}{ds}\right|}$$

Formula for Calculating Curvature

$$\boxed{\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|}$$

Example 1 Find the curvature of a circle

$$\underline{r}(t) = (a\cos t)\underline{i} + (a\sin t)\underline{j}$$

solution

$$\underline{v} = \frac{d\underline{r}}{dt} = -a\sin t\,\underline{i} + a\cos t\,\underline{j}$$

$$|\underline{v}| = \sqrt{(-a\sin t)^2 + (a\cos t)^2} = \sqrt{a^2} = a$$

$$\underline{T} = \frac{\underline{v}}{|\underline{v}|} = -\sin t\,\underline{i} + \cos t\,\underline{j}$$

$$\frac{d\underline{T}}{dt} = -\cos t\,\underline{i} - \sin t\,\underline{j}$$

$$\left|\frac{d\underline{T}}{dt}\right| = \sqrt{(-\cos t)^2 + (-\sin t)^2} = 1$$

$$\therefore \kappa = \frac{1}{|\underline{v}|}\left|\frac{d\underline{T}}{dt}\right|$$

$$= \frac{1}{a}(1)$$

$$= \frac{1}{a}$$ ##

radius

principal unit normal vector for a smooth curve

$$\underline{N} = \frac{1}{\kappa}\frac{d\underline{T}}{ds}$$

$$= \frac{d\underline{T}/dt}{|d\underline{T}/dt|}$$

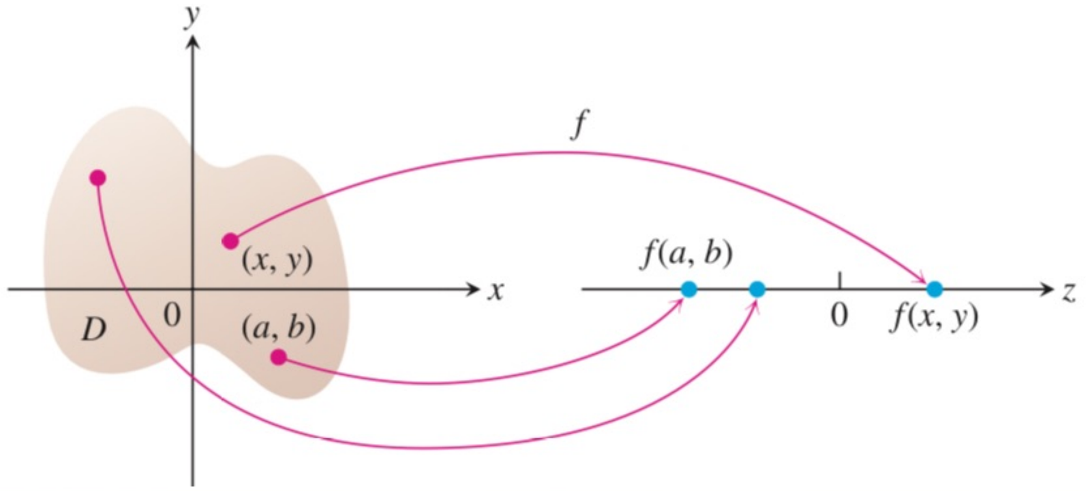
Function of Several Variables

Unit 7

DEFINITIONS Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \dots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

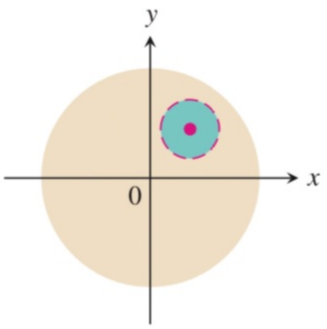
$$w = f(x_1, x_2, \dots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

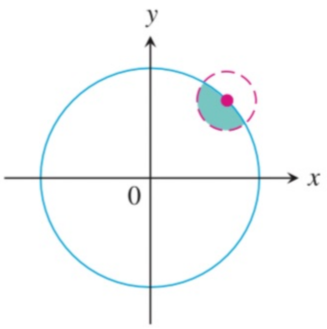


DEFINITIONS A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure 14.2). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

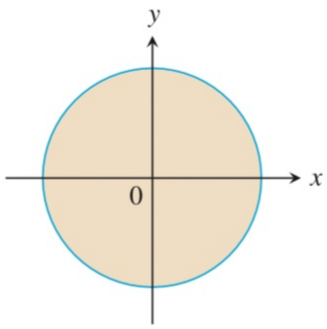
The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.3).



