Dafalias–Manzari Sand Model

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The implementation details of the Dafalias–Manzari sand model are documented. Compared to the original model, the Lode angle dependency is removed. Some constants are omitted.

The continuum mechanics convention (tension positive) is adopted for consistency.

1 Hyperelasticity

The hyperelastic response is defined as

$$G=G_0\frac{(2.97-e)^2}{1+e}\sqrt{pp_{at}},\qquad K=\frac{2}{3}\frac{1+\nu}{1-2\nu}G. \tag{1}$$

The corresponding derivatives are

$$\frac{\partial G}{\partial e}=G_0\sqrt{pp_{at}}\frac{e^2+2e-14.7609}{(1+e)^2},\qquad \frac{\partial G}{\partial p}=G_0\frac{(2.97-e)^2}{1+e}\frac{1}{2}\sqrt{\frac{p_{at}}{p}}. \tag{2}$$

The void ratio can be associated to strain so that

$$e=e_0+(1+e_0)\operatorname{tr}\varepsilon^{tr}. \tag{3}$$

The strain increment can be decomposed into elastic and plastic parts.

$$\boldsymbol{\varepsilon}^{tr}=\boldsymbol{\varepsilon}_n+\Delta\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_n+\Delta\boldsymbol{\varepsilon}^e+\Delta\boldsymbol{\varepsilon}^p. \tag{4}$$

As such, the stress increment can be expressed accordingly,

$$\boldsymbol{\sigma}=\boldsymbol{\sigma}_n+\Delta\boldsymbol{\sigma}=\boldsymbol{\sigma}_n+2G\left(\Delta\boldsymbol{e}-\Delta\boldsymbol{e}^p\right)+K\left(\Delta\varepsilon_v-\Delta\varepsilon_v^p\right)\boldsymbol{I}. \tag{5}$$

In derivatoric and spherical components,

$$\boldsymbol{\sigma}=\boldsymbol{s}+p\boldsymbol{I}, \tag{6}$$
$$p=p_n+K\left(\Delta\varepsilon_v-\Delta\varepsilon_v^p\right), \tag{7}$$
$$\boldsymbol{s}=\boldsymbol{s}_n+2G\left(\Delta\boldsymbol{e}-\Delta\boldsymbol{e}^p\right), \tag{8}$$

with

$$\Delta\boldsymbol{\varepsilon}=\Delta\boldsymbol{e}+\frac{1}{3}\Delta\varepsilon_v\boldsymbol{I}, \tag{9}$$

where $\boldsymbol{s}=\operatorname{dev}\boldsymbol{\sigma}$ is the deviatoric stress, $p=\dfrac{1}{3}\operatorname{tr}\boldsymbol{\sigma}$ is the hydrostatic stress.

2 Elastic Local Iteration

The trial state shall be computed assuming there is no plasticity. In which case,

$$p = p_n + K\Delta\varepsilon_v, \tag{10}$$
$$\boldsymbol{s} = \boldsymbol{s}_n + 2G\Delta\boldsymbol{e}. \tag{11}$$

The independent variables are chosen to be $\boldsymbol{x} = \begin{bmatrix} p & \boldsymbol{s} \end{bmatrix}^\mathrm{T}$, then the local residual is

$$\boldsymbol{R} = \left\{\begin{array}{l} p - p_n - K\Delta\varepsilon_v, \\ \boldsymbol{s} - \boldsymbol{s}_n - 2G\Delta\boldsymbol{e}. \end{array}\right. \tag{12}$$

The Jacobian can be expressed as

$$\boldsymbol{J} = \begin{bmatrix} 1 - \Delta\varepsilon_v \dfrac{2}{3}\dfrac{1+\nu}{1-2\nu}\dfrac{\partial G}{\partial p} & \mathbf{0} \\ -2\Delta\boldsymbol{e}\dfrac{\partial G}{\partial p} & \mathbf{I} \end{bmatrix}. \tag{13}$$

3 Critical State

The critical state parameter is chosen as

$$\psi = e - e_0 + \lambda_c \left(\frac{p}{p_{at}}\right)^\xi. \tag{14}$$

The derivatives are

$$\frac{\partial\psi}{\partial e} = 1, \qquad \frac{\partial\psi}{\partial p} = \lambda_c\xi \left(\frac{p}{p_{at}}\right)^{\xi-1} \frac{1}{p_{at}}. \tag{15}$$

The dilatancy surface is defined as

$$\alpha^d = \alpha^c \exp\left(n^d\psi\right). \tag{16}$$

The bounding surface is defined as

$$\alpha^b = \alpha^c \exp\left(-n^b\psi\right). \tag{17}$$

The corresponding derivatives are

$$\frac{\partial\alpha^d}{\partial\psi} = n^d\alpha_d, \qquad \frac{\partial\alpha^b}{\partial\psi} = -n^b\alpha_b. \tag{18}$$

4 Yield Function

A wedge-like function is chosen to be the yield surface.

$$F = \left|\boldsymbol{s} + p\boldsymbol{\alpha}\right| + mp = \left|\boldsymbol{\eta}\right| + mp, \tag{19}$$

where α is the so called back stress ratio and m characterises the size of the wedge. For simplicity, m is assumed to be a constant in this model.

