

FinSetsForCAP

**The elementary topos of (skeletal) finite
sets**

2020.04.01

1 April 2020

Mohamed Barakat

Julia Mickisch

Fabian Zickgraf

Mohamed Barakat

Email: mohamed.barakat@uni-siegen.de

Homepage: <http://www.mathematik.uni-kl.de/~barakat/>

Address: Walter-Flex-Str. 3
57068 Siegen
Germany

Julia Mickisch

Email: julia.mickisch@student.uni-siegen.de

Homepage: <https://github.com/juliamick/>

Address: Walter-Flex-Str. 3
57068 Siegen
Germany

Fabian Zickgraf

Email: fabian.zickgraf@uni-siegen.de

Homepage: <https://github.com/zickgraf/>

Address: Walter-Flex-Str. 3
57068 Siegen
Germany

Contents

1	The category of finite sets	3
1.1	GAP Categories	3
1.2	Attributes	3
1.3	Constructors	4
1.4	Tools	7
1.5	Examples	10
2	The category of skeletal finite sets	22
2.1	Skeletal GAP Categories	22
2.2	Skeletal Attributes	22
2.3	Skeletal Constructors	23
2.4	Skeletal Tools	24
2.5	Skeletal Examples	25
	Index	38

Chapter 1

The category of finite sets

1.1 GAP Categories

1.1.1 IsFiniteSet (for IsCapCategoryObject)

▷ `IsFiniteSet(object)` (filter)
Returns: true or false
The GAP category of objects in the category of finite sets.

1.1.2 IsFiniteSetMap (for IsCapCategoryMorphism)

▷ `IsFiniteSetMap(object)` (filter)
Returns: true or false
The GAP category of morphisms in the category of finite sets.

1.2 Attributes

1.2.1 AsList (for IsFiniteSet)

▷ `AsList(M)` (attribute)
Returns: a GAP set
The GAP set of the list used to construct a finite set S , i.e., `AsList(FinSet(L)) = Set(L)`.

1.2.2 Length (for IsFiniteSet)

▷ `Length(M)` (attribute)
Returns: an integer
The length of the GAP set of the list used to construct a finite set S , i.e., `Length(FinSet(L)) = Length(Set(L))`.

1.2.3 AsList (for IsFiniteSetMap)

▷ `AsList(f)` (attribute)
Returns: a list

The relation underlying a map between finite sets, i.e., `AsList(MapOfFinSets(S, G, T)) = G`.

1.3 Constructors

1.3.1 FinSet (for IsList)

▷ `FinSet(L)` (operation)

Returns: a CAP object

Construct a finite set out of the list L , i.e., an object in the CAP category `FinSets`. The GAP operation `Set` must be applicable to L without throwing an error. Equality is determined as follows: `FinSet(L1) = FinSet(L2)` iff `IsEqualForElementsOfFinSets(Immutable(Set(L1)), Immutable(Set(L2)))`. Warning: all internal operations use `FinSetNC` (see below) instead of `FinSet`. Thus, this notion of equality is only valid for objects created by calling `FinSet` explicitly. Internally, `FinSet(L)` is an alias for `FinSetNC(Set(L))` and equality is determined as for `FinSetNC`. Thus, `FinSet(L1) = FinSetNC(L2)` iff `IsEqualForElementsOfFinSets(Immutable(Set(L1)), Immutable(L2))` and `FinSetNC(L1) = FinSet(L2)` iff `IsEqualForElementsOfFinSets(Immutable(L1), Immutable(Set(L2)))`.

Example

```
gap> S := FinSet( [ 1, 3, 2, 2, 1 ] );
<An object in FinSets>
gap> Display( S );
[ 1, 2, 3 ]
gap> L := AsList( S );
[ 1, 2, 3 ]
gap> Q := FinSet( L );
<An object in FinSets>
gap> S = Q;
true
gap> FinSet( [ 1, 2 ] ) = FinSet( [ 2, 1 ] );
true
gap> M := FinSetNC( [ , 1, 2, 3 ] );
<An object in FinSets>
gap> IsWellDefined( M );
false
gap> M := FinSetNC( [ 1, 2, 3, 3 ] );
<An object in FinSets>
gap> IsWellDefined( M );
false
```

1.3.2 FinSetNC (for IsList)

▷ `FinSetNC(L)` (operation)

Returns: a CAP object

Construct a finite set out of the duplicate-free (w.r.t. `IsEqualForElementsOfFinSets`) and dense list L , i.e., an object in the CAP category `FinSets`. Equality is determined as follows: `FinSetNC(L1) = FinSetNC(L2)` iff `IsEqualForElementsOfFinSets(Immutable(L1), Immutable(L2))`.

Example

```

gap> S := FinSetNC( [ 1, 3, 2 ] );
<An object in FinSets>
gap> Display( S );
[ 1, 3, 2 ]
gap> L := AsList( S );
[ 1, 3, 2 ]
gap> Q := FinSetNC( L );
<An object in FinSets>
gap> S = Q;
true
gap> FinSetNC( [ 1, 2 ] ) = FinSetNC( [ 2, 1 ] );
false

```

1.3.3 MapOfFinSets (for IsFiniteSet, IsList, IsFiniteSet)

▷ MapOfFinSets(S , G , T)

(operation)

Returns: a CAP morphism

Construct a map $\phi : S \rightarrow T$ of the finite sets S and T , i.e., a morphism in the CAP category FinSets, where G is a list of pairs in $S \times T$ describing the graph of ϕ .

Example

```

gap> S := FinSet( [ 1, 3, 2, 2, 1 ] );
<An object in FinSets>
gap> T := FinSet( [ "a", "b", "c" ] );
<An object in FinSets>
gap> G := [ [ 1, "b" ], [ 3, "b" ], [ 2, "a" ] ];
gap> phi := MapOfFinSets( S, G, T );
<A morphism in FinSets>
gap> IsWellDefined( phi );
true
gap> phi( 1 );
"b"
gap> phi( 2 );
"a"
gap> phi( 3 );
"b"
gap> List( S, phi );
[ "b", "a", "b" ]
gap> psi := [ [ 1, "b" ], [ 2, "a" ], [ 3, "b" ] ];
gap> psi := MapOfFinSets( S, psi, T );
<A morphism in FinSets>
gap> IsWellDefined( psi );
true
gap> phi = psi;
true
gap> psi := MapOfFinSetsNC( S, [ , [ 1, "b" ], [ 3, "b" ], [ 2, "a" ] ], T );
<A morphism in FinSets>
gap> IsWellDefined( psi );
false
gap> psi := MapOfFinSets( S, [ [ 1, "d" ], [ 3, "b" ] ], T );
<A morphism in FinSets>

```

```

gap> IsWellDefined( psi );
false
gap> psi := MapOfFinSets( S, [ 1, 2, 3 ], T );
<A morphism in FinSets>
gap> IsWellDefined( psi );
false
gap> psi := MapOfFinSets( S, [ [ 1, "b" ], [ 3, "b" ], [ 2, "a", "b" ] ], T );
<A morphism in FinSets>
gap> IsWellDefined( psi );
false
gap> psi := MapOfFinSets( S, [ [ 5, "b" ], [ 3, "b" ], [ 2, "a" ] ], T );
<A morphism in FinSets>
gap> IsWellDefined( psi );
false
gap> psi := MapOfFinSets( S, [ [ 1, "d" ], [ 3, "b" ], [ 2, "a" ] ], T );
<A morphism in FinSets>
gap> IsWellDefined( psi );
false
gap> psi := MapOfFinSets( S, [ [ 1, "b" ], [ 2, "b" ], [ 2, "a" ] ], T );
<A morphism in FinSets>
gap> IsWellDefined( psi );
false

```

1.3.4 MapOfFinSetsNC (for IsFiniteSet, IsList, IsFiniteSet)

▷ MapOfFinSetsNC(S , G , T)

(operation)

Returns: a CAP morphism

Construct a map $\phi : S \rightarrow T$ of the finite sets S and T , i.e., a morphism in the CAP category FinSets, where G is a duplicate-free and dense list of pairs in $S \times T$ describing the graph of ϕ .

Example

```

gap> S := FinSetNC( [ 1, 3, 2 ] );
<An object in FinSets>
gap> T := FinSetNC( [ "a", "b", "c" ] );
<An object in FinSets>
gap> G := [ [ 1, "b" ], [ 3, "b" ], [ 2, "a" ] ];
gap> phi := MapOfFinSetsNC( S, G, T );
<A morphism in FinSets>
gap> IsWellDefined( phi );
true
gap> phi( 1 );
"b"
gap> phi( 2 );
"a"
gap> phi( 3 );
"b"
gap> List( S, phi );
[ "b", "b", "a" ]
gap> psi := [ [ 1, "b" ], [ 2, "a" ], [ 3, "b" ] ];
gap> psi := MapOfFinSetsNC( S, psi, T );
<A morphism in FinSets>
gap> IsWellDefined( psi );

```

```

true
gap> phi = psi;
true

```

1.4 Tools

1.4.1 IsEqualForElementsOfFinSets (for IsObject, IsObject)

▷ IsEqualForElementsOfFinSets(a, b) (operation)

Returns: a boolean

Compares two arbitrary objects using the following rules:

- integers, strings and chars are compared using the operation =
- lists and records are compared recursively
- CAP category objects are compared using IsEqualForObjects (if available)
- CAP category morphisms are compared using IsEqualForMorphismsOnMor (if available)
- other objects are compared using IsIdenticalObj

Note: if CAP category objects or CAP category morphisms are compared using IsEqualForObjects or IsEqualForMorphismsOnMor, respectively, the result must not be fail.