$\{(x, y) \mid x^2 + y^2 < 1\}$
Open unit disk. Every point an interior point.



$\{(x, y) \mid x^2 + y^2 = 1\}$
Boundary of unit disk. (The unit circle.)



$\{(x, y) \mid x^2 + y^2 \leq 1\}$
Closed unit disk. Contains all boundary points.

DEFINITIONS A region in the plane is **bounded** if it lies inside a disk of fixed radius. A region is **unbounded** if it is not bounded.

Graph, Level Curves, and Contours of functions of two Variables

DEFINITIONS The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** $z = f(x, y)$.

Function of Three variable

DEFINITION The set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x, y, z) = c$ is called a **level surface** of f.

Limit and Continuity in Higher Dimension

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

THEOREM 1—Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x, y) \to (x_0, y_0)} g(x, y) = M.$$

1. *Sum Rule:* $\displaystyle\lim_{(x, y) \to (x_0, y_0)} (f(x, y) + g(x, y)) = L + M$
2. *Difference Rule:* $\displaystyle\lim_{(x, y) \to (x_0, y_0)} (f(x, y) - g(x, y)) = L - M$
3. *Constant Multiple Rule:* $\displaystyle\lim_{(x, y) \to (x_0, y_0)} kf(x, y) = kL$ (any number k)
4. *Product Rule:* $\displaystyle\lim_{(x, y) \to (x_0, y_0)} (f(x, y) \cdot g(x, y)) = L \cdot M$
5. *Quotient Rule:* $\displaystyle\lim_{(x, y) \to (x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\displaystyle\lim_{(x, y) \to (x_0, y_0)} [f(x, y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\displaystyle\lim_{(x, y) \to (x_0, y_0)} \sqrt[n]{f(x, y)} = \sqrt[n]{L} = L^{1/n}$,
   n a positive integer, and if n is even, we assume that $L > 0$.

DEFINITION A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. $\displaystyle\lim_{(x, y) \to (x_0, y_0)} f(x, y)$ exists,
3. $\displaystyle\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

Partial derivative

Partial Derivative of a Function of Two variables

DEFINITION The **partial derivative of $f(x, y)$ with respect to x** at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

DEFINITION The **partial derivative of $f(x, y)$ with respect to y** at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y = y_0} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.

Example 1 Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1$$

solution ①
$$\frac{\partial f}{\partial x} = \frac{\partial (x^2 + 3xy + y - 1)}{\partial x}$$

$$= 2x + 3y + 0 - 0$$

$$= 2x + 3y$$

$$\therefore \left.\frac{\partial f}{\partial x}\right|_{x=4,\, y=-5} = 2(4) + 3(-5) = -7$$

②
$$\frac{\partial f}{\partial y} = \frac{\partial (x^2 + 3xy + y - 1)}{\partial y}$$

$$= 3x + 1$$

$$\therefore \left.\frac{\partial f}{\partial y}\right|_{x=4,\, y=-5} = 3(4) + 1 = 13$$

Hint ! diff เทียบตัวไหน ให้มองตัวแปรอีกตัวเป็น ค่าคงที่

Partial Derivative and Continuity

ถ้า $\frac{\partial f}{\partial x}$ และ $\frac{\partial f}{\partial y}$ ต่อเนื่อง $f(x,y)$ จะต่อเนื่อง

Second-order partial derivatives

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Example 2 If $f(x,y) = x\cos y + ye^x$ find the second-order derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$

