A List of Interesting Math Problems

Ming

https://tlc-1924.github.io/Interesting-Math-Questions/

1) Prove by induction $6^{4n} + 38^n - 2$ is divisible by 7.

2) Find/Derive a general equation or formula for the shortest distance between a quadratic curve and a straight line in 2D. Can you do it for a plane and a line in 3D?

3) The cubic equation: $x^3 + px^2 + qx + r = 0$ has roots α, β, γ. Given $\alpha = \beta\gamma$, show $r = pq - 1$

4) Find:

$$\int \frac{4x^3}{x^2+3}\,dx$$

5) Find:

$$\int \frac{x^2(x^4+1)}{(x^4+1)^{\frac{1}{4}}}\,dx$$

6) Find the Maclaurin's series for $ln(1+e^x)$ up to and including the term in x^2.

7) Sketch the polar curve: $r = 3\cos\theta\sin^2\theta$

8) Solve (using both trigonometric simplification and complex numbers separately):

$$\int \sin^6\theta\,d\theta$$

9) Find a general equation for:

$$\frac{d^2y}{dx^2} + 3\frac{d^2y}{dx^2} - 4y = xe^x$$

10) Find I:

$$I = \lim_{a\to\infty}\left[\int_0^a \frac{x^\omega}{x^{2\omega\lambda}} - \frac{\cos(\omega x)}{x^\omega}dx\right]$$

where $\sqrt{\omega}$ is the first known irrational number, and where λ is such that:

$$\lambda = \left(\lim_{f\to 0}\frac{e^{5f}-1}{f}\right) - \left(\int_0^\pi 5\sin(2t)\,dt\right) + \left(\lim_{n\to\infty}\frac{e^{n\ln\left(1+\frac{1}{n}\right)}}{\sqrt[3]{e^{\frac{3}{2}}}^{2}}\right) - 5$$

11) Find x if:

$$x^4 - x^3 - x + 1 = 0$$

12) Find the exact value of I:

$$I = \int_0^\infty \frac{ln^2 x}{\sqrt{x}(1-x)^2} dx$$

13) The line l_1 has equation: $\frac{x-2}{4} = \frac{y-4}{-2} = \frac{z+6}{1}$. The plane Π has equation: $x - 2y + z = 6$. The line l_2 is the reflection of l_1 in Π. Please find a vector equation of l_2.

14) Evaluate:

$$\lim_{\delta x \to 0} \sum_{x=1}^{4} \frac{e^{2\sqrt{x}}}{\sqrt{x}} \, \delta x$$

15) Find a general solution for:

$$\frac{d^2 y}{dx^2} = y^2$$

16) Find: $\cos 6\theta$ and $\frac{\sin 6\theta}{\sin\theta}$ in terms of $\cos 6\theta$

17) Find:

$$\int \frac{1}{(x+1)\sqrt{x^2+1}} dx$$

18) Prove that if a quartic equation has roots $\alpha, \beta, \gamma, \delta$ then $\alpha^2+\beta^2+\gamma^2+\delta^2 \equiv (\sum \alpha)^2 - 2\sum \alpha\beta$.

19) Find:

$$\int \frac{e^{-4x} + 3e^{-2x}}{e^{-4x} - 9} dx$$

20) Find the derivative of the function f(x):

$$f(x) = \sin(\cos(\ln(\frac{1}{x}))), \quad x > 0$$

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21) Evaluate:

$$\int_0^1 \frac{(x^2+1)\, lnx}{x^2} \, ln(x^2+1)\, dx$$

22) What is h, if: $3^h - 2^h = 65$

23) Evaluate:

$$\lim_{x\to\infty} (\sqrt{x+1} - \sqrt{x})^{\frac{1}{\ln(\ln x)}}$$

24) Let x be "smaller than every positive number yet not negative, the only place where intuition politely resigns.". Hence, explain and find the values of: x^x and $\sqrt[x]{x}$

25) Find:

$$\int \cos x \sin^2 x \tan^3 x \ dx$$

26) Find:

$$\int_0^\infty (\frac{1-e^{-k}}{k})^2 \, dk$$

27) Solve:

$$\int \frac{z-1}{(z+1)^3} e^z \, dz$$

28) Solve for x if:

$$(2+\sqrt{3})^x - (2-\sqrt{3})^x = 4$$

29) Solve:

$$\int (\frac{\cot x}{\csc x})^7 dx$$

30) Find I if:

$$I = \int \frac{1}{(1-x^2)\sqrt[4]{2x^2-1}} dx$$

31) Prove the Pythagorean theorem.

