18th International Conference on Information Fusion
Washington, DC - July 6-9, 2015

A Generalised Labelled Multi-Bernoulli Filter for Extended Multi-target Tracking

Michael Beard*†, Stephan Reuter‡, Karl Granström§, Ba-Tuong Vo†, Ba-Ngu Vo†, Alexander Scheel‡
*Maritime Division, Defence Science and Technology Organisation, Rockingham, WA, Australia
†Department of Electrical and Computer Engineering, Curtin University, Bentley, WA, Australia
‡Institute of Measurement, Control and Microtechnology, Ulm University, Ulm, Germany
§Department of Electrical and Computer Engineering, University of Connecticut, Storrs, CT, USA
michael.beard@dsto.defence.gov.au, stephan.reuter@uni-ulm.de, karl@engr.uconn.edu,
ba-tuong.vo@curtin.edu.au, ba-ngu.vo@curtin.edu.au, alexander.scheel@uni-ulm.de

***Abstract*—This paper addresses extended multi-target tracking in clutter, i.e. tracking targets that may produce more than one measurement on each scan. We propose a new algorithm for solving this problem, that is capable of initiating and maintaining labelled estimates of the target kinematics, measurement rates and extents. Our proposed technique is based on modelling the multi-target state as a generalised labelled multi-Bernoulli (GLMB), combined with the gamma Gaussian inverse Wishart (GGIW) distribution for a single extended target. Previously, probability hypothesis density (PHD) and cardinalised PHD (CPHD) filters based on GGIW mixtures have been proposed to solve the extended target tracking problem. Although these are computationally cheaper, they involve significant approximations, as well as lacking the ability to maintain target tracks over time. Here, we compare our proposed GLMB-based approach to the extended target PHD/CPHD filters, and show that the GLMB has improved performance.**

***Index Terms*—Multi-target tracking, extended targets, random finite sets, inverse Wishart**

I. Introduction

The goal of multiple target tracking is to estimate the number of targets and their states, based on noisy sensor measurements in the presence of missed detections and false alarms. To achieve this, tracking algorithms require models that describe how the observed measurements are related to the underlying target states. In most cases, the standard 'point target' model is used, which imposes the constraints that each measurement originates from at most one target, and each target gives rise to at most one measurement on each scan. This model greatly simplifies the development of multi-target tracking algorithms, but it is often unrealistic in practical situations.

Non-standard measurement models are capable of handling more realistic measurement generation processes by relaxing the aforementioned constraints, usually at the expense of increased computation. One example of this is when a group of targets gives rise to a single measurement, known as an unresolved target (or merged measurement) model [1]. Such a model is useful when dealing with low-resolution sensors that are incapable of generating seperate detections for closely spaced targets. On the other hand, in the case of higher resolution sensors, each target may give rise to multiple measurements on each scan. This is known as an extended target model [2], and is the subject of this paper.

An extended target measurement model usually requires two basic ingredients; a model for the number of measurements generated by each target, and a model for how those measurements are distributed. Clearly, these models will depend strongly on the type of targets being tracked. For example, in a radar application, some targets may posess many scatter points, each of which generates a distinct detection. However, other targets may be more stealthy and reflect most of the energy away from the receiver, therefore generating very few detections, or none at all. In general, when the targets are far enough away from the sensor, the detections from a single target can usually be characterised as a cluster of points that exhibits no specific geometric structure. In this case, the number of measurements is usually modelled as Poisson, which was the approach taken in [2] and [3].

Despite the lack of any specific target structure, it is still possible to estimate the size and shape of a target (known as the target extent), based on the spatial distribution of the measurements it generates. An approach to this, which assumes an elliptical target extent, was proposed by Koch in [4]. In that work, the extent was modelled as a multivariate Gaussian, parameterised by a random covariance matrix with an inverse Wishart distribution. This was termed as a Gaussian inverse Wishart (GIW), and this approach enables on-line estimation of the target extent, rather than requiring it to be specified a-priori. Further applications and improvements to this technique have appeared in [5], [6] and [7]. Alternative methods for estimating target extent have also been proposed, see for example [8]–[10].

The GIW method has been applied using filters based on the random finite set (RFS) framework. A probability hypothesis density (PHD) filter [11] for extended multi-target filtering was originally proposed by Mahler in [12], and an implementation based on the GIW model (GIW-PHD filter) was developed in [13]. This was generalised in [14] to the cardinalised PHD (CPHD) filter, in which the GIW approach was modified using a technique developed in [15], that enables estimation of target measurement rates. This method treats the rate parameter of the Poisson pdf (which characterises the number of measurements generated by a target) as a random variable. The distribution of this random variable is modelled as a gamma pdf, and the resulting algorithm was called the gamma Gaussian inverse Wishart CPHD (GGIW-CPHD) filter. Extended target PHD and CPHD filter have also been presented in [29], [30].

The (C)PHD filters use approximations that drastically reduce the complexity of the Bayes multi-target filter. While such approximations avoid the data association problem, which leads to fast implementations, they also induce a number of limitations. One such limitation is that they do not produce target tracks, thus in applications requiring tracks, it is necessary to perform post-processing of the filter output. The PHD filter is also known to involve a significant approximation of the multi-target posterior, which gives rise to highly uncertain cardinality estimates [11], [17]. Another limitation is the so-called 'spooky' effect [18], in which a target misdetection may cause a false estimate to spontaneously appear in a different part of the state space. An extended object Bernoulli filter was proposed in [19], which does not suffer from these issues, however, it is limited to the case of a single target in clutter.

These limitations can be alleviated by a method called the generalised labelled multi-Bernoulli (GLMB) filter, which was proposed in [20], [21]. This algorithm has been shown to outperform both PHD and CPHD, with the added advantage of producing labelled track estimates, albeit with a higher computational cost. An approximate version of this filter that is computationally cheaper was proposed in [22], called the labelled multi-Bernoulli filter (LMB). Also, the first GLMB filter for a non-standard measurement model was developed in [1]. This filter used a model that includes measurement merging, a problem which can be viewed as the dual of the extended target tracking problem.

In this paper, we develop a GLMB filter based on the GGIW extended target model. To our knowledge, this is the first time that an extended target model has been applied using a tracking algorithm based on labelled random finite sets. The resulting algorithm (GGIW-GLMB) is capable of estimating the kinematics and extents of multiple extended targets in clutter, with the advantage of producing full target tracks.

II. Background: Labelled RFS-based Tracking with Standard Sensor Model

The goal of multi-object Bayesian estimation is to estimate a finite set of states $X_k \subset \mathbb{X}$, called the *multi-object state*, at each time k. The multi-object states X_k and *multi-object observations* $Z_k \subset \mathbb{Z}$ are modelled as random finite sets (RFS), and finite set statistics (FISST) is a framework for working with RFSs [23] based on a notion of integration/density that is consistent with point process theory [24].