Solution

① $\frac{\partial f}{\partial x} = \frac{\partial (x\cos y + ye^x)}{\partial x}$

$= \cos y + ye^x$

② $\frac{\partial f}{\partial y} = \frac{\partial (x\cos y + ye^x)}{\partial y}$

$= -x\sin y + e^x$

③ $\frac{\partial^2 f}{\partial x^2} = \frac{\partial (\cos y + ye^x)}{\partial x}$

$= ye^x$

④ $\frac{\partial^2 f}{\partial y^2} = \frac{\partial (-x\sin y + e^x)}{\partial y}$

$= -x\cos y$

⑤ $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial(\cos y + ye^x)}{\partial y}$

$= -\sin y + e^x$

⑥ $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial(-x\sin y + e^x)}{\partial x}$

$= -\sin y + e^x$ ※

Mixed Derivative Theorem

THEOREM 2—The Mixed Derivative Theorem If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Example 3 Find $\dfrac{\partial^2 w}{\partial x \partial y}$ if $w = xy + \dfrac{e^y}{y^2+1}$

Solution ① $\dfrac{\partial w}{\partial y} = x + \dfrac{(y^2+1)e^y - e^y(2y)}{(y^2+1)^2}$

$\dfrac{\partial^2 w}{\partial x \partial y} = 1$

② $\dfrac{\partial w}{\partial x} = y$

$\dfrac{\partial^2 w}{\partial x \partial y} = 1$

วิธีนี้ง่ายกว่าเยอะ!!

Partial derivative of higher orders

- ส่วนใหญ่ในการใช้งานจริงจะใช้แค่ 1st และ 2nd order
- Higher order like f_{yyx}, f_{xyy}, f_{xxyy} can be calculated
- If all derivatives are continuous, order does not matter

Example 4 Find f_{yxyz} of $f(x,y,z) = 1 - 2xy^2z + x^2y$

Solution

$f_y = -4xyz + x^2$

$f_{yx} = -4yz + 2x$

$f_{yxy} = -4z$

$f_{yxyz} = 4$ ※

Differentiable

THEOREM 3—The Increment Theorem for Functions of Two Variables Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0). Then the change

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

in the value of f that results from moving from (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in R satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$.

DEFINITION A function $z = f(x, y)$ is **differentiable at** (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and Δz satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$. We call f **differentiable** if it is differentiable at every point in its domain, and say that its graph is a **smooth surface**.

COROLLARY OF THEOREM 3 If the partial derivatives f_x and f_y of a function $f(x, y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

THEOREM 4—Differentiability Implies Continuity If a function $f(x, y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

Limit Continuity

$$\lim_{t \to t_0} r(t) = \left\langle \left(\lim_{t \to t_0} f(t)\right), \left(\lim_{t \to t_0} g(t)\right), \lim_{t \to t_0} h(t) \right\rangle$$

$$r(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$$

$$\lim_{t \to t_0} r(t) = \cos(t_0)\,\vec{i} + \sin(t_0)\,\vec{j} + t_0\,\vec{h}$$

Derivative

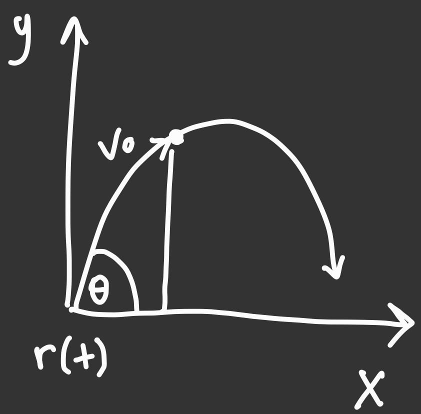
- $\dfrac{\partial r(t)}{\partial t} = \left\langle \dfrac{\partial f(t)}{\partial t}, \dfrac{\partial g(t)}{\partial t}, \dfrac{\partial h(t)}{\partial t} \right\rangle$

integral of vector

$$\int r(t)\, dt = R(t) + C$$

$$\int_a^b 6(t)\, dt = \int_a^b f(t)\,\vec{i}\, dt + \int_a^b g(t)\,\vec{j}\, dt + \int_a^b h(t)\,\vec{k}\, dt$$