32) Solve:

$$\int \frac{1}{x^2 \sqrt[4]{(x^4+1)^3}} dx$$

33) Find in terms of a:

$$\int x^{\frac{5}{2}} \sqrt{a-x} \ dx$$

34) Find m if:

$$2^m + 2^{3m} = 16$$

35) Solve:

$$\int \frac{7^x}{8^x} \cos x \, dx$$

36) Find m, where:

$$\sqrt[3]{m-\sqrt{m^2+8}} + \sqrt[3]{m+\sqrt{m^2+8}} - 8 = 0$$

37) Find:

$$\int \ln(\tanh^3(z))\, dz$$

38) Find x if: $x^2 = 4^x$

39) Solve:

$$\int \sinh x \cosh x \tanh x \ dx$$

40) Solve:

$$\int \frac{3^u}{e^u \ln u} du$$

41) Let:

$$x = \sqrt[3]{\sqrt{108}+10} \ - \ \sqrt[3]{\sqrt{108}-10}$$

Hence prove: $x^3 + 6x - 20 = 0$

42) Simplify: $\sqrt[3]{208+20\sqrt{108}} + \sqrt[3]{208-20\sqrt{108}}$

43) Find or simplify:

$$\left(\frac{1+\sqrt{5}}{2}\right)^{12}$$

44) Prove the following is irreducible:

$$\frac{21n+4}{14n+3}$$

45) Evaluate:

$$\int_1^2 \frac{1}{\sqrt{1+2x-x^2}}\, dx$$

46) Write $\sin^5\theta$ in multiple/compound angles.

47) Let O be the circumcentre, and **Ω** be the circumcircle of an acute-angled triangle ABC . Let P be an arbitrary point on **Ω**, distinct from A, B , C, and their antipodes in **Ω**. Denote the circumcentres of the triangles AOP , BOP , and COP by O_A, O_B, and O_C , respectively. The lines l_A, l_B, and l_C perpendicular to BC , CA, and AB pass through O_A, O_B, and O_C, respectively. Prove that the circumcircle of the triangle formed by l_A, l_B, and l_C is tangent to the line OP .

48) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that:

$$f(f(x)) = 4x\,, \quad \forall x \in \mathbb{R}.$$

49) Find:

$$\int \frac{1}{y(2+\ln y)^3} dy$$

50) Find constants a, b, c if: $2^a + 2^b + 2^c = 148$

51) It is given that:

$$a_{n+1} = a_n - \frac{1}{1+\sum_{i=1}^{n} \frac{1}{a_i}}$$

$$b_{n+1} = b_n + \frac{1}{1+\sum_{i=1}^{n} \frac{1}{b_i}}$$

So if $a_{100}b_{100} = a_{101}b_{101}$, find the value of $a_1 - b_1$

52) Solve for x if: $3^x + 9^x + 27^x = 14$

53)

$$\sum_{k=1}^{13} \frac{1}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right) \sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$$

54) Find x if: $x^6 = (x-6)^6$

55) Solve or simplify: $\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}} = 5$

56) Find $\mathcal{R}$ if: $(\frac{12}{\sqrt{3}})^{\mathcal{R}} = 144$

57) Calculate:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^2(x) + \frac{1}{2}}{1+e^{\sin x}} \, dx$$

58) Evaluate:

$$\int_{-1}^{1} \frac{1}{x} \sqrt{\frac{1+x}{1-x}} \ln\left(\frac{2x^2+2x+1}{2x^2-2x+1}\right) \, dx$$

59) Solve :

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\left(1+e^{x\cos x}\right)\left(\sin^4 x + \cos^4 x\right)} dx$$

60) Find:

$$\int_{0}^{\infty} \frac{1-\cos\left(x\sqrt{e-1}\right)}{xe^x} \, dx$$

61) Solve the equation:

$$\begin{vmatrix} x+1 & 0 & x & 0 & x-1 & 0 \\ 0 & x & 0 & x-1 & 0 & x+1 \\ x & 0 & x-1 & 0 & x+1 & 0 \\ 0 & x-1 & 0 & x+1 & 0 & x \\ x-1 & 0 & x+1 & 0 & x & 0 \\ 0 & x+1 & 0 & x & 0 & x-1 \end{vmatrix} = 9$$

62) Evaluate:

$$\int_{-2}^{2} (x^3 \cos(\frac{x}{2}) + \frac{1}{2})\sqrt{4-x^2}\, dx$$

63) Evaluate:

$$\int_0^1 \frac{\ln(x+1)}{x^2+1} dx$$

64) Evaluate:

$$\int_0^{\pi/2} \frac{1}{1+(\tan x)^{\pi}} dx$$

65) Find:

$$\int \frac{x^2+3x}{\sqrt{4-x^2}} dx$$

66) Evaluate:

$$\int_0^{\pi} \frac{\sin^6 x}{\sin^6 x + \cos^6 x} dx$$

67) Prove a kite has perpendicular diagonals.

68) Prove hexagons enclose more area per unit perimeter than any other regular tiling shape.

