ASTRONOMY 121 - FALL 2025
Homework 3
DUE by Tuesday, November 4, 2025
370 points

NOTE: Be sure to show all calculations clearly, and box/circle/highlight your final result.

1) Calculate the values of the mean molecular weights under the following circumstances: (a) all Hydrogen, i.e., X=1 and Y=Z=0; both completely neutral and completely ionized gas; (b) all Helium, i.e., X=Z=0 and Y=1; both completely neutral and completely ionized gas; (c) all heavy elements, i.e., X=Y=0 and Z=1; both completely neutral and completely ionized gas; (d) solar abundance, i.e., X=0.7, Y=0.28, and Z=0.02; both completely neutral and completely ionized gas. *NOTE:* $< n_{\mathrm{z}}/A_{\mathrm{z}} >$ *is roughly equal to (i)* $\sim 1/15.5$ *for the neutral case at solar metallicity, and (ii)* $z + 1/(2z+2) \approx 1/2$ *for the fully ionized case.* [80 points]

2) Calculate the mean molecular weight per electron, μ_{e}, for completely ionized conditions of (a) all Hydrogen (X=1 and Y=Z=0); (b) all Helium (X=Z=0, Y=1); and all heavy elements, i.e., X=Y=0 and Z=1. *NOTE:* $< n_{\mathrm{z}-1}/A_{\mathrm{z}} >$*=0.5 for the fully ionized case.* [30 points]

3) The center of a certain star contains 70% of Hydrogen by weight and 28% of Helium by weight (i.e., X=0.7, Y=0.28, Z=0.02). Evaluate numerically the equation of state (incuding both gas pressure and radiation pressure). What is the pressure at the center of the star if the density there is 50 g cm^{-3} and the temperature is 15×10^6 K? How do the gas and radiation pressures compare, i.e., what is $P_{\mathrm{gas}}/P_{\mathrm{rad}} = (1-\beta)/\beta$? [40 points]

4) Consider the formula for the pressure of completely degenerate, relativistic electrons:

$$P_{\mathrm{e,r}} = \frac{\pi m_{\mathrm{o}}^4 c^5}{3h^3} f(x), \tag{1}$$

with $x = p_o/(m_{\mathrm{o}}c)$ and $f(x)$ given by

$$f(x) = x(2x^2-3)(x^2+1)^{1/2} + 3sinh^{-1}x \tag{2}$$

The limit of small x, i.e., $p_o << m_{\mathrm{o}}c$, must correspond to non-relativistic particles. Show that $f(x) \approx (8/5)x^5 - (4/7)x^7 + ...$ for $x \to 0$, and confirm that the pressure obtained from this limiting value of $f(x)$ reduces to the completely degenerate non-relativistic electron pressure determined in class in Eq.

$$P_{\mathrm{e,nr}} = \frac{8\pi}{15mh^3} p_o^5. \tag{3}$$

[50 points]. *TIP: Taylor expand with 4 terms both* $(x^2+1)^{1/2}$ *and* $sinh^{-1}x$ *for* $x \to 0$.

5) The limit of large x of Eq. 2 above must correspond to highly relativistic degeneracy. Show that $f(x) \approx 2x^4 - 2x^2 + ...$ for $x \to \infty$. Show that the pressure obtained by inserting this limiting value of $f(x)$ into the equation above for P_{e} is identical to that obtained by letting $v_{\mathrm{p}} = c$ in the pressure integral [50 points]. *TIP: Taylor expand with 3 terms both* $(x^2+1)^{1/2} = x(1+\frac{1}{x^2})^{1/2}$ *and* $sinh^{-1}x = ln[x + \sqrt{1+x^2}]$ *for* $x \to \infty$.

6) Show that the electron pressure is twice that of the Maxwellian electron-gas formula when $\rho/\mu_e = 5 \times 10^{-8} T^{3/2}$ g cm^{-3}. [50 points]

7) A gas composed of ^{12}C and ^{16}O has a density of $2.5{\times}10^5$ g cm^{-3} at 10^8 K. Is this gas in the degenerate or non-degenerate region of the equation of state (i.e., in the plane $\log T$ vs $\log \rho$)? Assuming the degeneracy is complete, is it completely non-relativistic, partially relativistic, or extremely relativistic? Calculate the electron pressure from the given Table (i.e, $f(x)$ vs x). Assuming that the degeneracy is incomplete and non-relativistic, calculate the electron pressure from the Table (i.e., using the Fermi-Dirac functions). Compare the derived electron pressures using the different methods and comment. Which numerical answer is more correct for the present problem? Why? What is the ratio of the electron pressure to the ion pressure? [70 points]