An instantiation algorithm for TLA+ expressions

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1 Overview

TLA+ has two kinds of substitution: instantiation of modules, which preserves validity, and beta-reduction of lambda-expressions, which does not necessarily preserve validity. Following the notions of SANY, we distinguish three kinds of variables: temporal constants and temporal variables are bound by instantiation whereas formal parameters are bound by lambda abstraction. Furthermore, ENABLED binds all temporal variables within its argument which are in the context of the next state operator '.

In the presence of unresolved instantiations or substitutions, it is often unclear, which operator binds the occurrence of a particular temporal variable: when we consider the expression $(\lambda\ fp : ENABLED\ (fp \# fp'))(x)$, it seems that the variable x is bound in the module. But in the beta-reduced form $ENABLED\ (x \# x')$ of this expression, only the first occurrence of x is bound by the module but its second (primed) occurrence is bound by ENABLED[1].

Furthermore, the two substitutions do not commute. For example, let us consider modules Foo and Bar:

−−−− **MODULE** Foo −−−−	−−−− **MODULE** Bar −−−−
VARIABLE x	**VARIABLE** y

E(u) == x' # u'
D(u) == **ENABLED** (E(u))
	I == **INSTANCE** Foo with x <− y

THEOREM T1 == D(x)	**THEOREM** T2 == I!D(y)
====	====

It looks like I!T1 and T2 talk about the same formula D(y), but this is not the case: the first can be read as I!(D(y))[2] and the second as (I!D)(y). In other words, it makes a difference if we beta-reduce first or if we instantiate first. Reducing D(x) first leads to ENABLED (x' # x'), which − following the renaming instuctions in "Specifying Systems" − becomes ENABLED ($x' # $x') by the instantiation of x with y, because primed occurrences of $x are bound by their enclosing ENABLED, not by the instantiation.

Instantiating first keeps the occurrence of the formal parameter u intact, leading to I!D(u) == ENABLED (u # $x') but reducing the application I!D(y) leads to ENABLED (y' # $x'). Now it is clear that I!(D(y)) is unsatisfiable while (I!D)(y) is valid.[3].

In the following, we will develop algorithms for both kinds of substitutions. Since the occurrence of a lambda reducible expression (redex) as a subterm of an instantiation may block the evaluation of the latter, inner substitutions (i.e. those closer to the leaves of the term tree) must be evaluated before outer ones. Definitions will also need special consideration because we allow some of them to stay folded. But since a definition can contain substitutions, they can not be treated like temporal constants.[4]

[1] To prevent complications from renaming, we will not assume alpha equivalence but will handle the renaming of bound variables explicitly.

[2] This is not valid TLA+ syntax.

[3] Since one of the axioms of TLA+ is that (TRUE # FALSE), we can always find a domain element different from x. In fact, the axioms of set-theory even enforce denumerable models.

[4] Actually, already a definition D(x) == e contains a substitution because it is equivalent to D == LAMBDA x : e.

2 New Attempt: Explicit substitutions

The original idea here is to represent both beta-reduction and instantiation explicitly in the term graph. Then the two formulas in the introduction could be written as D(u){u ↦ x}[x ↦ y] and D(u)[x ↦ y]{u ↦ y}, where reduction is denoted by curly braces and instantiation is denoted by square braces. Actually, the SANY data-structures allow to write reduction as application to an abstraction: D(u){u ↦ x} is then just (LAMBDA u: D(u))(x). Again, this is not legal TLA+ but allowed by SANY.

2.1 Datastructures

node	content	comment
Module	list of constants, list of variables, list of instances, list of definitions	
(Temporal) Constant	name, arity	Set of Constants CS
(Temporal) Variable	name	arity == 0, Set of Variables VS
(Formal)Parameter	name, arity	Set of Paramters FP
Definition	name, arity, expression body	Set of Definitions DS, all FPs in body are bound
Expression	constant **or** variable **or** parameter **or** definition **or** abstraction **or** application **or** substin **or** fpsubstin	
Application	head expression, argument expression list	head.arity == list length
Abstraction	parameter, expression body	
FPSubstIn	list of pairs formal parameter, expression	explicit substitution node ()
SubstIn	module, instantiation, expression body	the explicit instantiation node
Instantiation	list of assigments of variables/constants to expressions	

The definition and instantiation elements do not contain arguments since they can always be rewritten in terms of abstractions: D(x) == F is equivalent to D == LAMBDA x : F and I(x)!D is equivalent to LAMBDA x : I!D. We write abstraction as $\lambda x : F$ and instantiation as $(\rho M\ with x_1 \leftarrow t_1, \ldots, x_n \leftarrow t_n)!e$ where the variables / constants x_i are exactly those declared in the module M.