By denoting $\boldsymbol{\eta} = \boldsymbol{s} + p\boldsymbol{\alpha}$, the directional unit tensor is defined as

$$\boldsymbol{n} = \dfrac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|}. \tag{20}$$

According to tensor algebra, the following expressions can be derived.

$$\dfrac{\partial|\boldsymbol{\eta}|}{\partial} = \boldsymbol{n} : \dfrac{\partial\boldsymbol{\eta}}{\partial}, \qquad \dfrac{\partial\boldsymbol{n}}{\partial} = \dfrac{1}{|\boldsymbol{\eta}|}\left(\dfrac{\partial\boldsymbol{\eta}}{\partial} - \boldsymbol{n}\otimes\left(\boldsymbol{n} : \dfrac{\partial\boldsymbol{\eta}}{\partial}\right)\right).$$

Hence,

$$\begin{aligned}
\dfrac{\partial|\boldsymbol{\eta}|}{\partial p} &= \eta_p = \boldsymbol{n} : \boldsymbol{\alpha}, & \dfrac{\partial\boldsymbol{n}}{\partial p} &= n_p = \dfrac{1}{|\boldsymbol{\eta}|}\left(\boldsymbol{\alpha} - \left(\boldsymbol{n} : \boldsymbol{\alpha}\right)\boldsymbol{n}\right),\\
\dfrac{\partial|\boldsymbol{\eta}|}{\partial \boldsymbol{s}} &= \eta_s = \boldsymbol{n} : \mathbf{I}, & \dfrac{\partial\boldsymbol{n}}{\partial \boldsymbol{s}} &= n_s = \dfrac{1}{|\boldsymbol{\eta}|}\left(\mathbf{I} - \boldsymbol{n}\otimes\left(\boldsymbol{n} : \mathbf{I}\right)\right),\\
\dfrac{\partial|\boldsymbol{\eta}|}{\partial \boldsymbol{\alpha}} &= \eta_\alpha = p\boldsymbol{n} : \mathbf{I}, & \dfrac{\partial\boldsymbol{n}}{\partial \boldsymbol{\alpha}} &= n_\alpha = \dfrac{p}{|\boldsymbol{\eta}|}\left(\mathbf{I} - \boldsymbol{n}\otimes\left(\boldsymbol{n} : \mathbf{I}\right)\right).
\end{aligned}$$

5 Flow Rule

A non-associated plastic flow is used, the corresponding flow rule is defined as follows.

$$\Delta\boldsymbol{\varepsilon}^p = \gamma\left(\boldsymbol{n} + \dfrac{1}{3}D\boldsymbol{I}\right), \tag{21}$$

where D is the dilatancy parameter.

$$D = A_d\left(\alpha_d - m - \boldsymbol{\alpha} : \boldsymbol{n}\right) = A_0\left(1 + \langle\boldsymbol{z} : \boldsymbol{n}\rangle\right)\left(\alpha_d - m - \boldsymbol{\alpha} : \boldsymbol{n}\right). \tag{22}$$

For $\boldsymbol{z} : \boldsymbol{n} \geqslant 0$,

$$\dfrac{\partial D}{\partial p} = A_0\left(\alpha_d - m - \boldsymbol{\alpha} : \boldsymbol{n}\right)\left(\boldsymbol{z} : n_p\right) + A_0\left(1 + \boldsymbol{z} : \boldsymbol{n}\right)\left(\dfrac{\partial\alpha_d}{\partial p} - \boldsymbol{\alpha} : n_p\right), \tag{23}$$

$$\dfrac{\partial D}{\partial \boldsymbol{s}} = A_0\left(\alpha_d - m - \boldsymbol{\alpha} : \boldsymbol{n}\right)\left(\boldsymbol{z} : n_s\right) - A_0\left(1 + \boldsymbol{z} : \boldsymbol{n}\right)\left(\boldsymbol{\alpha} : n_s\right), \tag{24}$$

$$\dfrac{\partial D}{\partial \boldsymbol{\alpha}} = A_0\left(\alpha_d - m - \boldsymbol{\alpha} : \boldsymbol{n}\right)\left(\boldsymbol{z} : n_\alpha\right) - A_0\left(1 + \boldsymbol{z} : \boldsymbol{n}\right)\left(\boldsymbol{\alpha} : n_\alpha + \boldsymbol{n} : \mathbf{I}\right), \tag{25}$$

$$\dfrac{\partial D}{\partial \boldsymbol{z}} = A_0\left(\alpha_d - m - \boldsymbol{\alpha} : \boldsymbol{n}\right)\left(\boldsymbol{n} : \mathbf{I}\right). \tag{26}$$