Example

```

gap> IsEqualForElementsOfFinSets( 2, 2 );
true
gap> IsEqualForElementsOfFinSets( 2, "2" );
false
gap> IsEqualForElementsOfFinSets( [ 2 ], [ 2 ] );
true
gap> IsEqualForElementsOfFinSets( [ 2 ], [ 2, 3 ] );
false
gap> IsEqualForElementsOfFinSets( [ , 2 ], [ 2, 2 ] );
false
gap> IsEqualForElementsOfFinSets( rec( a := "a", b := "b" ),
>                                rec( b := "b", a := "a" )
>                                );
true
gap> IsEqualForElementsOfFinSets( rec( a := "a", b := "b" ),
>                                rec( a := "a" )
>                                );
false
gap> IsEqualForElementsOfFinSets( rec( a := "a", b := "b" ),
>                                rec( a := "a", b := "notb" )
>                                );
false
gap> M := FinSet( [ ] );;
gap> N := FinSet( [ ] );;
gap> m := FinSet( 0 );;
gap> id_M := IdentityMorphism( M );;
gap> id_N := IdentityMorphism( N );;

```



```

gap> id_m := IdentityMorphism( m );;
gap> IsEqualForElementsOfFinSets( M, N );
true
gap> IsEqualForElementsOfFinSets( M, m );
false
gap> IsEqualForElementsOfFinSets( id_M, id_N );
true
gap> IsEqualForElementsOfFinSets( id_M, id_m );
false
gap> IsEqualForElementsOfFinSets( FinSets, SkeletalFinSets );
false

```

1.4.2 $\backslash \text{in}$ (for IsObject, IsFiniteSet)

▷ $\backslash \text{in}(obj, M)$ (operation)

Returns: a boolean

Returns true if there exists an element in $\text{AsList}(M)$ which is equal to obj w.r.t. $\text{IsEqualForElementsOfFinSets}$ and false if not.

1.4.3 $[]$ (for IsFiniteSet, IsInt)

▷ $[](M, i)$ (operation)

Returns: an object

Returns the i -th entry of the GAP set of the list used to construct a finite set S , i.e., $\text{FinSet}(L)[i] = \text{Set}(L)[i]$.

1.4.4 Iterator (for IsFiniteSet)

▷ $\text{Iterator}(M)$ (operation)

Returns: an iterator

An iterator of the GAP set of the list used to construct a finite set S , i.e., $\text{Iterator}(\text{FinSet}(L)) = \text{Iterator}(\text{Set}(L))$.

1.4.5 UnionOfFinSets (for IsList)

▷ $\text{UnionOfFinSets}(L)$ (operation)

Returns: a CAP object

Compute the set-theoretic union of the elements of L , where L is a list of finite sets.

1.4.6 ListOp (for IsFiniteSet, IsFunction)

▷ $\text{ListOp}(M, f)$ (operation)

Returns: a list

Returns $\text{List}(\text{AsList}(M), f)$.

1.4.7 FilteredOp (for IsFiniteSet, IsFunction)

▷ $\text{FilteredOp}(M, f)$ (operation)

Returns: a list

Returns `FinSetNC(Filtered(AsList(M), f))`.

1.4.8 FirstOp (for IsFiniteSet, IsFunction)

- ▷ `FirstOp(M, f)` (operation)
Returns: a list
 Returns `First(AsList(M), f)`.

1.4.9 EmbeddingOfFinSets (for IsFiniteSet, IsFiniteSet)

- ▷ `EmbeddingOfFinSets(S, T)` (operation)
Returns: a CAP morphism
 Construct the embedding $\iota : S \rightarrow T$ of the finite sets S and T , where S must be subset of T .

1.4.10 ProjectionOfFinSets (for IsFiniteSet, IsFiniteSet)

- ▷ `ProjectionOfFinSets(S, T)` (operation)
Returns: a CAP morphism
 Construct the projection $\pi : S \rightarrow T$ of the finite sets S and T , where T is a partition of S .

1.4.11 Preimage (for IsFiniteSetMap, IsFiniteSet)

- ▷ `Preimage(f, T_)` (operation)
Returns: a CAP object
 Compute the preimage of $T_$ under the morphism f .

1.4.12 ImageObject (for IsFiniteSetMap, IsFiniteSet)

- ▷ `ImageObject(f, S_)` (operation)
Returns: a CAP object
 Compute the image of $S_$ under the morphism f .

1.4.13 CallFuncList (for IsFiniteSetMap, IsList)

- ▷ `CallFuncList(phi, L)` (operation)
Returns: a list
 Returns the image of $L[1]$ under the map phi assuming $L[1]$ is an element of `AsList(Source(phi))`.

1.4.14 ListOp (for IsFiniteSet, IsFiniteSetMap)

- ▷ `ListOp(F, phi)` (operation)
Returns: a list
 Returns `List(AsList(F), phi)`.

1.5 Examples

1.5.1 IsHomSetInhabited

Example

```
gap> L := FinSet( [ ] );
<An object in FinSets>
gap> M := FinSet( [ 2 ] );
<An object in FinSets>
gap> N := FinSet( [ 3 ] );
<An object in FinSets>
gap> IsHomSetInhabited( L, L );
true
gap> IsHomSetInhabited( M, L );
false
gap> IsHomSetInhabited( L, M );
true
gap> IsHomSetInhabited( M, N );
true
```

1.5.2 PreCompose

Example

```
gap> S := FinSet( [ 1, 2, 3 ] );
<An object in FinSets>
gap> T := FinSet( [ "a", "b" ] );
<An object in FinSets>
gap> phi := [ [ 1, "b" ], [ 2, "a" ], [ 3, "b" ] ];;
gap> phi := MapOfFinSets( S, phi, T );
<A morphism in FinSets>
gap> psi := [ [ "a", 3 ], [ "b", 1 ] ];;
gap> psi := MapOfFinSets( T, psi, S );
<A morphism in FinSets>
gap> alpha := PreCompose( phi, psi );
<A morphism in FinSets>
gap> List( S, alpha );
[ 1, 3, 1 ]
gap> IsOne( alpha );
false
```

1.5.3 IsEpimorphism and IsMonomorphism

Example

```
gap> S := FinSet( [ 1, 2, 3 ] );
<An object in FinSets>
gap> T := S;
<An object in FinSets>
gap> phi := [ [ 1, 2 ], [ 2, 3 ], [ 3, 1 ] ];;
gap> phi := MapOfFinSets( S, phi, T );
<A morphism in FinSets>
gap> I := ImageObject( phi );
<An object in FinSets>
gap> Length( I );
3
```

```

gap> IsMonomorphism( phi );
true
gap> IsSplitMonomorphism( phi );
true
gap> IsEpimorphism( phi );
true
gap> IsSplitEpimorphism( phi );
true
gap> iota := ImageEmbedding( phi );
<A monomorphism in FinSets>
gap> pi := CostrictionToImage( phi );
<An epimorphism in FinSets>
gap> PreCompose( pi, iota ) = phi;
true

```

1.5.4 Initial and Terminal Objects

Example

```

gap> M := FinSet( [ 1, 2, 3 ] );
<An object in FinSets>
gap> IsInitial( M );
false
gap> IsTerminal( M );
false
gap> I := InitialObject( M );
<An object in FinSets>
gap> IsInitial( I );
true
gap> IsTerminal( I );
false
gap> iota := UniversalMorphismFromInitialObject( M );
<A morphism in FinSets>
gap> Display( I );
[ ]
gap> T := TerminalObject( M );
<An object in FinSets>
gap> Display( T );
[ "*" ]
gap> IsInitial( T );
false
gap> IsTerminal( T );
true
gap> pi := UniversalMorphismIntoTerminalObject( M );
<A morphism in FinSets>
gap> IsIdenticalObj( Range( pi ), T );
true
gap> t := FinSet( [ "Julia" ] );
<An object in FinSets>
gap> pi_t := UniversalMorphismIntoTerminalObjectWithGivenTerminalObject( M, t );
<A morphism in FinSets>
gap> List( M, pi_t );
[ "Julia", "Julia", "Julia" ]

```

1.5.5 Projective and Injective Objects

Example

```
gap> I := FinSet( [ ] );
<An object in FinSets>
gap> T := FinSet( [ 1 ] );
<An object in FinSets>
gap> M := FinSet( [ 2 ] );
<An object in FinSets>
gap> IsProjective( I );
true
gap> IsProjective( T );
true
gap> IsProjective( M );
true
gap> IsOne( EpimorphismFromSomeProjectiveObject( I ) );
true
gap> IsOne( EpimorphismFromSomeProjectiveObject( M ) );
true
```