Ideal Projectile motion



t, Y_{max}, X_{max}

$r(t) = \langle V_0 \cos\theta t$

$$r(t) = \left\langle V_0 \cos\theta t, V_0 \sin\theta t - \frac{gt^2}{2} \right\rangle$$

$$a(t) = -g$$

$$\int a(t)\, dt = -gt + C = -gt + V_0$$

$$v(t) = -gt + v_0$$

$$\int v(t) = -\frac{gt^2}{2}\vec{j} + v_0 t + r_0 = r(t)$$

$$r_0 = x_0 i + y_0 j$$

$$r(t) = x_0 i + \left(y_0 - \frac{gt^2}{2}\right) j + v_0 t$$

$$v_0 t = (v_0 \cos \vec{j} + v_0 \sin \vec{j})t$$

$$r(t) = (v_0 \cos\theta t + x_0) i + (v_0 \sin t + y_0 - gt$$

$$= (v_0 \cos\theta t \vec{i}) + (v_0 \sin t - \frac{gt}{2} j)$$

---

y_{max} = จุดสูงสุด

$$\frac{dy}{dt} = 0 = \frac{d\left(v_0 \sin t - \frac{gt^2}{2}\right)}{dt} = 0$$

$$= v_0 \sin - gt = 0$$

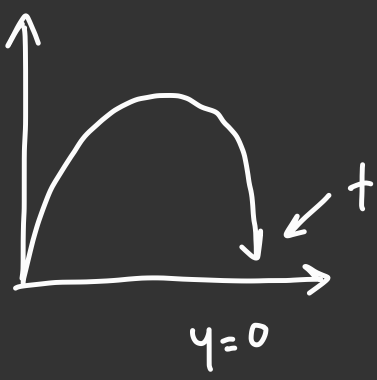
t ที่ y_0 ← $\boxed{t_{ymax} = \frac{v_0 \sin}{g}}$

$$y = v_0 \sin t - \frac{gt^2}{2} = v_0 \sin\left(\frac{v_0 \sin}{g}\right) - \frac{\cancel{g}\left(\frac{v\sin}{\cancel{g}}\right)^2}{2}$$

$$y_{max} = \frac{v_0^2 \sin^2}{g} - \frac{g\left(\frac{v_0^2 \sin^2}{g^2}\right)}{2}$$