At the previous time step $k-1$, the multi-object state is assumed to be distributed according to a *multi-object density* $\pi_{k-1}(\cdot|Z_{1:k-1})$, where $Z_{1:k-1}$ is an array of finite sets of measurements received up to time $k-1$. Each Z_k is assumed to be generated through a process of thinning of misdetected objects, Markov shifts of detected objects, and superposition of false measurements. The *multi-object prediction* to time k is given by the Chapman-Kolmogorov equation

$$\pi_{k|k-1}(X_k|Z_{1:k-1}) = \int f_{k|k-1}(X_k|X)\, \pi_{k-1}(X|Z_{1:k-1})\, \delta X, \quad (1)$$

where $f_{k|k-1}(X_k|X)$ is the multi-object transition kernel from time $k-1$ to time k, and the integral is the set integral,

$$\int f(X)\, \delta X = \sum_{i=0}^{\infty} \frac{1}{i!} \int_{\mathbb{X}^i} f(\{x_1, \ldots, x_i\})\, d(x_1, \ldots, x_i) \quad (2)$$

for any function f that takes $\mathcal{F}(\mathbb{X})$, the collection of all finite subsets of $\mathbb{X}$, to the real line. A new set of observations Z_k is received at time k, which is modelled by a *multi-object likelihood function* $g_k(Z_k|X_k)$. Thus the *multi-object posterior* at time k is given by Bayes rule

$$\pi_k(X_k|Z_{1:k}) = \frac{g_k(Z_k|X_k)\, \pi_{k|k-1}(X_k|Z_{1:k-1})}{\int g_k(Z_k|X)\, \pi_{k|k-1}(X|Z_{1:k-1})\, \delta X}. \quad (3)$$

Collectively, (1) and (3) are referred to as the *multi-object Bayes filter.* In general, computing the exact multi-object posterior is numerically intractable, and approximations are required to derive practical algorithms.

One of the first RFS-based algorithms to be proposed was the probability hypothesis density (PHD) filter [23], which tractably approximates the full multi-object Bayes recursion by propagating only the first moment of the density. This was followed by the cardinalised PHD (CPHD) filter [25], which propagates the probability distribution of the number of targets, in addition to the first moment. Another type of algorithm based on RFS techniques involves approximating the density as a multi-Bernoulli RFS, known as multi-object multi-Bernoulli (MeMBer) filtering [26]. Neither the PHD nor the multi-Bernoulli approaches require explicit data association, which has been a key reason for their popularity. However, they only provide a set of unlabelled point estimates at each time step, hence, for applications that require target trajectories, additional post-processing is necessary to produce the tracks. A recently proposed technique for addressing this problem is the concept of labelled random finite sets [20]. This technique involves assigning a distinct label to each element of the target set, so that the history of each object's trajectory can be naturally identified, without the requirement for post-processing.

In [20], an algorithm was proposed for solving the multi-object tracking problem under the standard point-detection likelihood model, based on a type of labelled RFS called 'generalised labelled multi-Bernoulli' (GLMB). We now review the main points of this technique, and in the section to follow we propose a generalisation which will enable it to handle extended targets.

We begin by introducing some notation and definitions relating to labelled random finite sets. The multi-object exponential of a real valued function h raised to a set X is defined as

$$[h(\cdot)]^X = \prod_{x \in X} h(x) \quad (4)$$

where $h^{\emptyset} = 1$, and the elements of X may be of any type such as scalars, vectors, or sets, provided that the function $h(\cdot)$ takes an argument of that type. The generalised Kronecker delta function is defined as

$$\delta_Y(X) = \begin{cases} 1, & \text{if } Y = X \\ 0, & \text{otherwise} \end{cases} \quad (5)$$

where again, X and Y may be of any type, such as scalars, vectors, or sets. Also, the set inclusion function is defined as

$$1_Y(X) \triangleq \begin{cases} 1, & \text{if } X \subseteq Y \\ 0, & \text{otherwise} \end{cases}. \quad (6)$$

Definition 1. A labelled RFS $\boldsymbol{X}$ with state space $\mathbb{X}$ and discrete label space $\mathbb{L}$, is an RFS on $\mathbb{X} \times \mathbb{L}$, such that the labels within each realisation are always distinct. That is, if $\mathcal{L}(\boldsymbol{X})$ is the set of unique labels in $\boldsymbol{X}$, and we define the distinct label indicator function as

$$\Delta(\boldsymbol{X}) = \begin{cases} 1, & \text{if } |\mathcal{L}(\boldsymbol{X})| = |\boldsymbol{X}| \\ 0, & \text{if } |\mathcal{L}(\boldsymbol{X})| \neq |\boldsymbol{X}| \end{cases} \quad (7)$$

then a labelled RFS $\boldsymbol{X}$ always satisfies $\Delta(\boldsymbol{X}) = 1$.

Definition 2. A generalised labelled multi-Bernoulli (GLMB) RFS is a labelled RFS with state space $\mathbb{X}$ and discrete label space $\mathbb{L}$, which satisfies the probability distribution

$$\boldsymbol{\pi}(\boldsymbol{X}) = \Delta(\boldsymbol{X}) \sum_{c \in \mathbb{C}} w^{(c)}(\mathcal{L}(\boldsymbol{X})) \left[p^{(c)}(\cdot)\right]^{\boldsymbol{X}} \quad (8)$$

where $\mathbb{C}$ is an arbitrary index set, and $w^{(c)}(\mathcal{L}(\boldsymbol{X}))$ and $p^{(c)}(x, l)$ satisfy

$$\sum_{c \in \mathbb{C}} w^{(c)}(L) = 1 \quad (9)$$

$$\int_{x \in \mathbb{X}} p^{(c)}(x, l)\, dx = 1. \quad (10)$$

A. Multi-object Transition Kernel

Let $\boldsymbol{X}$ be the labeled RFS of objects at the current time with label space $\mathbb{L}$. A particular object $(x, l) \in \boldsymbol{X}$ has probability $p_S(x, l)$ of surviving to the next time with state (x_+, l_+) distributed according to $f(x_+|x, l)\, \delta_l(l_+)$, and probability $q_S(x, l) = 1 - p_S(x, l)$ of being terminated. The set $\boldsymbol{S}$ of surviving objects at the next time is distributed according to

$$\boldsymbol{f}_S(\boldsymbol{S}|\boldsymbol{X}) = \Delta(\boldsymbol{S})\, \Delta(\boldsymbol{X})\, 1_{\mathcal{L}(\boldsymbol{X})}(\mathcal{L}(\boldsymbol{S}))\, [\Phi(\boldsymbol{S}; \cdot)]^{\boldsymbol{X}} \quad (11)$$