Example

```
gap> I := FinSet( [ ] );
<An object in FinSets>
gap> T := FinSet( [ 1 ] );
<An object in FinSets>
gap> M := FinSet( [ 2 ] );
<An object in FinSets>
gap> IsInjective( I );
false
gap> IsInjective( T );
true
gap> IsInjective( M );
true
gap> IsIsomorphism( MonomorphismIntoSomeInjectiveObject( I ) );
false
gap> IsMonomorphism( MonomorphismIntoSomeInjectiveObject( I ) );
true
gap> IsOne( MonomorphismIntoSomeInjectiveObject( M ) );
true
```

1.5.6 Product

Example

```
gap> S := FinSet( [ 1, 2, 3 ] );
<An object in FinSets>
gap> Length( S );
3
gap> T := FinSet( [ "a", "b" ] );
<An object in FinSets>
gap> Length( T );
2
gap> P := DirectProduct( S, T );
<An object in FinSets>
gap> Length( P );
6
```

```
gap> Display( P );
[ [ 1, "a" ], [ 1, "b" ], [ 2, "a" ], [ 2, "b" ], [ 3, "a" ], [ 3, "b" ] ]
```

1.5.7 Coproduct

Example

```
gap> S := FinSet( [ 1, 2, 3 ] );
<An object in FinSets>
gap> Length( S );
3
gap> T := FinSet( [ "a", "b" ] );
<An object in FinSets>
gap> Length( T );
2
gap> C := Coproduct( T, S );
<An object in FinSets>
gap> Length( C );
5
gap> Display( C );
[ [ 1, "a" ], [ 1, "b" ], [ 2, 1 ], [ 2, 2 ], [ 2, 3 ] ]
gap> M := FinSet( [ 1, 2, 3, 4, 5, 6, 7 ] );
<An object in FinSets>
gap> N := FinSet( [ 1, 2, 3 ] );
<An object in FinSets>
gap> P := FinSet( [ 1, 2, 3, 4 ] );
<An object in FinSets>
gap> C := Coproduct( M, N, P );
<An object in FinSets>
gap> AsList( C );
[ [ 1, 1 ], [ 1, 2 ], [ 1, 3 ], [ 1, 4 ], [ 1, 5 ], [ 1, 6 ],
  [ 1, 7 ], [ 2, 1 ], [ 2, 2 ], [ 2, 3 ], [ 3, 1 ], [ 3, 2 ],
  [ 3, 3 ], [ 3, 4 ] ]
gap> iota1 := InjectionOfCofactorOfCoproduct( [ M, N, P ], 1 );
<A morphism in FinSets>
gap> IsWellDefined( iota1 );
true
gap> AsList( iota1 );
[ [ 1, [ 1, 1 ] ], [ 2, [ 1, 2 ] ], [ 3, [ 1, 3 ] ], [ 4, [ 1, 4 ] ],
  [ 5, [ 1, 5 ] ], [ 6, [ 1, 6 ] ], [ 7, [ 1, 7 ] ] ]
gap> iota2 := InjectionOfCofactorOfCoproduct( [ M, N, P ], 2 );
<A morphism in FinSets>
gap> IsWellDefined( iota2 );
true
gap> AsList( iota2 );
[ [ 1, [ 2, 1 ] ], [ 2, [ 2, 2 ] ], [ 3, [ 2, 3 ] ] ]
gap> iota3 := InjectionOfCofactorOfCoproduct( [ M, N, P ], 3 );
<A morphism in FinSets>
gap> IsWellDefined( iota3 );
true
gap> AsList( iota3 );
[ [ 1, [ 3, 1 ] ], [ 2, [ 3, 2 ] ], [ 3, [ 3, 3 ] ], [ 4, [ 3, 4 ] ] ]
gap> psi := UniversalMorphismFromCoproduct( [ M, N, P ],
> [ iota1, iota2, iota3 ]
```

```

>                                     );
<A morphism in FinSets>
gap> psi = IdentityMorphism( Coproduct( [ M, N, P ] ) );
true

```

1.5.8 Image

Example

```

gap> S := FinSet( [ 1, 2, 3 ] );
<An object in FinSets>
gap> T := FinSet( [ "a", "b", "c" ] );
<An object in FinSets>
gap> phi := [ [ 1, "b" ], [ 2, "a" ], [ 3, "b" ] ];;
gap> phi := MapOfFinSets( S, phi, T );
<A morphism in FinSets>
gap> I := ImageObject( phi );
<An object in FinSets>
gap> Length( I );
2
gap> IsMonomorphism( phi );
false
gap> IsSplitMonomorphism( phi );
false
gap> IsEpimorphism( phi );
false
gap> IsSplitEpimorphism( phi );
false
gap> iota := ImageEmbedding( phi );
<A monomorphism in FinSets>
gap> pi := CostrictionToImage( phi );
<An epimorphism in FinSets>
gap> PreCompose( pi, iota ) = phi;
true

```

1.5.9 Coimage

Example

```

gap> S := FinSet( [ 1, 2, 3 ] );
<An object in FinSets>
gap> T := FinSet( [ "a", "b", "c" ] );
<An object in FinSets>
gap> phi := [ [ 1, "b" ], [ 2, "a" ], [ 3, "b" ] ];;
gap> phi := MapOfFinSets( S, phi, T );
<A morphism in FinSets>
gap> I := Coimage( phi );
<An object in FinSets>
gap> Length( I );
2
gap> IsMonomorphism( phi );
false
gap> IsSplitMonomorphism( phi );
false
gap> IsEpimorphism( phi );

```

```

false
gap> IsSplitEpimorphism( phi );
false
gap> iota := AstrictionToCoimage( phi );
<A monomorphism in FinSets>
gap> pi := CoimageProjection( phi );
<An epimorphism in FinSets>
gap> PreCompose( pi, iota ) = phi;
true

```

1.5.10 Equalizer

Example

```

gap> S := FinSet( [ 1 .. 5 ] );
<An object in FinSets>
gap> T := FinSet( [ 1 .. 3 ] );
<An object in FinSets>
gap> f1 := MapOfFinSets( S, [ [1,3],[2,3],[3,1],[4,2],[5,2] ], T );
<A morphism in FinSets>
gap> f2 := MapOfFinSets( S, [ [1,3],[2,2],[3,3],[4,1],[5,2] ], T );
<A morphism in FinSets>
gap> f3 := MapOfFinSets( S, [ [1,3],[2,1],[3,2],[4,1],[5,2] ], T );
<A morphism in FinSets>
gap> D := [ f1, f2, f3 ];
[ <A morphism in FinSets>, <A morphism in FinSets>, <A morphism in FinSets> ]
gap> Eq := Equalizer( D );
<An object in FinSets>
gap> Display( Eq );
[ 1, 5 ]
gap> iota := EmbeddingOfEqualizer( D );
gap> IsWellDefined( iota );
true
gap> Im := ImageObject( iota );
<An object in FinSets>
gap> Display( Im );
[ 1, 5 ]
gap> mu := MorphismFromEqualizerToSink( D );
gap> PreCompose( iota, f1 ) = mu;
true
gap> M := FinSet( [ "a" ] );
gap> phi := MapOfFinSets( M, [ [ "a", 5 ] ], S );
gap> IsWellDefined( phi );
true
gap> psi := UniversalMorphismIntoEqualizer( D, phi );
<A morphism in FinSets>
gap> IsWellDefined( psi );
true
gap> Display( psi );
[ [ "a" ], [ [ "a", 5 ] ], [ 1, 5 ] ]
gap> PreCompose( psi, iota ) = phi;
true

```


1.5.11 Pullback

Example

```
gap> M := FinSet( [ 1 .. 5 ] );
<An object in FinSets>
gap> N1 := FinSet( [ 1 .. 3 ] );
<An object in FinSets>
gap> iota1 := EmbeddingOfFinSets( N1, M );
<A monomorphism in FinSets>
gap> N2 := FinSet( [ 2 .. 5 ] );
<An object in FinSets>
gap> iota2 := EmbeddingOfFinSets( N2, M );
<A monomorphism in FinSets>
gap> D := [ iota1, iota2 ];
[ <A monomorphism in FinSets>, <A monomorphism in FinSets> ]
gap> int := FiberProduct( D );
<An object in FinSets>
gap> Display( int );
[ [ 2, 2 ], [ 3, 3 ] ]
gap> pi1 := ProjectionInFactorOfFiberProduct( D, 1 );
<A monomorphism in FinSets>
gap> int1 := ImageObject( pi1 );
<An object in FinSets>
gap> Display( int1 );
[ 2, 3 ]
gap> pi2 := ProjectionInFactorOfFiberProduct( D, 2 );
<A monomorphism in FinSets>
gap> int2 := ImageObject( pi2 );
<An object in FinSets>
gap> Display( int2 );
[ 2, 3 ]
```