For $\boldsymbol{z} : \boldsymbol{n} < 0$,

$$\frac{\partial D}{\partial p} = A_0 \left(\frac{\partial \alpha_d}{\partial p} - \boldsymbol{\alpha} : n_p \right), \tag{27}$$

$$\frac{\partial D}{\partial \boldsymbol{s}} = -A_0 \left(\boldsymbol{\alpha} : n_s \right), \tag{28}$$

$$\frac{\partial D}{\partial \boldsymbol{\alpha}} = -A_0 \left(\boldsymbol{\alpha} : n_\alpha + \boldsymbol{n} : \mathbf{I} \right), \tag{29}$$

$$\frac{\partial D}{\partial \boldsymbol{z}} = \mathbf{0}. \tag{30}$$

6 Hardening Rule

The evolution rate of the back stress ratio $\boldsymbol{\alpha}$ is defined in terms of a proper distance measure from the bounding surface,

$$\Delta \boldsymbol{\alpha} = \gamma h \left(\left(\alpha^b - m \right) \boldsymbol{n} - \boldsymbol{\alpha} \right), \tag{31}$$

where h controls the hardening rate,

$$h = b_0 \exp \left(h_1 \left(\boldsymbol{\alpha}_{in} : \boldsymbol{n} - \boldsymbol{\alpha} : \boldsymbol{n} \right) \right). \tag{32}$$

The constant $\boldsymbol{\alpha}_{in}$ is updated whenever load reversal occurs.

The parameter b_0 is defined as a function of current state,

$$b_0 = G_0 h_0 \left(1 - c_h e \right) \sqrt{\frac{p_{at}}{p}}. \tag{33}$$

The derivatives are

$$\frac{\partial b_0}{\partial e} = -G_0 h_0 c_h \sqrt{\frac{p_{at}}{p}}, \qquad \frac{\partial b_0}{\partial p} = -G_0 h_0 \left(1 - c_h e \right) \frac{\sqrt{p p_{at}}}{2p^2} = -\frac{b_0}{2p}. \tag{34}$$

Hence,

$$\frac{\partial h}{\partial p} = \frac{\partial b_0}{\partial p} \exp \left(h_1 \left(\boldsymbol{\alpha}_{in} : \boldsymbol{n} - \boldsymbol{\alpha} : \boldsymbol{n} \right) \right), \tag{35}$$

$$\frac{\partial h}{\partial \boldsymbol{s}} = h h_1 \left(\boldsymbol{\alpha}_{in} - \boldsymbol{\alpha} \right) : n_s, \tag{36}$$

$$\frac{\partial h}{\partial \boldsymbol{\alpha}} = h h_1 \left(\left(\boldsymbol{\alpha}_{in} - \boldsymbol{\alpha} \right) : n_a - \boldsymbol{n} : \mathbf{I} \right). \tag{37}$$

7 Fabric Effect

The fabric tensor changes when $\Delta \varepsilon_v^p$ is positive,

$$\Delta \boldsymbol{z} = -c_z \left\langle \Delta \varepsilon_v^p \right\rangle \left(z_m \boldsymbol{n} + \boldsymbol{z} \right) = -c_z \gamma \left\langle D \right\rangle \left(z_m \boldsymbol{n} + \boldsymbol{z} \right). \tag{38}$$

8 Residual

The are five local residual equations.

$$\boldsymbol{R}=\left\{\begin{array}{l}\left|\boldsymbol{\eta}\right|+mp,\\ p-p_n+K\left(\gamma D-\Delta\varepsilon_v\right),\\ \boldsymbol{s}-\boldsymbol{s}_n+2G\left(\gamma\boldsymbol{n}-\Delta\boldsymbol{e}\right),\\ \boldsymbol{\alpha}-\boldsymbol{\alpha}_n+\gamma h\left(\boldsymbol{\alpha}-\left(\alpha^b-m\right)\boldsymbol{n}\right),\\ \boldsymbol{z}-\boldsymbol{z}_n+c_z\gamma\left\langle D\right\rangle\left(z_m\boldsymbol{n}+\boldsymbol{z}\right).\end{array}\right. \tag{39}$$