where

$$\begin{aligned} \Phi(\boldsymbol{S}; x, l) = & \sum_{(x_+, l_+) \in \boldsymbol{S}} \delta_l(l_+)\, p_S(x, l)\, f(x_+|x, l) \\ & + \left(1 - 1_{\mathcal{L}(\boldsymbol{S})}(l)\right) q_S(x, l), \end{aligned} \quad (12)$$

and $f(x_+|x, l)$ is the single target transition kernel. Now let $\boldsymbol{B}$ be the labelled RFS of new born objects with label space $\mathbb{B}$, where $\mathbb{L} \cap \mathbb{B} = \emptyset$. Since the births must have distinct labels, and assuming that their states are independent, $\boldsymbol{B}$ is distributed according to

$$\boldsymbol{f}_B(\boldsymbol{B}) = \Delta(\boldsymbol{B})\, w_B(\mathcal{L}(\boldsymbol{B}))\, [p_B(\cdot)]^{\boldsymbol{B}} \quad (13)$$

where $p_B(\cdot)$ is the single target birth density, and $w_B(L)$ is the birth weight. The overall multi-object state at the next time step is the union of surviving and new born objects, i.e. $\boldsymbol{X}_+ = \boldsymbol{S} \cup \boldsymbol{B}$. The label spaces $\mathbb{L}$ and $\mathbb{B}$ are disjoint, and the states of new born objects are independent of surviving objects, hence $\boldsymbol{S}$ and $\boldsymbol{B}$ are independent. It was shown in [20] that the transition kernel for a labelled multi-object density is given by

$$\boldsymbol{f}(\boldsymbol{X}_+|\boldsymbol{X}) = \boldsymbol{f}_S(\boldsymbol{X}_+ \cap (\mathbb{X} \times \mathbb{L})\, |\boldsymbol{X})\, \boldsymbol{f}_B(\boldsymbol{X}_+ - \mathbb{X} \times \mathbb{L}), \quad (14)$$

and that a GLMB density of the form (8) is closed under the Chapman-Kolmogorov prediction equation with this transition kernel.

B. Standard Multi-object Observation Model

Let $\boldsymbol{X}$ be the labelled RFS of objects that exist at the observation time. A particular object $(x, l) \in \boldsymbol{X}$ has probability $p_D(x, l)$ of generating a detection z with likelihood $g(z|x)$, and probability $q_D(x, l) = 1 - p_D(x, l)$ of being misdetected. Since the detections are conditionally independent, we model the set of detections D as a multi-Bernoulli RFS. Using the notation $\{(r(x), p(x))\, | x \in X\}(\cdot)$ to denote a multi-Bernoulli density with existence probabilities r and single-object densities p, then the density of D can be expressed as

$$\pi_D(D|\boldsymbol{X}) = \{(p_D(x, l), g(\cdot|x)); (x, l) \in \boldsymbol{X}\}(D). \quad (15)$$

Let K be the set of clutter observations, which are independent of the target detections. We model K as a Poisson RFS with rate λ and spatial distribution $c(\cdot)$, hence K is distributed according to

$$\pi_K(K) = e^{-\lambda} [\lambda c(\cdot)]^K. \quad (16)$$

The overall multi-object observation is the union of target detections and clutter observations, i.e. $Z = D \cup K$. Since D and K are independent, it was shown in [23] that the multi-object likelihood function is given by

$$g(Z|\boldsymbol{X}) = e^{-\lambda} [\lambda c(\cdot)]^Z \sum_{\theta \in \Theta} [\psi_Z(\cdot; \theta)]^{\boldsymbol{X}} \quad (17)$$

where Θ is the set of all one-to-one mappings of labels in $\boldsymbol{X}$ to measurement indices in Z,

$$\Theta = \{\theta : \mathcal{L}(\boldsymbol{X}) \to \{0 : |Z|\}\} \quad (18)$$

such that $[\theta(i) = \theta(j) > 0] \Rightarrow [i = j]$, and $\psi_Z(\cdot; \theta)$ is given by

$$\psi_Z(x, l; \theta) = \begin{cases} \frac{p_D(x,l) g\left(z_{\theta(l)}|x,l\right)}{\lambda c\left(z_{\theta(l)}\right)}, & \theta(l) > 0 \\ q_D(x, l), & \theta(l) = 0 \end{cases} \quad (19)$$

It was shown in [20] that a GLMB density of the form (8) is closed under the Bayes update with likelihood function (17).

III. Labelled RFS-based Extended Target Tracking

In this section, we present an extended multi-target observation model, and a state space model for a single extended target, which includes the single target prediction and measurement update. Based on these models, we then propose a GLMB filter for tracking multiple extended targets in clutter.

A. Extended Multi-object Observation Model

Let $\boldsymbol{X} = \{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n\}$ be the labelled RFS of extended objects that exist at the observation time. A particular object $(\xi, l) \in \boldsymbol{X}$ has probability p_D of generating a set of detections W with likelihood $\tilde{g}(W|\xi, l)$, and probability $q_D = 1 - p_D$ of being misdetected. Let D be the set of target detections. As shown in [23] , D is distributed according to

$$\pi_D(D|\boldsymbol{X}) = \sum_{W_1 \uplus \cdots \uplus W_m = D} \pi(W_1|\boldsymbol{\xi}_1) \ldots \pi(W_n|\boldsymbol{\xi}_n), \quad (20)$$

where each $\pi(W|\boldsymbol{\xi}_i)$ is an RFS distribution defined by

$$\pi(W|\boldsymbol{\xi}_i) \propto \begin{cases} q_D, & W = \emptyset \\ p_D . \tilde{g}(W|\boldsymbol{\xi}_i), & \text{otherwise} \end{cases} \quad (21)$$

and the symbol $\uplus$ denotes that the summation is taken over all mutually disjoint subsets of D, such that $W_1 \cup \cdots \cup W_m = D$. The set K of clutter observations, which is independent of the target detections, is modelled as a Poisson RFS with rate λ and spatial distribution $c(\cdot)$, hence it is distributed according to (16).