1.5.12 Coequalizer

Example

```
gap> N := FinSet( [1,3] );
<An object in FinSets>
gap> M := FinSet( [1,2,4] );
<An object in FinSets>
gap> f := MapOfFinSets( N, [ [1,1], [3,2] ], M );
<A morphism in FinSets>
gap> g := MapOfFinSets( N, [ [1,2], [3,4] ], M );
<A morphism in FinSets>
gap> C := Coequalizer( f, g );
<An object in FinSets>
gap> AsList( C );
[ [ 1, 2, 4 ] ]
gap> A := FinSet( [ 1, 2, 3, 4 ] );
<An object in FinSets>
gap> B := FinSet( [ 1, 2, 3, 4, 5, 6, 7, 8 ] );
<An object in FinSets>
gap> f1 := MapOfFinSets( A, [ [ 1, 1 ], [ 2, 2 ], [ 3, 3 ], [ 4, 8 ] ], B );
<A morphism in FinSets>
gap> f2 := MapOfFinSets( A, [ [ 1, 2 ], [ 2, 3 ], [ 3, 8 ], [ 4, 5 ] ], B );
```

```

<A morphism in FinSets>
gap> f3 := MapOfFinSets( A, [ [ 1, 4 ], [ 2, 2 ], [ 3, 3 ], [ 4, 8 ] ], B );
<A morphism in FinSets>
gap> C1 := Coequalizer( [ f1, f3 ] );
<An object in FinSets>
gap> AsList( C1 );
[ [ 1, 4 ], [ 2 ], [ 3 ], [ 5 ], [ 6 ], [ 7 ], [ 8 ] ]
gap> C2 := Coequalizer( [ f1, f2, f3 ] );
<An object in FinSets>
gap> AsList( C2 );
[ [ 1, 2, 3, 8, 5, 4 ], [ 6 ], [ 7 ] ]
gap> S := FinSet( [ 1 .. 5 ] );
<An object in FinSets>
gap> T := FinSet( [ 1 .. 4 ] );
<An object in FinSets>
gap> f := MapOfFinSets( S, [ [1,2], [2,4], [3,4], [4,3], [5,4] ], T );
<A morphism in FinSets>
gap> g := MapOfFinSets( S, [ [1,2], [2,3], [3,4], [4,3], [5,4] ], T );
<A morphism in FinSets>
gap> C := Coequalizer( f, g );
<An object in FinSets>
gap> Display( C );
[ [ 1 ], [ 2 ], [ 4, 3 ] ]
gap> S := FinSet( [ 1, 2, 3, 4, 5 ] );
<An object in FinSets>
gap> T := FinSet( [ 1, 2, 3, 4 ] );
<An object in FinSets>
gap> G_f := [ [ 1, 3 ], [ 2, 4 ], [ 3, 4 ], [ 4, 2 ], [ 5, 4 ] ];
gap> f := MapOfFinSets( S, G_f, T );
<A morphism in FinSets>
gap> G_g := [ [ 1, 3 ], [ 2, 3 ], [ 3, 4 ], [ 4, 2 ], [ 5, 4 ] ];
gap> g := MapOfFinSets( S, G_g, T );
<A morphism in FinSets>
gap> D := [ f, g ];
[ <A morphism in FinSets>, <A morphism in FinSets> ]
gap> C := Coequalizer( D );
<An object in FinSets>
gap> AsList( C );
[ [ 1 ], [ 2 ], [ 3, 4 ] ]
gap> pi := ProjectionOntoCoequalizer( D );
<An epimorphism in FinSets>
gap> AsList( pi );
[ [ 1, [ 1 ] ], [ 2, [ 2 ] ], [ 3, [ 3, 4 ] ], [ 4, [ 3, 4 ] ] ]
gap> mu := MorphismFromSourceToCoequalizer( D );
gap> PreCompose( f, pi ) = mu;
true
gap> G_tau := [ [ 1, 2 ], [ 2, 1 ], [ 3, 2 ], [ 4, 2 ] ];
gap> tau := MapOfFinSets( T, G_tau, FinSet( [ 1, 2 ] ) );
<A morphism in FinSets>
gap> phi := UniversalMorphismFromCoequalizer( D, tau );
<A morphism in FinSets>
gap> AsList( phi );

```

```

[ [ [ 1 ], 2 ], [ [ 2 ], 1 ], [ [ 3, 4 ], 2 ] ]
gap> PreCompose( pi, phi ) = tau;
true
gap> S := FinSet( [ 1, 2, 3, 4, 5 ] );
<An object in FinSets>
gap> T := FinSet( [ 1, 2, 3, 4 ] );
<An object in FinSets>
gap> G_f := [ [ 1, 2 ], [ 2, 3 ], [ 3, 3 ], [ 4, 2 ], [ 5, 4 ] ];
gap> f := MapOfFinSets( S, G_f, T );
<A morphism in FinSets>
gap> G_g := [ [ 1, 2 ], [ 2, 3 ], [ 3, 2 ], [ 4, 2 ], [ 5, 4 ] ];
gap> g := MapOfFinSets( S, G_g, T );
<A morphism in FinSets>
gap> D := [ f, g ];
[ <A morphism in FinSets>, <A morphism in FinSets> ]
gap> C := Coequalizer( D );
<An object in FinSets>
gap> AsList( C );
[ [ 1 ], [ 2, 3 ], [ 4 ] ]
gap> pi := ProjectionOntoCoequalizer( D );
<An epimorphism in FinSets>
gap> AsList( pi );
[ [ 1, [ 1 ] ], [ 2, [ 2, 3 ] ], [ 3, [ 2, 3 ] ], [ 4, [ 4 ] ] ]
gap> PreCompose( f, pi ) = PreCompose( g, pi );
true
gap> mu := MorphismFromSourceToCoequalizer( D );
gap> PreCompose( f, pi ) = mu;
true
gap> G_tau := [ [ 1, 1 ], [ 2, 2 ], [ 3, 2 ], [ 4, 1 ] ];
gap> tau := MapOfFinSets( T, G_tau, FinSet( [ 1, 2 ] ) );
<A morphism in FinSets>
gap> phi := UniversalMorphismFromCoequalizer( D, tau );
<A morphism in FinSets>
gap> AsList( phi );
[ [ [ 1 ], 1 ], [ [ 2, 3 ], 2 ], [ [ 4 ], 1 ] ]
gap> PreCompose( pi, phi ) = tau;
true
gap> A := FinSet( [ "A" ] );
<An object in FinSets>
gap> B := FinSet( [ "B" ] );
<An object in FinSets>
gap> M := FinSetNC( [ A, B ] );
<An object in FinSets>
gap> f := MapOfFinSetsNC( M, [ [ A, A ], [ B, A ] ], M );
<A morphism in FinSets>
gap> g := IdentityMorphism( M );
<An identity morphism in FinSets>
gap> C := Coequalizer( [ f, g ] );
<An object in FinSets>
gap> Length( C );
1
gap> Length( AsList( C )[ 1 ] );

```

```

2
gap> Display( AsList( C )[ 1 ][ 1 ] );
[ "A" ]
gap> Display( AsList( C )[ 1 ][ 2 ] );
[ "B" ]

```

1.5.13 Pushout

Example

```

gap> M := FinSet( [ 1 .. 5 ] );
<An object in FinSets>
gap> N1 := FinSet( [ 1, 2, 4 ] );
<An object in FinSets>
gap> iota1 := EmbeddingOfFinSets( N1, M );
<A monomorphism in FinSets>
gap> N2 := FinSet( [ 2, 3 ] );
<An object in FinSets>
gap> iota2 := EmbeddingOfFinSets( N2, M );
<A monomorphism in FinSets>
gap> D := [ iota1, iota2 ];
[ <A monomorphism in FinSets>, <A monomorphism in FinSets> ]
gap> int := FiberProduct( D );
<An object in FinSets>
gap> Display( int );
[ [ 2, 2 ] ]
gap> pi1 := ProjectionInFactorOfFiberProduct( D, 1 );
<A monomorphism in FinSets>
gap> pi2 := ProjectionInFactorOfFiberProduct( D, 2 );
<A monomorphism in FinSets>
gap> UU := Pushout( pi1, pi2 );
<An object in FinSets>
gap> Display( UU );
[ [ [ 1, 1 ] ], [ [ 1, 2 ] ], [ [ 2, 2 ] ], [ [ 1, 4 ] ], [ [ 2, 3 ] ] ]
gap> iota := UniversalMorphismFromPushout( [ pi1, pi2 ], [ iota1, iota2 ] );
<A morphism in FinSets>
gap> U := ImageObject( iota );
<An object in FinSets>
gap> Display( U );
[ 1, 2, 4, 3 ]
gap> UnionOfFinSets( [ N1, N2 ] ) = U;
true

```

1.5.14 Cartesian lambda introduction

Example

```

gap> S := FinSet( [ 1 .. 3 ] );
<An object in FinSets>
gap> R := FinSet( [ 1 .. 2 ] );
<An object in FinSets>
gap> f := MapOfFinSets( S, [ [1,2],[2,2],[3,1] ], R );
<A morphism in FinSets>
gap> IsWellDefined( f );
true

```

```

gap> T := TerminalObject( f );
<An object in FinSets>
gap> IsTerminal( T );
true
gap> lf := CartesianLambdaIntroduction( f );
<A split monomorphism in FinSets>
gap> Source( lf ) = T;
true
gap> Length( Range( lf ) );
8
gap> lf( T[1] ) = f;
true