rewrite for explicit fp substs

To allow for more flexibility, we state the algorithm as a set of rewrite rules. The goal is to permute applications to abstractions further inside the term and resolve them at the leaves of the term tree (rules 1-3, 8 and 9) or, if the leaf is a folded definition, let the substitutions accumulate at the leaf (rules 4 and 10). Beta-reduction readily distributes over application (rule 5), but it needs special treatment when it is applied to an abstraction: in case there are no name clashes, $\lambda\, x$ can be evaluated before $\lambda\, y$ (rule 6), but if there are name clashes, we have to rename y to a fresh variable z first (rule 7).

insert operations on substitutions

A substitution σ is a function which maps formal parameters to expression and which differs from the identity function id only at a finite number of points. We write $\sigma = \{x \leftarrow e\}$ for the function

$$\sigma(y) = \begin{cases} e & \text{if } y = x \\ y & otherwise \end{cases}$$

. The composition $f \circ g$ of two substitutions is simply function composition $f(g(x))$. The removal operator $\sigma \backslash S$ is defined as:

$$\sigma(y) \backslash S = \begin{cases} y & \text{if } y \in S \\ \sigma(y) & otherwise \end{cases}$$

Describe instantiations.

We assume we have a set $Unfolded$ of definitions which are unfolded and a context tracing if we are inside ENABLED. When we denote meta-variables standing for any term in boldface, arbitrary FP substins as σ, and an assignment of $CS(M) \cup VS(M)$ to arbirary terms as ρ_M, then the rewrite rules are:

	rewrite rule			side conditions	
1	$c\sigma$	$\rightarrow$	c	$c \in CS \cup VS$	
2	$x\sigma$	$\rightarrow$	$\sigma(x)$	$x \in FP$	
3	D	$\rightarrow$	b	$D \in DS, b = D.body, D \in Unfolded$	
4	$(\mathbf{f}(\mathbf{g})))\sigma$	$\rightarrow$	$(\mathbf{f}\sigma)(\mathbf{g}\sigma)$		
5	$(\lambda x : \mathbf{s})(\mathbf{t})$	$\rightarrow$	$s(\{x \leftarrow t\})$	$x \notin FV(\mathbf{t})$	
6	$(\lambda x : \mathbf{s})(\mathbf{t})$	$\rightarrow$	$(\lambda z : \mathbf{s}(\{x \leftarrow z\}))(\mathbf{t})$	$x \in FV(\mathbf{t}), z \notin FV(\mathbf{s}) \cup FV(\mathbf{t})$	
7	$(\lambda x : \mathbf{s})\sigma$	$\rightarrow$	$\lambda x : \mathbf{s}(\sigma \backslash \{x\})$		
8	$\mathbf{t}\sigma_1\sigma_2$	$\rightarrow$	$\mathbf{t}(\sigma_1 \circ \sigma_2)$		
9	$[M : \rho_M]!c$	$\rightarrow$	$\rho_M(c)$	$c \in CS \cup VS$	
10	$[M : \rho_M]!x$	$\rightarrow$	x	$x \in FP$	
11	$[M : \rho_M]!D$	$\rightarrow$	$[M : \rho_M]!(b)$	$D \in DS, D \in Unfolded, b = D.body$	
12	$[M : \rho_M]!(\mathbf{f}(\mathbf{g}))$	$\rightarrow$	$[M : \rho_M]!(\mathbf{f})($ $[M : \rho_M]!(\mathbf{g}))$	$\mathbf{f} \neq'$ or $\mathbf{f}$ outside of EN	
13	$[M : \rho_M]!(\mathbf{g}')$	$\rightarrow$	$([M : c \leftarrow \$c]!(\mathbf{g}))'$	$'$ inside of $EN, \$c \notin CS \cup VS$	*

Remarks:

* this is unclean since it doesn't capture the difference between $EN(x \neq x' \wedge x = x')$ and $EN(x \neq x') \wedge EN(x = x')$ well

FP-substitution stops at folded definitions and CS/VS substitutions
CS/VS-substitution stops at folded definitions and FP-substitutions

2.2 Confluence

We consider all critical pairs:

- Rule 4 and rule 5 at position (1):
  For this to happen we can assume that $x \notin FV(t)$.
  Starting rewriting at ϵ, we derive:
  $$\begin{aligned} ((\lambda x : s)t)\sigma &\rightarrow ((\lambda x : s)\sigma)(t\sigma) \\ &\rightarrow (\lambda x : s(\sigma \backslash \{x\}))(t\sigma) \qquad x \notin t\sigma \\ &\rightarrow (s(\sigma \backslash \{x\}))\{x \leftarrow t\sigma\} \\ &\rightarrow s(\sigma \backslash \{x\} \circ \{x \leftarrow t\sigma\}) \end{aligned}$$
  Starting rewriting at (1), we derive:
  $$\begin{aligned} ((\lambda x : s)t)\sigma &\rightarrow (s\{x \leftarrow t\})\sigma) \\ &\rightarrow s(\{x \leftarrow t\} \circ \sigma) \end{aligned}$$
  But $(\sigma \backslash \{x\} \circ \{x \leftarrow t\sigma\}) = (\{x \leftarrow t\} \circ \sigma)$:
  $$(\sigma \backslash \{x\} \circ \{x \leftarrow t\sigma\})(y) = \begin{cases} t\sigma & \text{if } y = x \\ y\sigma & otherwise \end{cases} \qquad (\{x \leftarrow t\} \circ \sigma)(y) = \begin{cases} t\sigma & \text{if } y = x \\ y\sigma & otherwise \end{cases}$$
  handle $x \in t\sigma$ case

- Rule 4 and rule 6 at position (1): Starting rewriting at ϵ we derive:
  $$\begin{aligned} ((\lambda x : s)t)\sigma &\rightarrow ((\lambda x : s)\sigma)(t\sigma) \\ &\rightarrow (\lambda x : s(\sigma \backslash \{x\}))(t\sigma) \end{aligned}$$
  Starting rewriting at ϵ we derive:
  $$((\lambda x : s)t)\sigma \rightarrow ((\lambda z : s\{x \leftarrow z\})t)\sigma$$
  finish

2.3 Termination

We define a lexicographic order on the following measures:

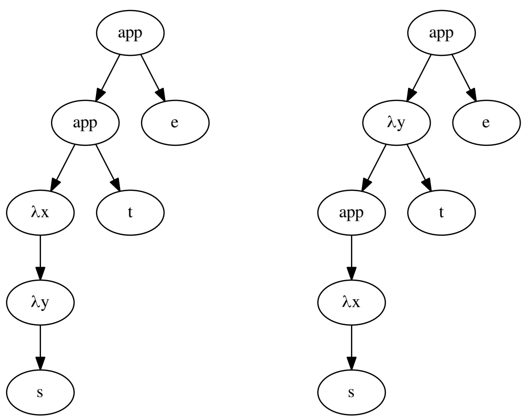
1. Variable name clashes:
   - $clash(c) = clash(v) = clash(fp) = 0$
   - $clash(s(t)) = clash(s) + clash(t)$ where s is not abstraction
   - $clash(\lambda x.s)t = clash(s)$ if $x \notin FV(t)$
   - $clash(\lambda x.s)t = clash(s) + 1$ if $x \in FV(t)$
   - $clash(\lambda x.s) = clash(s)$ for the cases not covered above

2. Deepest term under an abstraction which can be applied:
   - $d(v) = d(c) = d(fp) = 0$
   - $d(\lambda x.s) = 1 + d(s)$
   - $d(s(t)) = 1 + max(d(s), d(t))$

   - $da(v) = da(c) = da(fp) = 0$
   - $da(\lambda x.s) = 1 + da(s)$
   - $da(s(t)) = \begin{cases} 1 + d(s) + max(da(s), da(t)) & \text{if } s = \lambda x.r \\ max(da(s), da(t)) & \text{otherwise} \end{cases}$

3 Open Problems

- Confluence: I only checked overlaps of root position vs root position, non-root overlaps are possible
- The ordering da does not decrease in some cases (e.g.: wrap the redex of the *** rule into an application redex(e)). The reason is that outer pattern matches weigh heavier than inner ones:



The weight of the abstraction on x decreases as expected, but at the same time the weight of the abstraction on y increases. Since y is higher, its weight contributes more.

4 Excursion: parametrized instantiations in SANY

In SANY, parametrized instantuitions have a representation where the parameter is shifted over the instantiation. Let us consider the modules Foo and Bar:

```
---- MODULE Foo ----

VARIABLE a

D(u) == u' # a'
E(u) == D(u) \/ ENABLED D(u)

====
```

```
---- MODULE Bar ----
EXTENDS Naturals

VARIABLE x

I(v) == INSTANCE Foo WITH a <- x+v

====
```

The term I(x)!E(x) is represented as I!E(x,x) but they are not equivalent. In the meta-notation, they would be E{u $\mapsto$ x}[a $\mapsto$ x+v]{v $\mapsto$ x} vs. E{u $\mapsto$ x, v $\mapsto$ x}[a $\mapsto$ x+v] with the same consequences as in the introduction.

compute the set of unfolded defs

$(ENABLED\ A) \Leftrightarrow C$ used instantiated