The overall multi-object observation is the union of target detections and clutter observations, i.e. $Z = D \cup K$. Since D and K are independent, the multi-object likelihood is given by the convolution

$$g(Z|\boldsymbol{X}) = \sum_{D \subseteq Z} \pi_D(D|\boldsymbol{X}) \pi_K(Z - D). \quad (22)$$

This function can be equivalently expressed as a double summation over partitions of Z up to size $|\boldsymbol{X}| + 1$, and mappings of measurement groups to targets as follows

$$g(Z|\boldsymbol{X}) = e^{-\lambda} [\lambda c(\cdot)]^Z \sum_{i=1}^{|\boldsymbol{X}|+1} \sum_{\substack{\mathcal{U}(Z) \in \mathcal{P}_i(Z) \\ \theta \in \Theta(\mathcal{U}(Z))}} \left[\psi_{\mathcal{U}(Z)}(\cdot; \theta)\right]^{\boldsymbol{X}}, \quad (23)$$

where $\mathcal{P}_i(Z)$ is the set of all partitions[1] of Z containing exactly i groups, and $\Theta(\mathcal{U}(Z))$ is the set of all one-to-one mappings $\theta : \mathcal{L}(\boldsymbol{X}) \to \{0, 1, \ldots, |\mathcal{U}(Z)|\}$ taking the labels in $\boldsymbol{X}$ to either a group of measurements in $\mathcal{U}(Z)$, or a misdetection. Finally $\psi_{\mathcal{U}(Z)}(\boldsymbol{\xi}; \theta)$ is given by

$$\psi_{\mathcal{U}(Z)}(\xi, l; \theta) = \begin{cases} \frac{p_D \cdot \tilde{g}\left(\mathcal{U}(Z)_{\theta(l)} | \xi, l\right)}{[\lambda c(\cdot)]^{\mathcal{U}_{\theta(l)}(Z)}}, & \theta(l) > 0 \\ q_D, & \theta(l) = 0 \end{cases} \quad (24)$$

where $\mathcal{U}_{\theta(l)}(Z)$ is the group of measurements in partition $\mathcal{U}(Z)$ that was assigned to label l under the mapping θ, and $\tilde{g}(W|\xi, l)$ is the likelihood that a single extended target with state (ξ, l) generates measurement set W.

In general, computing the exact likelihood function in (23) will be numerically intractable, as the sets of measurement partitions and group-to-target mappings can potentially become extremely large. However, it has been shown that in many practical situations, it is only necessary to consider a small subset of these partitions to achieve good performance [13], [16]. In addition, the set of group-to-target mappings can be significantly reduced using a ranked assignment algorithm, thereby cutting down the number of terms in the likelihood even further.

B. Extended Target State-space Model

In this section we describe a family of distributions which can be used to model a single extended target. We begin by introducing the following notation:

- $\mathbb{R}^+$ is the space of positive reals
- $\mathbb{R}^n$ is the space of real n-dimensional vectors
- $\mathbb{S}^n_{++}$ is the space of $n \times n$ positive definite matrices
- $\mathbb{S}^n_{+}$ is the space of $n \times n$ positive semi-definite matrices
- $\mathcal{GAM}(\gamma; \alpha, \beta)$ is the gamma pdf defined on $\gamma > 0$, with shape $\alpha > 0$, and inverse scale $\beta > 0$:
  $$\mathcal{GAM}(\gamma; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \gamma^{\alpha-1} e^{-\beta\gamma}$$
- $\mathcal{N}(x; m, P)$ is the multivariate Gaussian pdf defined on $x \in \mathbb{R}^n$, with mean $m \in \mathbb{R}^n$ and covariance $P \in \mathbb{S}^n_+$
  $$\mathcal{N}(x; m, P) = \frac{1}{\sqrt{(2\pi)^n |P|}} e^{-\frac{1}{2}(x-m)^T P^{-1}(x-m)}$$
- $\mathcal{IW}_d(\chi; v, V)$ is the inverse Wishart distribution defined on $\chi \in \mathbb{S}^d_{++}$, with degrees of freedom $v > 2d$, and scale matrix $V \in \mathbb{S}^d_{++}$ [31]
  $$\mathcal{IW}_d(\chi; v, V) = \frac{2^{-\frac{v-d-1}{2}} |V|^{\frac{v-d-1}{2}}}{\Gamma_d\left(\frac{v-d-1}{2}\right) |\chi|^{\frac{v}{2}}} e^{-\frac{1}{2}\mathrm{tr}\left(V\chi^{-1}\right)}$$
  where $\Gamma_d(\cdot)$ is the multivariate gamma function, and $\mathrm{tr}(\cdot)$ takes the trace of a matrix.
- $A \otimes B$ is the Kronecker product of matrices A and B

We model the extended target state as the triple

$$\xi_k = (\gamma_k, x_k, \chi_k) \in \mathbb{R}^+ \times \mathbb{R}^{n_x} \times \mathbb{S}^d_{++}, \quad (25)$$

where $\gamma_k \in \mathbb{Z}^+$ is the Poisson rate of the number of measurements generated by the target, and $x_k \in \mathbb{R}^{n_x}$ and $\chi_k \in \mathbb{S}^d_{++}$ are the mean and covariance of the Gaussian spatial distribution of these measurements. The distribution of the state of an extended target is a gamma Gaussian inverse Wishart (GGIW) density on $\mathbb{R}^+ \times \mathbb{R}^{n_x} \times \mathbb{S}^d_{++}$, given by

$$\begin{aligned} p(\xi_k|Z_{1:k}) &= p(\gamma_k|Z_{1:k}) p(x_k|\chi_k, Z_{1:k}) p(\chi_k|Z_{1:k}) \\ &= \mathcal{GAM}\left(\gamma_k; \alpha_{k|k}, \beta_{k|k}\right) \\ &\quad \times \mathcal{N}\left(x_k; m_{k|k}, P_{k|k} \otimes \chi_k\right) \\ &\quad \times \mathcal{IW}_d\left(\chi_k; v_{k|k}, V_{k|k}\right) \\ &\triangleq \mathcal{GGIW}\left(\xi_k; \zeta_{k|k}\right) \end{aligned} \quad (26)$$

[1] A partition of an arbitrary set A is defined as a disjoint collection of non-empty sets, whose union is equal to A.

where $\zeta_{k|k} = \left(\alpha_{k|k}, \beta_{k|k}, m_{k|k}, P_{k|k}, v_{k|k}, V_{k|k}\right)$ is an array containing the GGIW density parameters. In what follows, we describe the prediction and update procedures for a single extended target with a GGIW distribution.

1) Prediction : To compute the predicted extended target density $p\left(\xi_{k+1}|Z_{1:k}\right)$, we need to solve the following Champan-Kolmogorov equation

$$p\left(\xi_{k+1}|Z_{1:k}\right) = \int f\left(\xi_{k+1}|\xi_k\right) p\left(\xi_k|Z_{1:k}\right) d\xi_k, \tag{27}$$

where $p\left(\xi_k|Z_{1:k}\right)$ is the posterior density at time k, and $f\left(\xi_{k+1}|\xi_k\right)$ is the transition density. However, this has no closed form solution for the GGIW defined in (26). To obtain a tractable approximation, we start by making the simplifying assumption that the transition density can be expressed as the product [14]

$$\begin{aligned} p\left(\xi_{k+1}|\xi_k\right) \approx \\ p_\gamma\left(\gamma_{k+1}|\gamma_k\right) p_x\left(x_{k+1}|\chi_{k+1}, x_k\right) p_\chi\left(\chi_{k+1}|\chi_k\right), \end{aligned} \tag{28}$$

which leads to the following expression for the predicted GGIW

$$\begin{aligned} p\left(\xi_{k+1}|Z_{1:k}\right) &\approx \int \mathcal{GAM}\left(\gamma_k; \alpha_{k|k}, \beta_{k|k}\right) p_\gamma\left(\gamma_{k+1}|\gamma_k\right) d\gamma_k \\ &\times \int \mathcal{N}\left(x_k; m_{k|k}, P_{k|k} \otimes \chi_{k+1}\right) p_x\left(x_{k+1}|\chi_{k+1}, x_k\right) dx_k \\ &\times \int \mathcal{IW}_d\left(\chi_k; v_{k|k}, V_{k|k}\right) p_X\left(\chi_{k+1}|\chi_k\right) d\chi_k. \end{aligned} \tag{29}$$