```

1.5.15 Topos properties

Example

```

gap> M := FinSet( [ 1 .. 5 ] );
gap> N := FinSet( [ 1, 2, 4 ] );
gap> P := FinSet( [ 1, 4, 8, 9 ] );
gap> G_f := [ [ 1, 1 ], [ 2, 2 ], [ 3, 1 ], [ 4, 2 ], [ 5, 4 ] ];
gap> f := MapOfFinSets( M, G_f, N );
gap> IsWellDefined( f );
true
gap> G_g := [ [ 1, 4 ], [ 2, 4 ], [ 3, 2 ], [ 4, 2 ], [ 5, 1 ] ];
gap> g := MapOfFinSets( M, G_g, N );
gap> IsWellDefined( g );
true
gap> DirectProduct( M, N );
gap> DirectProductOnMorphisms( f, g );
gap> CartesianAssociatorLeftToRight( M, N, P );
gap> CartesianAssociatorRightToLeft( M, N, P );
gap> TerminalObject( FinSets );
gap> CartesianLeftUnitor( M );
gap> CartesianLeftUnitorInverse( M );
gap> CartesianRightUnitor( M );
gap> CartesianRightUnitorInverse( M );
gap> CartesianBraiding( M, N );
gap> CartesianBraidingInverse( M, N );
gap> ExponentialOnObjects( M, N );
gap> ExponentialOnMorphisms( f, g );
gap> CartesianEvaluationMorphism( M, N );
gap> CartesianCoevaluationMorphism( M, N );
gap> DirectProductToExponentialAdjunctionMap( M, N,
>   UniversalMorphismIntoTerminalObject( DirectProduct( M, N ) )
> );
gap> ExponentialToDirectProductAdjunctionMap( M, N,
>   UniversalMorphismFromInitialObject( ExponentialOnObjects( M, N ) )
> );
gap> M := FinSet( [ 1, 2 ] );
gap> N := FinSet( [ "a", "b" ] );
gap> P := FinSet( [ "*" ] );
gap> Q := FinSet( [ "$", "&" ] );
gap> CartesianPreComposeMorphism( M, N, P );

```

```
gap> CartesianPostComposeMorphism( M, N, P );;
gap> DirectProductExponentialCompatibilityMorphism( M, N, P, Q );;
```

1.5.16 Subobject Classifier

Example

```
gap> S := FinSet([1,2,3,4,5]);
<An object in FinSets>
gap> A := FinSet([1,5]);
<An object in FinSets>
gap> m := MapOfFinSets(A, List(AsList(A), x -> [x,x]), S);
<A morphism in FinSets>
gap> Display(TruthMorphismIntoSubobjectClassifier(FinSets));
[ [ "*" ], [ [ "*", "true" ] ], [ "true", "false" ] ]
gap> Display(ClassifyingMorphismOfSubobject(m));
[ [ 1, 2, 3, 4, 5 ], [ [ 1, "true" ], [ 2, "false" ], [ 3, "false" ],
[ 4, "false" ], [ 5, "true" ] ], [ "true", "false" ] ]
```

Chapter 2

The category of skeletal finite sets

2.1 Skeletal GAP Categories

2.1.1 IsCategoryOfSkeletalFinSets (for IsCapCategory)

▷ `IsCategoryOfSkeletalFinSets(object)` (filter)
Returns: true or false
The GAP category of categories of skeletal finite sets.

2.1.2 IsSkeletalFiniteSet (for IsCapCategoryObject and IsCellOfSkeletalCategory)

▷ `IsSkeletalFiniteSet(object)` (filter)
Returns: true or false
The GAP category of objects in the category of skeletal finite sets.

2.1.3 IsSkeletalFiniteSetMap (for IsCapCategoryMorphism and IsCellOfSkeletalCategory)

▷ `IsSkeletalFiniteSetMap(object)` (filter)
Returns: true or false
The GAP category of morphisms in the category of skeletal finite sets.

2.2 Skeletal Attributes

2.2.1 Length (for IsSkeletalFiniteSet)

▷ `Length(M)` (attribute)
Returns: an integer
The integer defining the skeletal finite set M , i.e., `Length(FinSet(n)) = n`.

2.2.2 AsList (for IsSkeletalFiniteSet)

▷ `AsList(M)` (attribute)
Returns: a list
The list associated to a skeletal finite set, i.e., `AsList(FinSet(n)) = [1 .. n]`.

2.3 Skeletal Constructors

2.3.1 CategoryOfSkeletalFinSets

- ▷ `CategoryOfSkeletalFinSets()` (operation)
Returns: a CAP category
 Construct a category of skeletal finite sets. Accepts the options `overhead` (default: `true`) and `FinalizeCategory` (default: `true`).

2.3.2 SkeletalFinSets

- ▷ `SkeletalFinSets` (global variable)

The default instance of the category of skeletal finite sets. It is automatically created while loading this package.

2.3.3 FinSet (for IsInt)

- ▷ `FinSet(n)` (operation)
Returns: a CAP object
 Construct a skeletal finite set residing in the default instance of the category of skeletal finite sets `SkeletalFinSets` of order given by the nonnegative integer n .

Example

```
gap> m := FinSet( 7 );
<An object in SkeletalFinSets>
gap> IsWellDefined( m );
true
gap> n := FinSet( -2 );
<An object in SkeletalFinSets>
gap> IsWellDefined( n );
false
gap> n := FinSet( 3 );
<An object in SkeletalFinSets>
gap> IsWellDefined( n );
true
gap> p := FinSet( 4 );
<An object in SkeletalFinSets>
gap> IsWellDefined( p );
true
```

2.3.4 FinSet (for IsCategoryOfSkeletalFinSets, IsInt)

- ▷ `FinSet(C, n)` (operation)
Returns: a CAP object
 Construct a skeletal finite set residing in the given category of skeletal finite sets C of order given by the nonnegative integer n .

2.3.5 MapOfFinSets (for IsSkeletalFiniteSet, IsList, IsSkeletalFiniteSet)

▷ MapOfFinSets(s , G , t) (operation)

Returns: a CAP morphism

Construct a map $\phi : s \rightarrow t$ of the skeletal finite sets s and t , i.e., a morphism in the CAP category of s , where G is a list of integers in t describing the graph of ϕ .

Example

```
gap> s := FinSet( 3 );
<An object in SkeletalFinSets>
gap> t := FinSet( 7 );
<An object in SkeletalFinSets>
gap> phi := MapOfFinSets( s, [7, 5, 5], t );
<A morphism in SkeletalFinSets>
gap> IsWellDefined( phi );
true
gap> Display( phi );
[ 3, [ 7, 5, 5 ], 7 ]
```

2.4 Skeletal Tools

2.4.1 ListOp (for IsSkeletalFiniteSet, IsFunction)

▷ ListOp(s , f) (operation)

Returns: a list

Returns List(AsList(s), f).

2.4.2 EmbeddingOfFinSets (for IsSkeletalFiniteSet, IsSkeletalFiniteSet)

▷ EmbeddingOfFinSets(s , t) (operation)

Returns: a CAP morphism

Construct the embedding $\iota : s \rightarrow t$ of the finite sets s and t , where s must be subset of t .

2.4.3 Preimage (for IsSkeletalFiniteSetMap, IsList)

▷ Preimage(ϕ , t) (operation)

Returns: a CAP object

Compute the Preimage of t under the morphism ϕ .

2.4.4 ImageObject (for IsSkeletalFiniteSetMap, IsSkeletalFiniteSet)

▷ ImageObject(ϕ , s_*) (operation)

Returns: a CAP object

Compute the image of s_* under the morphism ϕ .

2.4.5 CallFuncList (for IsSkeletalFiniteSetMap, IsList)

▷ CallFuncList(ϕ , L) (operation)

Returns: a list

Returns the image of $L[1]$ under the map ϕ assuming $L[1]$ is a positive integer smaller or equal to $\text{Length}(\text{Source}(\phi))$.

2.5 Skeletal Examples

2.5.1 SkeletalIsHomSetInhabited

Example

```
gap> L := FinSet( 0 );
<An object in SkeletalFinSets>
gap> M := FinSet( 2 );
<An object in SkeletalFinSets>
gap> N := FinSet( 3 );
<An object in SkeletalFinSets>
gap> IsHomSetInhabited( L, L );
true
gap> IsHomSetInhabited( M, L );
false
gap> IsHomSetInhabited( L, M );
true
gap> IsHomSetInhabited( M, N );
true
```

2.5.2 SkeletalWellDefined

Example

```
gap> s := FinSet( 7 );
<An object in SkeletalFinSets>
gap> t := FinSet( 4 );
<An object in SkeletalFinSets>
gap> psi := MapOfFinSets( s, [ 1, 3, 2, 3, 2, 4 ], t );
<A morphism in SkeletalFinSets>
gap> IsWellDefined( psi );
false
gap> psi := MapOfFinSets( s, [ 1, 3, 2, 3, 2, 4, -1 ], t );
<A morphism in SkeletalFinSets>
gap> IsWellDefined( psi );
false
gap> psi := MapOfFinSets( s, [ 2, 3, 2, 5, 3, 2, 4 ], t );
<A morphism in SkeletalFinSets>
gap> IsWellDefined( psi );
false
gap> psi := MapOfFinSets( s, [ 1, 3, 2, 4, 3, 2, 4 ], t );
<A morphism in SkeletalFinSets>
gap> IsWellDefined( psi );
true
```