$$= \frac{v_0^2 \sin^2}{g} - \frac{v_0^2 \sin^2}{2g}$$

$$\boxed{y_{max} = \frac{v_0^2 \sin^2}{2g}}$$



$$V_0 \sin t - \frac{gt^2}{2} = 0$$

$$2V_0 \sin t - gt^2 = 0$$

$$t = 0$$

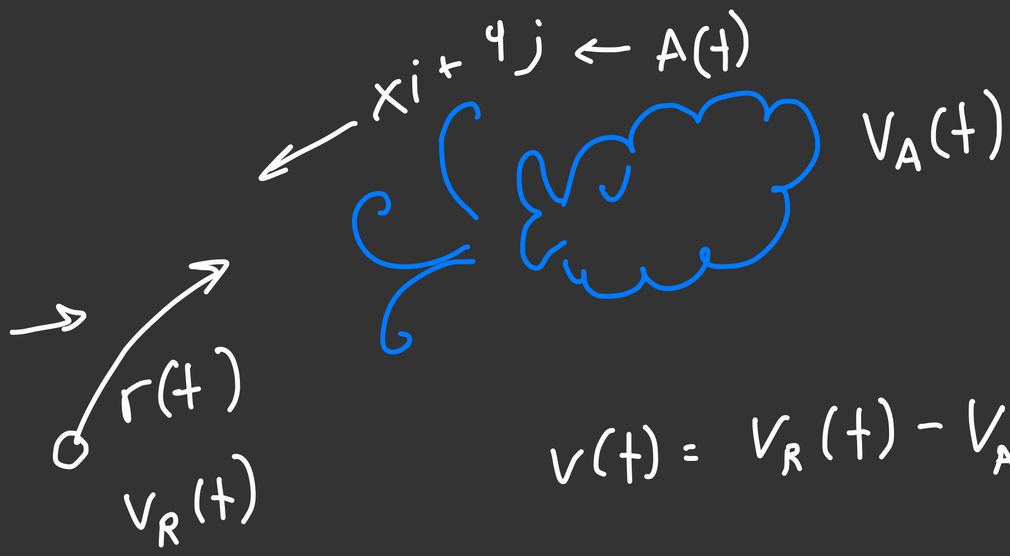
$$2V_0 \sin - gt = 0$$

$$\boxed{t_{x_{mas}} = \frac{2V_0 \sin}{g}}$$

$$t_{xmax_0} \rightarrow X \qquad x = V_0 \cos t$$

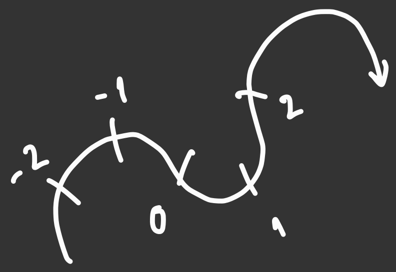
$$= V_0 \cos\left(\frac{2V_0 \sin}{g}\right)$$

$$X_{max} = \frac{V_0^2 \, 2 \sin \cos}{g} = \frac{V_0^2 \sin 2\theta}{g}$$



$$v(t) = V_R(t) - V_A t + V_B(t)$$

Arc length Along a Space Curve



$r(t) = \langle x, y, z \rangle$

$$|v| = \left|\frac{dr}{dt}\right|$$

$$L = \int_a^b \sqrt{(x')^2 + (y')^2 + (z')^2}$$

$$L = \int_a^b |v|\, dt$$

$$\int_{t_0}^{t} |v(\tau)|\, dt = v(\tau) + C$$

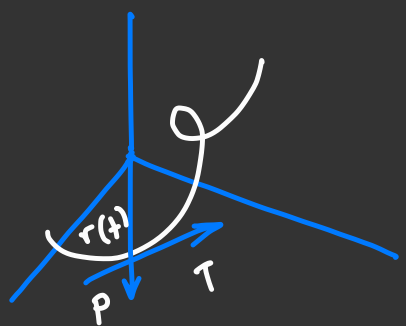
Speed on a smooth Curve

$$\frac{ds}{dt} = \frac{d \int |v(t)|\, dt}{dt}$$

$$=$$

Tangent Line

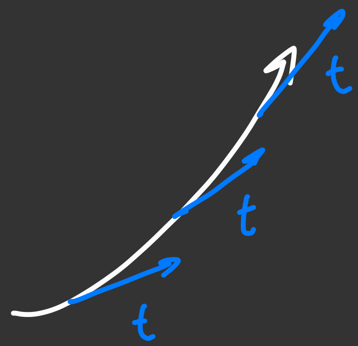
$$T = \frac{v}{|v|} = \frac{\frac{dr}{dt}}{\left|\frac{dr}{dt}\right|}$$



Curvature of a Plane Curve

↓

ความโค้งของ Curve

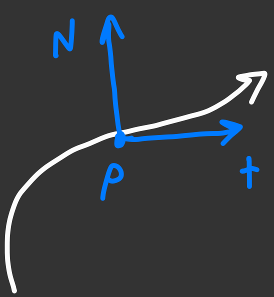


$$k = \left|\frac{dT}{ds}\right|$$

$$= \left|\frac{dT}{dt} \cdot \frac{dt}{ds}\right|$$

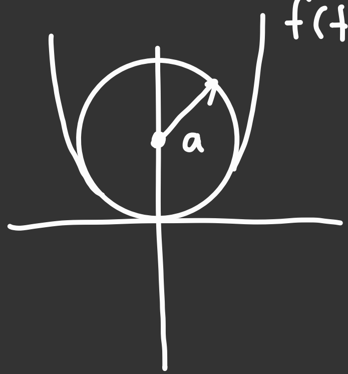
$$K = \frac{1}{|v|}\left|\frac{dT}{dt}\right|$$