Under a linear Gaussian motion model, the kinematic component can be solved in closed form:

$$\begin{aligned} \int \mathcal{N}(x_k; m_{k|k}, &P_{k|k} \otimes \chi_{k+1}) p_x\left(x_{k+1}|\chi_{k+1}, x_k\right) dx_k \\ &= \mathcal{N}\left(x_{k+1}; m_{k+1|k}, P_{k+k|k} \otimes \chi_{k+1}\right), \end{aligned} \tag{30}$$

$$m_{k+1|k} = \left(F_{k+1|k} \otimes I_d\right) m_{k|k}, \tag{31}$$

$$P_{k+1|k} = F_{k+1|k} P_{k|k} F_{k+1|k}^T + Q_{k+1|k}. \tag{32}$$

However, the measurement rate and target extension components still do not permit closed form solutions. As was done in [4] and [14], we make the following approximations:

$$\begin{aligned} \int \mathcal{GAM}\left(\gamma_k; \alpha_{k|k}, \beta_{k|k}\right) p_\gamma\left(\gamma_{k+1}|\gamma_k\right) d\gamma_k \\ \approx \mathcal{GAM}\left(\gamma_{k+1}; \alpha_{k+1|k}, \beta_{k+1|k}\right), \end{aligned} \tag{33}$$

$$\alpha_{k+1|k} = \frac{\alpha_{k|k}}{\mu_k}, \qquad \beta_{k+1|k} = \frac{\beta_{k|k}}{\mu_k}, \tag{34}$$

and

$$\begin{aligned} \int \mathcal{IW}_d\left(\chi_k; v_{k|k}, V_{k|k}\right) p_\chi\left(\chi_{k+1}|\chi_k\right) d\chi_k \\ \approx \mathcal{IW}_d\left(\chi_{k+1}; v_{k|k+1}, V_{k|k+1}\right), \end{aligned} \tag{35}$$

$$v_{k+1|k} = e^{-T/\tau} v_{k|k}, \tag{36}$$

$$V_{k+1|k} = \frac{v_{k+1|k} - d - 1}{v_{k|k} - d - 1} V_{k|k}. \tag{37}$$

In the above, $\mu_k = \frac{1}{1-1/w}$ is an exponential forgetting factor with window length $w > 1$, and τ is a temporal decay constant. This leads to an approximate predicted density in the form of a GGIW, $p_{k+1|k}\left(\xi_{k+1}|Z_{1:k}\right) \approx \mathcal{GGIW}\left(\xi_{k+1}; \zeta_{k|k+1}\right)$, where $\zeta_{k|k+1}$ is the array of predicted parameters defined by equations (31), (32), (34), (36) and (37).

2) Update: Each extended target will undergo measurement updates using subsets of the overall measurement set. When updating a target with predicted GGIW density $p_{k|k-1}$ (with parameters $\zeta_{k|k-1}$), using a particular subset $W \subseteq Z_k$, the posterior GGIW density for that target is computed as follows. We start by calculating the mean and scale matrix for the measurement subset, the innovation, innovation factor, innovation matrix and gain vector:

$$\bar{z}_k = \frac{1}{|W|} \sum_{z_k^{(i)} \in W} z_k^{(i)}, \tag{38}$$

$$\Psi_k = \sum_{z_k^{(i)} \in W} \left(z_k^{(i)} - \bar{z}_k\right) \left(z_k^{(i)} - \bar{z}_k\right)^T, \tag{39}$$

$$\epsilon_{k|k-1} = \bar{z}_k - \left(H_k \otimes I_d\right) m_{k|k-1}, \tag{40}$$

$$S_{k|k-1} = H_k P_{k|k-1} H_k^T + \frac{1}{|W|}, \tag{41}$$

$$N_{k|k-1} = \left(S_{k|k-1}\right)^{-1} \epsilon_{k|k-1} \epsilon_{k|k-1}^T, \tag{42}$$

$$K_{k|k-1} = P_{k|k-1} H_k^T \left(S_{k|k-1}\right)^{-1}. \tag{43}$$

The parameters of the posterior GGIW are now given by:

$$\alpha_{k|k} = \alpha_{k|k-1} + |W|, \tag{44}$$

$$\beta_{k|k} = \beta_{k|k-1} + 1, \tag{45}$$

$$m_{k|k} = m_{k|k-1} + \left(K_{k|k-1} \otimes I_d\right) \epsilon_{k|k-1}, \tag{46}$$

$$P_{k|k} = P_{k|k-1} - K_{k|k-1} S_{k|k-1} K_{k|k-1}^T, \tag{47}$$

$$v_{k|k} = v_{k|k-1} + |W|, \tag{48}$$

$$V_{k|k} = V_{k|k-1} + N_{k|k-1} + \Psi_k. \tag{49}$$

Finally, the Bayes normalising constant (which is required in the next section to compute the weights of the posterior GLMB density) is given by the product of the following two terms,

$$\eta_\gamma\left(W; p_{k|k-1}\right) = \frac{1}{|W|!} \frac{\Gamma\left(\alpha_{k|k}\right) \left(\beta_{k|k-1}\right)^{\alpha_{k|k-1}}}{\Gamma\left(\alpha_{k|k-1}\right) \left(\beta_{k|k}\right)^{\alpha_{k|k}}}, \tag{50}$$

$$\eta_{x,\chi}\left(W; p_{k|k-1}\right) = \frac{\left(\pi^{|W|} |W|\right)^{-\frac{d}{2}} \left|V_{k|k-1}\right|^{\frac{v_{k|k-1}}{2}} \Gamma_d\left(\frac{v_{k|k}}{2}\right)}{\left(S_{k|k-1}\right)^{\frac{d}{2}} \left|V_{k|k}\right|^{\frac{v_{k|k}}{2}} \Gamma_d\left(\frac{v_{k|k-1}}{2}\right)}. \tag{51}$$

Note that the measurement rate component (50) corresponds to a negative-binomial pdf, and the kinematics-extension component (51) is proportional to a matrix variate Generalized Beta type II pdf [13].