2.5.3 SkeletalPreCompose

Example

```
gap> m := FinSet( 3 );
<An object in SkeletalFinSets>
gap> n := FinSet( 5 );
```

```

<An object in SkeletalFinSets>
gap> p := FinSet( 7 );
<An object in SkeletalFinSets>
gap> psi := MapOfFinSets( m, [ 2, 5, 3 ], n );
<A morphism in SkeletalFinSets>
gap> phi := MapOfFinSets( n, [ 1, 4, 6, 6, 3 ], p );
<A morphism in SkeletalFinSets>
gap> alpha := PreCompose( psi, phi );
<A morphism in SkeletalFinSets>
gap> Display( alpha );
[ 3, [ 4, 3, 6 ], 7 ]

```

2.5.4 Skeletal Monomorphisms and Epimorphisms

Example

```

gap> m := FinSet( 3 );
<An object in SkeletalFinSets>
gap> n := FinSet( 5 );
<An object in SkeletalFinSets>
gap> p := FinSet( 7 );
<An object in SkeletalFinSets>
gap> psi := MapOfFinSets( m, [ 1, 3, 5 ], n );
<A morphism in SkeletalFinSets>
gap> IsEpimorphism( psi );
false
gap> IsSplitEpimorphism( psi );
false
gap> IsMonomorphism( psi );
true
gap> IsSplitMonomorphism( psi );
true
gap> psi := MapOfFinSets( p, [ 1, 3, 2, 3, 3, 2, 1 ], m );
<A morphism in SkeletalFinSets>
gap> IsEpimorphism( psi );
true
gap> IsSplitEpimorphism( psi );
true
gap> IsMonomorphism( psi );
false
gap> IsSplitMonomorphism( psi );
false

```

2.5.5 Skeletal Initial and Terminal Objects

Example

```

gap> m := FinSet( 8 );
<An object in SkeletalFinSets>
gap> IsInitial( m );
false
gap> IsTerminal( m );
false
gap> i := InitialObject( m );
<An object in SkeletalFinSets>

```

```

gap> IsInitial( i );
true
gap> IsTerminal( i );
false
gap> iota := UniversalMorphismFromInitialObject( m );
<A morphism in SkeletalFinSets>
gap> AsList( i );
[ ]
gap> t := TerminalObject( m );
<An object in SkeletalFinSets>
gap> AsList( t );
[ 1 ]
gap> IsInitial( t );
false
gap> IsTerminal( t );
true
gap> pi := UniversalMorphismIntoTerminalObject( m );
<A morphism in SkeletalFinSets>
gap> IsIdenticalObj( Range( pi ), t );
true
gap> pi_t := UniversalMorphismIntoTerminalObjectWithGivenTerminalObject( m, t );
<A morphism in SkeletalFinSets>
gap> AsList( pi_t );
[ 1, 1, 1, 1, 1, 1, 1, 1 ]
gap> pi = pi_t;
true

```

2.5.6 Projective and Injective Objects

Example

```

gap> I := FinSet( 0 );
<An object in SkeletalFinSets>
gap> T := FinSet( 1 );
<An object in SkeletalFinSets>
gap> M := FinSet( 2 );
<An object in SkeletalFinSets>
gap> IsProjective( I );
true
gap> IsProjective( T );
true
gap> IsProjective( M );
true
gap> IsOne( EpimorphismFromSomeProjectiveObject( I ) );
true
gap> IsOne( EpimorphismFromSomeProjectiveObject( M ) );
true

```

Example

```

gap> I := FinSet( 0 );
<An object in SkeletalFinSets>
gap> T := FinSet( 1 );
<An object in SkeletalFinSets>
gap> M := FinSet( 2 );

```

```

<An object in SkeletalFinSets>
gap> IsInjective( I );
false
gap> IsInjective( T );
true
gap> IsInjective( M );
true
gap> IsIsomorphism( MonomorphismIntoSomeInjectiveObject( I ) );
false
gap> IsMonomorphism( MonomorphismIntoSomeInjectiveObject( I ) );
true
gap> IsOne( MonomorphismIntoSomeInjectiveObject( M ) );
true

```

2.5.7 Skeletal Product

Example

```

gap> m := FinSet( 7 );
<An object in SkeletalFinSets>
gap> n := FinSet( 3 );
<An object in SkeletalFinSets>
gap> p := FinSet( 4 );
<An object in SkeletalFinSets>
gap> d := DirectProduct( [ m, n, p ] );
<An object in SkeletalFinSets>
gap> AsList( d );
[ 1 .. 84 ]
gap> pi1 := ProjectionInFactorOfDirectProduct( [ m, n, p ], 1 );
<A morphism in SkeletalFinSets>
gap> Display( pi1 );
[ 84,
  [ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2,
    2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4,
    4, 4, 4, 4, 4, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 6, 6, 6, 6,
    6, 6, 6, 6, 6, 6, 6, 6, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7 ], 7
]
gap> pi2 := ProjectionInFactorOfDirectProduct( [ m, n, p ], 2 );
<A morphism in SkeletalFinSets>
gap> Display( pi2 );
[ 84,
  [ 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3,
    3, 3, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 1, 1, 1, 1, 2, 2, 2,
    2, 3, 3, 3, 3, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 1, 1, 1, 1,
    2, 2, 2, 2, 3, 3, 3, 3, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3 ], 3
]
gap> pi3 := ProjectionInFactorOfDirectProduct( [ m, n, p ], 3 );
<A morphism in SkeletalFinSets>
gap> Display( pi3 );
[ 84,
  [ 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2,
    3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3,
    4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4,
    1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4 ], 4
]

```

```

]
gap> psi := UniversalMorphismIntoDirectProduct( [ m, n, p ], [ pi1, pi2, pi3 ] );
<A morphism in SkeletalFinSets>
gap> psi = IdentityMorphism( d );
true

```

2.5.8 Skeletal Coproduct

Example

```

gap> m := FinSet( 7 );
<An object in SkeletalFinSets>
gap> n := FinSet( 3 );
<An object in SkeletalFinSets>
gap> p := FinSet( 4 );
<An object in SkeletalFinSets>
gap> c := Coproduct( m, n, p );
<An object in SkeletalFinSets>
gap> AsList( c );
[ 1 .. 14 ]
gap> iota1 := InjectionOfCofactorOfCoproduct( [ m, n, p ], 1 );
<A morphism in SkeletalFinSets>
gap> IsWellDefined( iota1 );
true
gap> Display( iota1 );
[ 7, [ 1, 2, 3, 4, 5, 6, 7 ], 14 ]
gap> iota2 := InjectionOfCofactorOfCoproduct( [ m, n, p ], 2 );
<A morphism in SkeletalFinSets>
gap> IsWellDefined( iota2 );
true
gap> Display( iota2 );
[ 3, [ 8, 9, 10 ], 14 ]
gap> iota3 := InjectionOfCofactorOfCoproduct( [ m, n, p ], 3 );
<A morphism in SkeletalFinSets>
gap> IsWellDefined( iota3 );
true
gap> Display( iota3 );
[ 4, [ 11, 12, 13, 14 ], 14 ]
gap> psi := UniversalMorphismFromCoproduct( [ m, n, p ],
>                                           [ iota1, iota2, iota3 ]
>                                           );
<A morphism in SkeletalFinSets>
gap> psi = IdentityMorphism( Coproduct( [ m, n, p ] ) );
true

```

2.5.9 Skeletal Image

Example

```

gap> m := FinSet( 7 );
<An object in SkeletalFinSets>
gap> n := FinSet( 3 );
<An object in SkeletalFinSets>
gap> phi := MapOfFinSets( n, [7, 5, 5], m );
<A morphism in SkeletalFinSets>

```

```

gap> IsWellDefined( phi );
true
gap> ImageObject( phi );
<An object in SkeletalFinSets>
gap> Length( ImageObject( phi ) );
2
gap> s := FinSet( 2 );
<An object in SkeletalFinSets>
gap> I := ImageObject( phi, s );
<An object in SkeletalFinSets>
gap> Length( I );
2

```

Example

```

gap> m := FinSet( 7 );
<An object in SkeletalFinSets>
gap> n := FinSet( 3 );
<An object in SkeletalFinSets>
gap> phi := MapOfFinSets( n, [ 7, 5, 5 ], m );
<A morphism in SkeletalFinSets>
gap> pi := ImageEmbedding( phi );
<A monomorphism in SkeletalFinSets>
gap> Display( pi );
[ 2, [ 5, 7 ], 7 ]

```

Example

```

gap> m := FinSet( 7 );
<An object in SkeletalFinSets>
gap> n := FinSet( 3 );
<An object in SkeletalFinSets>
gap> phi := MapOfFinSets( n, [ 7, 5, 5 ], m );
<A morphism in SkeletalFinSets>
gap> IsWellDefined( phi );
true
gap> f := CostrictionToImage( phi );
<An epimorphism in SkeletalFinSets>
gap> Display( f );
[ 3, [ 2, 1, 1 ], 2 ]
gap> IsWellDefined( f );
true
gap> IsEpimorphism( f );
true
gap> IsSplitEpimorphism( f );
true
gap> m := FinSet( 77 );
<An object in SkeletalFinSets>
gap> n := FinSet( 4 );
<An object in SkeletalFinSets>
gap> phi := MapOfFinSets( n, [ 77, 2, 25, 2 ], m );
<A morphism in SkeletalFinSets>
gap> IsWellDefined( phi );
true
gap> iota := ImageEmbedding( phi );
<A monomorphism in SkeletalFinSets>