Principal unit normal vector



$$N = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|} = \frac{1}{k}\,dt$$

Circle Curvature



f(t)

$$\boxed{a = \frac{1}{k_{f(x)}}}$$

Partial derivatives

$f(x,y)$

Domain → plane

x เป็นอะไรบ้าง

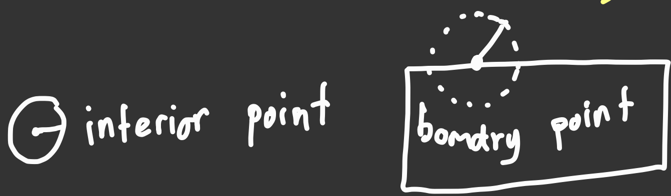
y เป็นอะไรบ้าง

$\boxed{z = \sin xy}$ → x,y ทั้งระนาบ

$\boxed{z = \sqrt{y - x^2}}$ → $y \geq x^2$

$\boxed{z = \dfrac{1}{xy}}$ → $xy \neq 0$

interior point / Boundary point



open / closed

↓

ไม่มี boundary point

มี boundary point ทั้งหมด

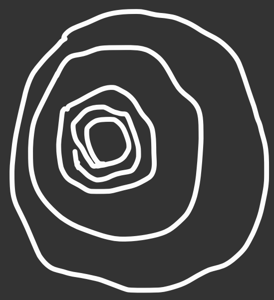
Bound / Unbound

↓

อยู่ในขอบเขตวงกลม
ที่กำหนด (ล้อมได้)

ล้อมไม่ได้

Level curve



$$\boxed{f(x,y) = c}$$

Sol. จับสมการ = c แล้วแก้สมการ

$f(x,y) = 100 - x^2 - y^2$ $\qquad f(x,y) = 0$

$$\downarrow$$

$$100 - x^2 - y^2 = 0$$

$$\boxed{x^2 + y^2 = 100} \rightarrow \begin{array}{l} \text{ศก. } (0,0) \\ r = 10 \end{array}$$

* ไม่วงกลมก็ พาราโบล่า

Limit & cont of high มิ

$$f(x,y) = \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} = \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{x - y}$$

$$= \frac{x(x-y)(\sqrt{x} + \sqrt{y})}{(x-y)}$$

$$= x(\sqrt{x} + \sqrt{y}) = 0$$

Continuity

1. f หาค่าได้ (x_0, y_0)

2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x)$ หาค่าได้

3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} &, (x,y) \neq (0,0) \\ 0 &, (x,y) = (0,0) \end{cases}$$

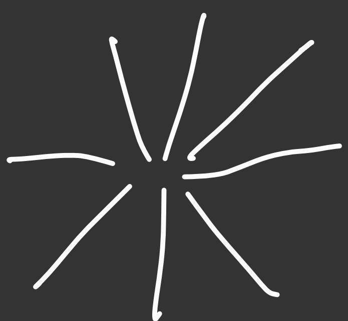
$(0,0)$

1 ✓

2. $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \dfrac{2xy}{x^2+y^2}$

$$\left.\begin{aligned} x \in I,\ y = x &= \frac{2x^2}{2x^2} = 1 \\ y \in I,\ x = y &= \frac{2y^3}{2y^2} = 1 \end{aligned}\right\} 1$$

2-Path test



$y = mx$

$$\frac{2x(mx)}{x^2y(mx)^2} = \frac{2m\cancel{x^2}}{\cancel{x^2}(1+m)} = \frac{2m}{(1+m)}$$

Ex. $\frac{2x^2y}{x^2+y^2}$ (0,0)

$y = kx^2$

$yz - \ln z = x + y$

$\frac{\partial z}{\partial x} \rightarrow y\left[\frac{\partial z}{\partial x}\right] - \frac{\partial \ln z}{\partial x} = \frac{\partial x}{\partial x}$