C. Extended Target GLMB Filter

We now present the extended target GLMB filter using the likelihood function described in the previous sections. Firstly, we note that the prediction is identical to that of the standard GLMB filter derived in [20], which we shall revisit here for the sake of completeness. For simplicity, we assume that the survival probability is state independent, i.e. $p_S(\xi,l)=p_S$. If the multi-object density at time $k-1$ is a GLMB of the form (8), then the predicted multi-object density at time k is a GLMB given by

$$\pi_k(\boldsymbol{X}_k)=\Delta(\boldsymbol{X}_k)\sum_{c\in\mathbb{C}}w_k^{(c)}(\mathcal{L}(\boldsymbol{X}_k))\left[p_k^{(c)}(\cdot)\right]^{\boldsymbol{X}_k} \quad (52)$$

where

$$w_k^{(c)}(L)=w_B(L-\mathbb{L})\,w_S^{(c)}(L\cap\mathbb{L}), \quad (53)$$

$$p_k^{(c)}(\xi,l)=1_{\mathbb{L}}(l)\,p_S^{(c)}(\xi,l)+(1-1_{\mathbb{L}}(l))\,p_B(\xi,l), \quad (54)$$

$$p_S^{(c)}(\xi,l)=\int f(\xi|\xi_{k-1},l)\,p^{(c)}(\xi_{k-1},l)\,d\xi_{k-1}, \quad (55)$$

$$w_S^{(c)}(J)=[p_S]^J\sum_{I\subseteq\mathbb{L}}1_I(J)\,[q_S]^{I-J}\,w^{(c)}(I), \quad (56)$$

The function $f(\xi|\xi_{k-1},l)$ is the single-object transition kernel, which in this case is the GGIW transition defined in Section III-B1.

Proposition 3. *If the prior is a GLMB of the form (8), then under the extended multi-object likelihood function (23), the posterior is also a GLMB, given by*

$$\pi(\boldsymbol{X}|Z)=\Delta(\boldsymbol{X})\sum_{c\in\mathbb{C}}\sum_{i=1}^{|\boldsymbol{X}|+1}\sum_{\substack{\mathcal{U}(Z)\in\mathcal{P}_i(Z)\\ \theta\in\Theta(\mathcal{U}(Z))}}w_{\mathcal{U}(Z)}^{(c,\theta)}(\mathcal{L}(\boldsymbol{X}))\times\left[p^{(c,\theta)}(\cdot|\mathcal{U}(Z))\right]^{\boldsymbol{X}} \quad (57)$$

where

$$w_{\mathcal{U}(Z)}^{(c,\theta)}(L)=\frac{w^{(c)}(L)\left[\eta_{\mathcal{U}(Z)}^{(c,\theta)}\right]^L}{\sum_{c\in\mathbb{C}}\sum_{J\subseteq\mathbb{L}}\sum_{i=1}^{|J|+1}\sum_{\substack{\mathcal{U}(Z)\in\mathcal{P}_i(Z)\\ \theta\in\Theta(\mathcal{U}(Z))}}w^{(c)}(J)\left[\eta_{\mathcal{U}(Z)}^{(c,\theta)}\right]^J}, \quad (58)$$

$$p^{(c,\theta)}(\xi,l|\mathcal{U}(Z))=\frac{p^{(c)}(\xi,l)\,\psi_{\mathcal{U}(Z)}(\xi,l;\theta)}{\eta_{\mathcal{U}(Z)}^{(c,\theta)}(l)}, \quad (59)$$

$$\eta_{\mathcal{U}(Z)}^{(c,\theta)}(l)=\frac{p_D\eta_\gamma\big(\mathcal{U}_{\theta(l)}(Z);p^{(c)}\big)\eta_{x,\chi}\big(\mathcal{U}_{\theta(l)}(Z);p^{(c)}\big)}{[\lambda c(\cdot)]^{\mathcal{U}_{\theta(l)}(Z)}}, \quad (60)$$

in which $\psi_{\mathcal{U}(Z)}(\xi,l;\theta)$, $\eta_\gamma(\cdot)$ and $\eta_{x,\chi}(\cdot)$ are given by (24), (50) and (51) respectively.

Note that Proposition 3 means that the GLMB can be regarded as a conjugate prior with respect to the extended multi-target measurement likelihood function.

IV. Implementation

To compute the predicted GLMB density, we first predict the individual target pdfs forward using (29)-(37). Then, using the target survival probabilities in combination with a k-shortest paths algorithm, we generate the GLMB density for surviving targets. This density is then multipled by the GLMB for spontaneous births, to arrive at the overall prediction. See [1], [21] for more details on these procedures.

The measurement update step begins with generating a feasible collection of partitions of the current measurement set. As is the case for the extended target (C)PHD filter, the main barrier to computing the full extended target GLMB posterior in (57), is the fact that it involves a sum over all possible partitions of the measurement set. Even when the set of measurements is relatively small, performing an exhaustive enumeration of the partitions is clearly infeasible, because the number of possibilities (which is given by the Bell number) grows combinatorially with the number of elements. Therefore, for the filter to be computationally tractable, it is imperative that the number of partitions is reduced to a more managable level. To minimise the truncation error in the posterior density, it is necessary to ensure that the retained partitions are those that will give rise to GLMB components with the highest posterior weights. Although it is difficult to establish a method that can guarantee this, the use of clustering techniques to generate the most likely partitions has been shown to produce favourable results [13], [16]. In our implementation of the GGIW-GLMB filter, we use a combination of distance-based clustering and the expecation-maximisation algorithm to generate a set of feasible partitions of the measurements, in a similar manner to [13] and [16].

After generating the feasible partitions, each group of measurements that appears in a partition is then used to update the single target pdfs in the GLMB density, using (38)-(49). We then use Murty's algorithm to create new posterior components, by generating highly weighted assignments of measurement groups to targets. After computing the posterior components, labelled target estimates from the posterior GLMB density are extracted and used to update the track output. Finally, the posterior GLMB is pruned, retaining the top N components with highest weights.

Due to length restrictions, we have not provided pseudocode for the GGIW-GLMB filter in this paper. However, this will be presented in future work.

V. Simulation Results

The proposed GGIW-GLMB filter is compared to GGIW-mixture implementations of an extended target PHD filter [13], [15] and an extended target CPHD filter [14]. The cardinality estimation error (estimated cardinality minus true cardinality) and the optimal subpattern assignment (OSPA) metric [28] are computed for all three filters. The OSPA used in this study is a weighted combination of three metrics; the absolute value for the measurement rates, the Euclidean norm for the kinematic vectors, and the Frobenius norm for the extension matrices. Further implementation details can be found in [14, Section VI].

Two scenarios were simulated, both of which were used in [14] to compare the performance of the GGIW-PHD filter and the GGIW-CPHD filter. Scenario 1 is 200 time steps long and has four targets that appear/disappear from the surveillance area at different times. The measurements were simulated with probability of detection $p_D = 0.8$ and the Poisson rate for the clutter measurements was set to 30 per time step. This scenario is primarily used to compare the three filters' cardinality estimation, as targets appearing/disappearing, lower probability of detection, and higher clutter rate, are all challenging for the cardinality estimation. The true target tracks are shown in Figure 1.