```

```

gap> pi := CostrictionToImage( phi );
<An epimorphism in SkeletalFinSets>
gap> Display( pi );
[ 4, [ 3, 1, 2, 1 ], 3 ]
gap> IsWellDefined( pi );
true
gap> IsEpimorphism( pi );
true
gap> IsSplitEpimorphism( pi );
true
gap> PreCompose( pi, iota ) = phi;
true

```

2.5.10 Skeletal Preimage

Example

```

gap> m := FinSet( 7 );
<An object in SkeletalFinSets>
gap> n := FinSet( 3 );
<An object in SkeletalFinSets>
gap> phi := MapOfFinSets( n, [7, 5, 5], m );
<A morphism in SkeletalFinSets>
gap> P := Preimage( phi, [2] );
[ ]
gap> P := Preimage( phi, AsList( FinSet( 5 ) ) );
[ 2, 3 ]

```

2.5.11 Skeletal Equalizer

Example

```

gap> S := FinSet( 5 );
<An object in SkeletalFinSets>
gap> T := FinSet( 3 );
<An object in SkeletalFinSets>
gap> f1 := MapOfFinSets( S, [ 3, 3, 1, 2, 2 ], T );
<A morphism in SkeletalFinSets>
gap> f2 := MapOfFinSets( S, [ 3, 2, 3, 1, 2 ], T );
<A morphism in SkeletalFinSets>
gap> f3 := MapOfFinSets( S, [ 3, 1, 2, 1, 2 ], T );
<A morphism in SkeletalFinSets>
gap> D := [ f1, f2, f3 ];
gap> Eq := Equalizer( D );
<An object in SkeletalFinSets>
gap> Length( Eq );
2
gap> iota := EmbeddingOfEqualizer( D );
<A monomorphism in SkeletalFinSets>
gap> Display( iota );
[ 2, [ 1, 5 ], 5 ]
gap> phi := MapOfFinSets( FinSet( 2 ), [ 5, 1 ], S );
gap> IsWellDefined( phi );
true
gap> psi := UniversalMorphismIntoEqualizer( D, phi );

```



```

<A morphism in SkeletalFinSets>
gap> IsWellDefined( psi );
true
gap> Display( psi );
[ 2, [ 2, 1 ], 2 ]
gap> PreCompose( psi, iota ) = phi;
true
gap> D := [ f2, f3 ];
[ <A morphism in SkeletalFinSets>, <A morphism in SkeletalFinSets> ]
gap> Eq := Equalizer( D );
<An object in SkeletalFinSets>
gap> Length( Eq );
3
gap> psi := EmbeddingOfEqualizer( D );
<A monomorphism in SkeletalFinSets>
gap> Display( psi );
[ 3, [ 1, 4, 5 ], 5 ]

```

2.5.12 Skeletal Pullback

Example

```

gap> m := FinSet( 5 );
<An object in SkeletalFinSets>
gap> n1 := FinSet( 3 );
<An object in SkeletalFinSets>
gap> iota1 := EmbeddingOfFinSets( n1, m );
<A monomorphism in SkeletalFinSets>
gap> Display( iota1 );
[ 3, [ 1, 2, 3 ], 5 ]
gap> n2 := FinSet( 4 );
<An object in SkeletalFinSets>
gap> iota2 := EmbeddingOfFinSets( n2, m );
<A monomorphism in SkeletalFinSets>
gap> Display( iota2 );
[ 4, [ 1, 2, 3, 4 ], 5 ]
gap> D := [ iota1, iota2 ];
[ <A monomorphism in SkeletalFinSets>, <A monomorphism in SkeletalFinSets> ]
gap> Fib := FiberProduct( D );
<An object in SkeletalFinSets>
gap> Display( Fib );
3
gap> pi1 := ProjectionInFactorOfFiberProduct( D, 1 );
<A monomorphism in SkeletalFinSets>
gap> Display( pi1 );
[ 3, [ 1, 2, 3 ], 3 ]
gap> int1 := ImageObject( pi1 );
<An object in SkeletalFinSets>
gap> Display( int1 );
3
gap> pi2 := ProjectionInFactorOfFiberProduct( D, 2 );
<A monomorphism in SkeletalFinSets>
gap> Display( pi2 );
[ 3, [ 1, 2, 3 ], 4 ]

```

```

gap> int2 := ImageObject( pi2 );
<An object in SkeletalFinSets>
gap> Display( int2 );
3
gap> omega1 := PreCompose( pi1, iota1 );
<A monomorphism in SkeletalFinSets>
gap> omega2 := PreCompose( pi2, iota2 );
<A monomorphism in SkeletalFinSets>
gap> omega1 = omega2;
true
gap> Display( omega1 );
[ 3, [ 1, 2, 3 ], 5 ]

```

2.5.13 Skeletal Coequalizer

Example

```

gap> s := FinSet( 5 );
<An object in SkeletalFinSets>
gap> t := FinSet( 4 );
<An object in SkeletalFinSets>
gap> f := MapOfFinSets( s, [ 3, 4, 4, 2, 4 ], t );
<A morphism in SkeletalFinSets>
gap> g := MapOfFinSets( s, [ 3, 3, 4, 2, 4 ], t );
<A morphism in SkeletalFinSets>
gap> D := [ f, g ];
[ <A morphism in SkeletalFinSets>, <A morphism in SkeletalFinSets> ]
gap> C := Coequalizer( D );
<An object in SkeletalFinSets>
gap> Length( C );
3
gap> pi := ProjectionOntoCoequalizer(D);
<An epimorphism in SkeletalFinSets>
gap> Display( pi );
[ 4, [ 1, 2, 3, 3 ], 3 ]
gap> tau := MapOfFinSets( t, [2, 1, 2, 2], FinSet( 2 ) );
<A morphism in SkeletalFinSets>
gap> phi := UniversalMorphismFromCoequalizer( D, tau );
<A morphism in SkeletalFinSets>
gap> Display( phi );
[ 3, [ 2, 1, 2 ], 2 ]
gap> PreCompose( pi, phi ) = tau;
true
gap> s := FinSet( 5 );
<An object in SkeletalFinSets>
gap> t := FinSet( 4 );
<An object in SkeletalFinSets>
gap> f := MapOfFinSets( s, [ 2, 3, 3, 2, 4 ], t );
<A morphism in SkeletalFinSets>
gap> g := MapOfFinSets( s, [ 2, 3, 2, 2, 4 ], t );
<A morphism in SkeletalFinSets>
gap> D := [ f, g ];
[ <A morphism in SkeletalFinSets>, <A morphism in SkeletalFinSets> ]
gap> C := Coequalizer( D );

```

```

<An object in SkeletalFinSets>
gap> Length( C );
3
gap> pi := ProjectionOntoCoequalizer( D );
<An epimorphism in SkeletalFinSets>
gap> Display( pi );
[ 4, [ 1, 2, 2, 3 ], 3 ]
gap> PreCompose( f, pi ) = PreCompose( g, pi );
true
gap> tau := MapOfFinSets( t, [1, 2, 2, 1 ], FinSet( 2 ) );
<A morphism in SkeletalFinSets>
gap> phi := UniversalMorphismFromCoequalizer( D, tau );
<A morphism in SkeletalFinSets>
gap> Display( phi );
[ 3, [ 1, 2, 1 ], 2 ]
gap> PreCompose( pi, phi ) = tau;
true
gap> s := FinSet( 2 );;
gap> t := FinSet( 3 );;
gap> f := MapOfFinSets( s, [ 1, 2 ], t );;
gap> IsWellDefined( f );
true
gap> g := MapOfFinSets( s, [ 2, 3 ], t );;
gap> IsWellDefined( g );
true
gap> C := Coequalizer( [ f, g ] );
<An object in SkeletalFinSets>
gap> Length( C );
1

```

2.5.14 Skeletal Pushout

Example

```

gap> M := FinSet( 5 );
<An object in SkeletalFinSets>
gap> N1 := FinSet( 3 );
<An object in SkeletalFinSets>
gap> iota1 := EmbeddingOfFinSets( N1, M );
<A monomorphism in SkeletalFinSets>
gap> Display( iota1 );
[ 3, [ 1, 2, 3 ], 5 ]
gap> N2 := FinSet( 2 );
<An object in SkeletalFinSets>
gap> iota2 := EmbeddingOfFinSets( N2, M );
<A monomorphism in SkeletalFinSets>
gap> Display( iota2 );
[ 2, [ 1, 2 ], 5 ]
gap> D := [ iota1, iota2 ];
[ <A monomorphism in SkeletalFinSets>, <A monomorphism in SkeletalFinSets> ]
gap> Fib := FiberProduct( D );
<An object in SkeletalFinSets>
gap> Display( Fib );
2