69) Find the general equation of:

$$\frac{d^3x}{dt^3} = \frac{d^2y}{dx^2} + \frac{dy}{dx}$$

70) Solve:

$$\int \frac{1}{x^5+1} dx$$

71) You're building a GPS navigation system for a city where intersections are nodes and roads have travel-time weights. Given a weighted graph and a starting intersection, you need to compute the shortest travel time to every other intersection. Write the general formula and/or code that implements an algorithm to solve this problem efficiently.

72) Evaluate:

$$\int \frac{2g-7}{(g^2+4)(g-1)}dg$$

73) Let ABC be an equilateral triangle with side length x. Let O be the centroid of $\triangle ABC$. A line segment YZ is drawn such that it passes through O, with Y on side BC and Z on side AC. Find the minimum possible length of the segment YZ, in terms of x.

74) Find x and y if: $3^x - 3^y = 234$

75) What is the maximum consecutive number of squares a horse/knight can cover on a standard 8 by 8 chess board (each square can be jumped only once, and cannot be jumped upon again). The piece starts at any corner of the board. Can you formulate this for an n by n chess board?

76) Find all integer solutions to the equation $x^2 - y^2 = 2025$

77) Solve:

$$I = \int \frac{1}{\sqrt{1 - csch^2 x}}dx$$

78) Solve:

$$\int_1^\infty \frac{1}{(p+1)\sqrt{p^2+2p-2}}dp$$

79) Evaluate:

$$\int_{-\infty}^\infty \frac{1}{\cosh h + \cos h}dh$$

80) Let $f(x) = (1+x^2)e^x$. Show that $f'(x) \geq 0$ and sketch the graph of $f(x)$. Hence, show that this equation:

$$(1+x^2)e^x = k,$$

where k is a constant, has exactly one real root if $k > 0$ and no real roots if $k \leq 0$.

81) Find the general solution of the equation:

$$y''' - 2y'' - 4y' + 8y = 2e^{-2x} + 16x + 16\cos x$$

82) Let n be a positive integer such that $n \geq 3$. The n roots of the equation $z^n - 1 = 0$ are denoted by $\omega_k = e^{i\frac{2\pi k}{n}}$, where $k = 0, 1, 2, \ldots, n-1$. Prove the identity:

$$\sum_{k=1}^{n-1} \frac{1}{(2-\omega_k)} = \frac{n2^{n-1} - 2^n + 1}{2^n - 1}$$

83) Evaluate:

$$\int_0^{\infty} \frac{2y}{\left(y^2+1+\lfloor 2y \rfloor\right)^2}\, dy$$

84) Find:

$$\int \frac{x^2(x-1)}{\cos x} + x^{1/4}\; dx$$

85) Find z if $z^i = -1$

86) The point P lies on the ellipse with polar equation: $r(5-3\cos\theta) = 8\,,\; 0 \le \theta < 2\pi$ The ellipse has foci at the origin $(0,0)$ and at T $(3,0)$. Show that $|OP| + |PT|$ is constant for all positions of P.

87) Find the stationary points of the function: $V(x,y) = \frac{1}{3}y^3 + x^2y - y$ and classify them by using the Hessian.

88) Evaluate:

$$\int_0^{\infty} \frac{m(e^{-m}+1)}{e^m - e^0}\, dm$$

89) Find all the possible basis $b > 6$ such that the number 5654 in base b represents a power of a prime number.

90) Consider the set G of 2×2 matrices with entries from the set of integers modulo 3 ($\mathbb{Z}_3$), defined by:

$$G = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \{1, 2\} \right\}$$

under the operation of matrix multiplication modulo 3. Assume that matrix multiplication is associative. Does G form an Abelian group? Determine the order of each element in G.

91) A 3×3 matrix $\mathbf{M}$ has distinct eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$.
Consider the matrix $\mathbf{B}$ defined by:

$$\mathbf{B} = \left(\mathbf{M}^3 - 2\mathbf{I}\right)^{-1}$$

where $\mathbf{I}$ is the 3×3 identity matrix. Determine the characteristic equation of matrix $\mathbf{B}$. Give your answer in the form with integer coefficients, $a, b, c, d \in \mathbb{Z}$:

$$a\lambda^3 + b\lambda^2 + c\lambda + d = 0$$

92) A curve C is defined by the parametric equations

$$x = a(\cos t + t\sin t), \quad y = a(\sin t - t\cos t)$$

where a is a positive constant and t is a parameter in the interval $0 \le t \le \pi$. The arc of the curve C is rotated through 2π radians about the x-axis to form a surface of revolution. Determine the exact area of this surface, giving your answer in terms of a and π.

93) Explain why $x^n \to 0$, $|x| < 1$, $n \to \infty$ (using Bernoulli's inequality or formal $\epsilon - N$).