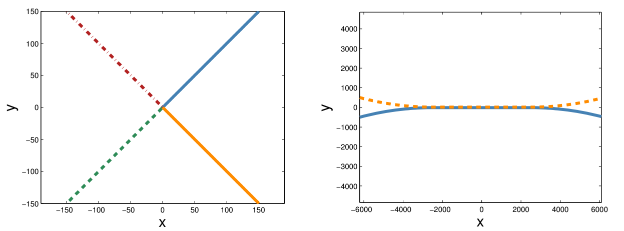


Figure 1. Simulated true target tracks. In scenario 1 (left) all tracks start in the origin. In scenario 2 (right) the tracks start on the left.

Scenario 2 is 100 time steps long and has two targets that are present during all 100 time steps. The two targets are spatially separated at the beginning, then move in parallel at close distance, before separating again. Measurements were simulated with probability of detection $p_D = 0.98$ and clutter Poisson rate of 10. This scenario is primarily used to compare how the three filters handle targets that are spatially close. Spatially close targets are challenging because the true target measurements will also be spatially close, appearing as a single cluster in the sensor data, rather than two separate clusters which is typically the case for spatially separated targets. Previous work has shown that tracking spatially close targets can be challenging [13], [16].

For each scenario, 1000 Monte Carlo runs were carried out. The mean OSPA metrics are shown in Figure 2, and the mean cardinality errors are shown in Figure 3. For the sake of clarity, uncertainty regions (mean ± one standard deviation) are only shown for the CPHD and GLMB filters.

For both scenarios, the PHD filter has the worst performance, both in terms of the OSPA metric and the cardinality error. The PHD filter's cardinality estimate has a high variance, and the filter is therefore sensitive to missed detections and clutter, which explains the larger cardinality error. The larger OSPA metric that can be seen in scenario 2 is due to the fact that a missed detection typically leads to a lost target estimate, and in subsequent time steps the two spatially close targets are treated as a single target. This does not happen for the GLMB and CPHD filters because they have cardinality estimates with lower variance, and hence do not lose estimates.

Comparing the CPHD and GLMB filters, for scenario 2 the mean OSPA metrics are approximately equal, however for scenario 1 the GLMB filter gives a smaller OSPA metric. The mean cardinality errors are approximately equal in both scenarios, however the uncertainty region for the GLMB filter is smaller than for the CPHD filter, which indicates that the GLMB filter's cardinality estimate has lower variance.

For scenario 1, the algorithm execution times (mean ± one standard deviation) are 5.89 ± 4.53s for the GLMB filter, 2.20 ± 0.47s for the CPHD filter, and 0.73 ± 0.31s for the PHD filter. The simulation study shows that, compared to the PHD and CPHD filter, the GLMB filter has a better cardinality estimate (lower variance), and shows that when the probability of detection is lower and the clutter rate is higher, the GLMB filter has lower OSPA metric. The improved estimation performance comes at the cost of higher computational complexity.

VI. Conclusion

In this paper we have developed an adaptation to the generalised labelled multi-Bernoulli filter, enabling it to track multiple extended targets in clutter. The proposed filter is based on modelling the target kinematics and their extents using gamma Gaussian inverse Wishart distributions. A simulation study showed that the proposed filter outperforms prior work on multiple extended target tracking.

Acknowledgement

The authors would like to acknowledge the ATN-DAAD (German Academic Exchange Service): Joint Research Co-operation Scheme, for their support of this work, under the project entitled “Random Finite Set Based Extended Object Tracking with Application to Vehicle Environment Perception”. This project is also supported by the Australian Research Council under projects DE120102388 and DP130104404.

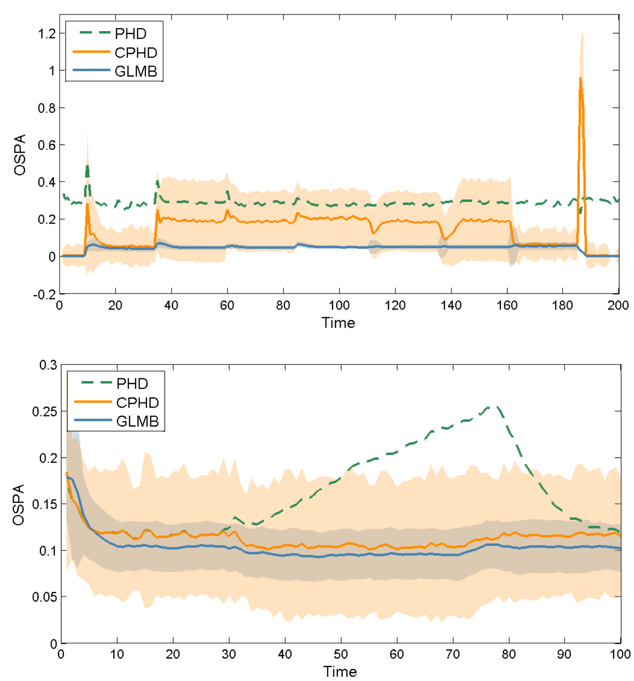
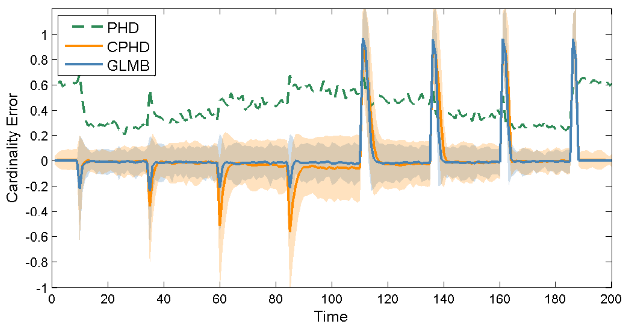


Figure 2. OSPA metric for scenario 1 (top) and scenario 2 (bottom). The thick lines show the Monte Carlo mean, the colored regions show the Monte Carlo mean ± one standard deviation.



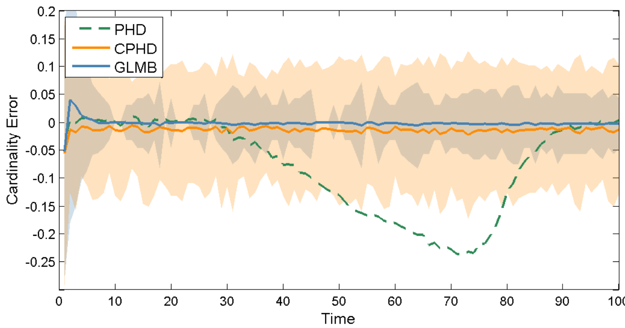


Figure 3. Cardinality errors for scenario 1 (top) and scenario 2 (bottom). The thick lines show the Monte Carlo mean, the colored regions show the Monte Carlo mean ± one standard deviation.