```

```

gap> pi1 := ProjectionInFactorOfFiberProduct( D, 1 );
<A monomorphism in SkeletalFinSets>
gap> Display( pi1 );
[ 2, [ 1, 2 ], 3 ]
gap> pi2 := ProjectionInFactorOfFiberProduct( D, 2 );
<A monomorphism in SkeletalFinSets>
gap> Display( pi2 );
[ 2, [ 1, 2 ], 2 ]
gap> ## The easy way
>
> D := [ pi1, pi2 ];
[ <A monomorphism in SkeletalFinSets>, <A monomorphism in SkeletalFinSets> ]
gap> UU := Pushout( D );
<An object in SkeletalFinSets>
gap> Display( UU );
3
gap> kappa1 := InjectionOfCofactorOfPushout( D, 1 );
<A morphism in SkeletalFinSets>
gap> Display( kappa1 );
[ 3, [ 1, 2, 3 ], 3 ]
gap> kappa2 := InjectionOfCofactorOfPushout( D, 2 );
<A morphism in SkeletalFinSets>
gap> Display( kappa2 );
[ 2, [ 1, 2 ], 3 ]
gap> PreCompose( pi1, kappa1 ) = PreCompose( pi2, kappa2 );
true
gap> ## The long way
>
> Co := Coproduct( N1, N2 );
<An object in SkeletalFinSets>
gap> Display( Co );
5
gap> iota_1 := InjectionOfCofactorOfCoproduct( [ N1, N2 ], 1 );
<A morphism in SkeletalFinSets>
gap> Display( iota_1 );
[ 3, [ 1, 2, 3 ], 5 ]
gap> iota_2 := InjectionOfCofactorOfCoproduct( [ N1, N2 ], 2 );
<A morphism in SkeletalFinSets>
gap> Display( iota_2 );
[ 2, [ 4, 5 ], 5 ]
gap> alpha := PreCompose( pi1, iota_1 );
<A morphism in SkeletalFinSets>
gap> Display( alpha );
[ 2, [ 1, 2 ], 5 ]
gap> beta := PreCompose( pi2, iota_2 );
<A morphism in SkeletalFinSets>
gap> Display( beta );
[ 2, [ 4, 5 ], 5 ]
gap> Cq := Coequalizer( [ alpha, beta ] );
<An object in SkeletalFinSets>
gap> Display( Cq );
3

```

```

gap> psi := ProjectionOntoCoequalizer( [ alpha, beta ] );
<An epimorphism in SkeletalFinSets>
gap> Display( psi );
[ 5, [ 1, 2, 3, 1, 2 ], 3 ]
gap> Display( PreCompose( iota_1, psi ) );
[ 3, [ 1, 2, 3 ], 3 ]
gap> Display( PreCompose( iota_2, psi ) );
[ 2, [ 1, 2 ], 3 ]
gap> PreCompose( alpha, psi ) = PreCompose( beta, psi );
true

```

2.5.15 Skeletal Lift

Example

```

gap> m := FinSet( 5 );
<An object in SkeletalFinSets>
gap> n := FinSet( 4 );
<An object in SkeletalFinSets>
gap> f := MapOfFinSets( m, [ 2, 2, 1, 1, 3 ], n );
<A morphism in SkeletalFinSets>
gap> g := MapOfFinSets( m, [ 5, 5, 4, 4, 5 ], m );
<A morphism in SkeletalFinSets>
gap> IsColiftable( f, g );
true
gap> chi := Colift( f, g );
<A morphism in SkeletalFinSets>
gap> Display( chi );
[ 4, [ 4, 5, 5, 1 ], 5 ]
gap> PreCompose( f, Colift( f, g ) ) = g;
true
gap> IsColiftable( g, f );
false
gap> Colift( g, f );
fail

```

Example

```

gap> m := FinSet( 3 );
<An object in SkeletalFinSets>
gap> n := FinSet( 4 );
<An object in SkeletalFinSets>
gap> f := MapOfFinSets( m, [ 2, 2, 1 ], m );
<A morphism in SkeletalFinSets>
gap> g := MapOfFinSets( n, [ 3, 2, 1, 2 ], m );
<A morphism in SkeletalFinSets>
gap> IsLiftable( f, g );
true
gap> chi := Lift( f, g );
<A morphism in SkeletalFinSets>
gap> Display( chi );
[ 3, [ 2, 2, 3 ], 4 ]
gap> PreCompose( Lift( f, g ), g ) = f;
true
gap> IsLiftable( g, f );
false

```

```

gap> Lift( g, f );
fail
gap> k := FinSet( 100000 );
<An object in SkeletalFinSets>
gap> h := ListWithIdenticalEntries( Length( k ) - 3, 3 );;
gap> h := Concatenation( h, [ 2, 1, 2 ] );;
gap> h := MapOfFinSets( k, h, m );
<A morphism in SkeletalFinSets>
gap> IsLiftable( f, h );
true
gap> IsLiftable( h, f );
false

```

2.5.16 Skeletal Colift

2.5.17 Skeletal topos properties

Example

```

gap> M := FinSet( 4 );;
gap> N := FinSet( 3 );;
gap> P := FinSet( 4 );;
gap> G_f := [ 1, 2, 1, 3 ];;
gap> f := MapOfFinSets( M, G_f, N );;
gap> IsWellDefined( f );
true
gap> G_g := [ 3, 3, 2, 1 ];;
gap> g := MapOfFinSets( M, G_g, N );;
gap> IsWellDefined( g );
true
gap> DirectProduct( M, N );;
gap> DirectProductOnMorphisms( f, g );;
gap> CartesianAssociatorLeftToRight( M, N, P );;
gap> CartesianAssociatorRightToLeft( M, N, P );;
gap> TerminalObject( FinSets );;
gap> CartesianLeftUnitor( M );;
gap> CartesianLeftUnitorInverse( M );;
gap> CartesianRightUnitor( M );;
gap> CartesianRightUnitorInverse( M );;
gap> CartesianBraiding( M, N );;
gap> CartesianBraidingInverse( M, N );;
gap> ExponentialOnObjects( M, N );;
gap> ExponentialOnMorphisms( f, g );;
gap> CartesianEvaluationMorphism( M, N );;

```

Index

- `[]`
 - for `IsFiniteSet`, `IsInt`, 8
- `AsList`
 - for `IsFiniteSet`, 3
 - for `IsFiniteSetMap`, 3
 - for `IsSkeletalFiniteSet`, 22
- `\in`
 - for `IsObject`, `IsFiniteSet`, 8
- `CallFuncList`
 - for `IsFiniteSetMap`, `IsList`, 9
 - for `IsSkeletalFiniteSetMap`, `IsList`, 24
- `CategoryOfSkeletalFinSets`, 23
- `EmbeddingOfFinSets`
 - for `IsFiniteSet`, `IsFiniteSet`, 9
 - for `IsSkeletalFiniteSet`, `IsSkeletalFiniteSet`, 24
- `FilteredOp`
 - for `IsFiniteSet`, `IsFunction`, 8
- `FinSet`
 - for `IsCategoryOfSkeletalFinSets`, `IsInt`, 23
 - for `IsInt`, 23
 - for `IsList`, 4
- `FinSetNC`
 - for `IsList`, 4
- `FirstOp`
 - for `IsFiniteSet`, `IsFunction`, 9
- `ImageObject`
 - for `IsFiniteSetMap`, `IsFiniteSet`, 9
 - for `IsSkeletalFiniteSetMap`, `IsSkeletalFiniteSet`, 24
- `IsCategoryOfSkeletalFinSets`
 - for `IsCapCategory`, 22
- `IsEqualForElementsOfFinSets`
 - for `IsObject`, `IsObject`, 7
- `IsFiniteSet`
 - for `IsCapCategoryObject`, 3
- `IsFiniteSetMap`
 - for `IsCapCategoryMorphism`, 3
- `IsSkeletalFiniteSet`
 - for `IsCapCategoryObject` and `IsCellOfSkeletalCategory`, 22
- `IsSkeletalFiniteSetMap`
 - for `IsCapCategoryMorphism` and `IsCellOfSkeletalCategory`, 22
- `Iterator`
 - for `IsFiniteSet`, 8
- `Length`
 - for `IsFiniteSet`, 3
 - for `IsSkeletalFiniteSet`, 22
- `ListOp`
 - for `IsFiniteSet`, `IsFiniteSetMap`, 9
 - for `IsFiniteSet`, `IsFunction`, 8
 - for `IsSkeletalFiniteSet`, `IsFunction`, 24
- `MapOfFinSets`
 - for `IsFiniteSet`, `IsList`, `IsFiniteSet`, 5
 - for `IsSkeletalFiniteSet`, `IsList`, `IsSkeletalFiniteSet`, 24
- `MapOfFinSetsNC`
 - for `IsFiniteSet`, `IsList`, `IsFiniteSet`, 6
- `Preimage`
 - for `IsFiniteSetMap`, `IsFiniteSet`, 9
 - for `IsSkeletalFiniteSetMap`, `IsList`, 24
- `ProjectionOfFinSets`
 - for `IsFiniteSet`, `IsFiniteSet`, 9
- `SkeletalFinSets`, 23
- `UnionOfFinSets`
 - for `IsList`, 8