REFERENCES

[1] M. Beard, B.-T. Vo, B.-N. Vo, "Bayesian multi-target tracking with merged measurements using labelled random finite sets," *IEEE Transactions on Signal Processing*, vol. 63, no. 6, pp. 1433-1447, March 2015.

[2] K. Gilholm and D. Salmond, "Spatial distribution model for tracking extended objects," *IEE Proceedings Radar, Sonar and Navigation*, vol. 152, no. 5, pp. 364–371, October 2005.

[3] K. Gilholm, S. Godsill, S. Maskell, and D. Salmond, "Poisson models for extended target and group tracking," *Proceedings of Signal and Data Processing of Small Targets*, vol. 5913, pp. 230–241, San Diego, CA, August 2005.

[4] J. W. Koch, "Bayesian approach to extended object and cluster tracking using random matrices," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 44, no. 3, pp. 1042–1059, July 2008.

[5] W. Wieneke, J. W. Koch, "Probabilistic tracking of multiple extended targets using random matrices," *Proc. SPIE Signal and Data Processing of Small Targets*, Orlando, FL, USA, April 2010.

[6] M. Feldmann, D. Fränken, and J. W. Koch, "Tracking of extended objects and group targets using random matrices," *IEEE Transactions on Signal Processing*, vol. 59, no. 4, pp. 1409–1420, April 2011.

[7] J. W. Koch, M. Feldmann, "Cluster tracking under kinematical constraints using random matrices," *Robotics and Autonomous Systems*, vol. 57, no. 3, pp. 296–309, March 2009.

[8] M. Baum, B. Noack, and U. D. Hanebeck, "Extended object and group tracking with elliptic random hypersurface models," *Proc. 13th International Conference on Information Fusion*, Edinburgh, UK, July 2010.

[9] M. Baum and U. D. Hanebeck, "Shape tracking of extended objects and group targets with star-convex RHMs," *Proc. 14th International Conference on Information Fusion*, Chicago, IL, USA, July 2011.

[10] C. Lundquist, K. Granström, U. Orguner, "Estimating the shape of targets with a PHD filter," *Proc. 14th International Conference on Information Fusion*, Chicago, IL, USA, July 2011.

[11] R. Mahler, "Multitarget Bayes filtering via first-order multitarget moments", *IEEE Transactions on Aerospace and Elecronic Syst*ems, vol. 39, no. 4, pp. 1152-1178, October 2003.

[12] R. Mahler, "PHD filters for nonstandard targets, I: Extended targets," *Proc. 12th International Conference on Information Fusion*, Seattle, WA, USA, July 2009.

[13] K. Granström, U. Orguner, "A PHD filter for tracking multiple extended targets using random matrices," *IEEE Transactions on Signal Processing*, vol. 60, no. 11, pp. 5657–5671, November 2012.

[14] C. Lundquist, K. Granström, U. Orguner, "An extended target CPHD filter and a gamma Gaussian inverse Wishart implementation," *IEEE Journal on Selected Topics in Signal Processing*, vol. 7, no. 3, pp. 472-483, February 2013.

[15] K. Granström, U. Orguner, "Estimation and maintenance of measurement rates for multiple extended target tracking," *Proc. 15th International Conference on Information Fusion*, Singapore, July 2012.

[16] K. Granström, C. Lundquist, U. Orguner, "Extended Target Tracking using a Gaussian-Mixture PHD Filter ", *IEEE Transactions on Aerospace and Elecronic Syst*ems, vol. 48, no. 4, pp. 3268 - 3286, October 2012.

[17] B.-T. Vo, B.-N. Vo, A. Cantoni, "Analytic implementations of the cardinalized probability hypothesis density filter," *IEEE Transactions on Signal Processing*, vol. 55, no. 7, pp. 3553-3567, July 2007.

[18] B.-T. Vo, B.-N. Vo, "The para-normal Bayes multi-target filter and the spooky effect," *Proc. 15th International Conference on Information Fusion*, Singapore, July 2012.

[19] B. Ristic, J. Sherrah, "Bernoulli filter for joint detection and tracking of an extended object in clutter," *IET Radar, Sonar and Navigation*, vol. 7, no. 1, pp. 26-35, January 2013.

[20] B.-T. Vo and B.-N. Vo, "Labeled random finite sets and multi-object conjugate priors," *IEEE Transactions on Signal Processing*, vol. 61, no. 13, pp. 3460-3475, July 2013.

[21] B.-N. Vo; B.-T. Vo, D. Phung, "Labeled random finite sets and the Bayes multi-target tracking filter," *IEEE Transactions on Signal Processing*, vol. 62, no. 24, pp. 6554-6567, December 2014.

[22] S. Reuter, B.-T. Vo, B.-N. Vo, K. Dietmayer, "The labeled multi-Bernoulli filter," *IEEE Transactions on Signal Processing*, vol.62, no.12, pp. 3246,3260, June 2014.

[23] R. P. S. Mahler, *Statistical Multisource-Multitarget Information Fusion*, Artech House, 2007.

[24] B.-N. Vo, S. Singh, and A. Doucet, "Sequential Monte Carlo methods for Bayesian multi-target filtering with random finite sets", *IEEE Transactions on Aerospace and Electronic Syst*ems, vol. 41, no. 4, pp. 1224-1245, October 2005.

[25] R. Mahler, "PHD filters of higher order in target number", *IEEE Transactions on Aerospace and Electronic Syst*ems, vol. 43, no. 4, pp.1523-1543, October 2007.

[26] B.-T. Vo, B.-N. Vo, A. Cantoni. "The cardinality balanced multi-target multi-Bernoulli filter and its implementations", *IEEE Transactions on Signal Process*ing, vol. 57, no. 2, pp. 409-423, February 2009.

[27] K. G. Murty, "An algorithm for ranking all the assignments in increasing order of cost," *Operations Research*, vol. 16, no. 3, pp. 682-678, 1968.

[28] D. Schuhmacher, B.-T. Vo, B.-N. Vo, "A consistent metric for performance evaluation of multi-object filters," *IEEE Transactions on Signal Processing*, vol. 56, no. 8, pp. 3447-3457, August 2008.

[29] A. Swain and D. Clark, "Extended object filtering using spatial independent cluster processes," in *Proc. 13th International Conference on Information Fusion*, Edinburgh, UK, July 2010.

[30] A. Swain and D. Clark, "The PHD filter for extended target tracking with estimable shape parameters of varying size," *Proc. 15th International Conference on Information Fusion*, Singapore, July 2012.

[31] A. Gupta and D. Nagar, *Matrix Variate Distributions*, Chapman & Hall, 2000.