9 Local Iteration

By choosing $\boldsymbol{x}=\begin{bmatrix}\gamma & p & \boldsymbol{s} & \boldsymbol{\alpha} & \boldsymbol{z}\end{bmatrix}^\mathrm{T}$, the Jacobian consists of the entries that can be listed as follows.

$$\begin{bmatrix}
\cdot & \eta_p+m & \eta_s & \eta_\alpha & \cdot\\
DK & 1+\dfrac{\partial K}{\partial p}\left(\gamma D-\Delta\varepsilon_v\right)+K\gamma\dfrac{\partial D}{\partial p} & \gamma K\dfrac{\partial D}{\partial \boldsymbol{s}} & \gamma K\dfrac{\partial D}{\partial \boldsymbol{\alpha}} & \gamma K\dfrac{\partial D}{\partial \boldsymbol{z}}\\
2G\boldsymbol{n} & 2\dfrac{\partial G}{\partial p}\left(\gamma\boldsymbol{n}-\Delta\boldsymbol{e}\right)+2G\gamma n_p & \mathbf{I}+2G\gamma n_s & 2G\gamma n_\alpha & \cdot\\
h\left(\boldsymbol{\alpha}-\alpha^{bm}\boldsymbol{n}\right) & \gamma\dfrac{\partial h}{\partial p}\left(\boldsymbol{\alpha}-\alpha^{bm}\boldsymbol{n}\right)-\gamma h\left(\dfrac{\partial\alpha^b}{\partial p}\boldsymbol{n}+\alpha^{bm}n_p\right) & \gamma\left(\boldsymbol{\alpha}-\alpha^{bm}\boldsymbol{n}\right)\dfrac{\partial h}{\partial \boldsymbol{s}}-\gamma h\alpha^{bm}n_s & (1+\gamma h)\,\mathbf{I}+\gamma\left(\boldsymbol{\alpha}-\alpha^{bm}\boldsymbol{n}\right)\dfrac{\partial h}{\partial \boldsymbol{\alpha}}-\gamma h\alpha^{bm}n_a & \cdot\\
c_z\left\langle D\right\rangle\boldsymbol{z}_z & c_z\gamma\dfrac{\partial\left\langle D\right\rangle}{\partial p}\boldsymbol{z}_z+c_z\gamma\left\langle D\right\rangle z_m n_p & c_z\gamma\boldsymbol{z}_z\dfrac{\partial\left\langle D\right\rangle}{\partial \boldsymbol{s}}+c_z\gamma\left\langle D\right\rangle z_m n_s & c_z\gamma\boldsymbol{z}_z\dfrac{\partial\left\langle D\right\rangle}{\partial \boldsymbol{\alpha}}+c_z\gamma\left\langle D\right\rangle z_m n_\alpha & \left(1+c_z\gamma\left\langle D\right\rangle\right)\mathbf{I}+c_z\gamma\boldsymbol{z}_z\dfrac{\partial\left\langle D\right\rangle}{\partial \boldsymbol{z}}
\end{bmatrix}.$$

In which,

$$\alpha^{bm}=\alpha^b-m, \tag{40}$$

$$\boldsymbol{z}_z=z_m\boldsymbol{n}+\boldsymbol{z}. \tag{41}$$

10 Consistent Tangent Operator

$$\dfrac{\partial R_2}{\partial\boldsymbol{\varepsilon}^{tr}}=\dfrac{\partial K}{\partial\boldsymbol{\varepsilon}^{tr}}\left(\gamma D-\Delta\varepsilon_v\right)+K\left(\gamma\dfrac{\partial D}{\partial\boldsymbol{\varepsilon}^{tr}}-I\right), \tag{42}$$

$$\dfrac{\partial R_3}{\partial\boldsymbol{\varepsilon}^{tr}}=2\dfrac{\partial G}{\partial\boldsymbol{\varepsilon}^{tr}}\left(\gamma\boldsymbol{n}-\Delta\boldsymbol{e}\right)-2GI_d, \tag{43}$$

$$\dfrac{\partial R_4}{\partial\boldsymbol{\varepsilon}^{tr}}=\gamma\left(\boldsymbol{\alpha}-\alpha^{bm}\boldsymbol{n}\right)\dfrac{\partial h}{\partial\boldsymbol{\varepsilon}^{tr}}-\gamma h\boldsymbol{n}\dfrac{\partial\alpha^b}{\partial\boldsymbol{\varepsilon}^{tr}}, \tag{44}$$

$$\dfrac{\partial R_5}{\partial\boldsymbol{\varepsilon}^{tr}}=c_z\gamma\left(z_m\boldsymbol{n}+\boldsymbol{z}\right)\dfrac{\partial\left\langle D\right\rangle}{\partial\boldsymbol{\varepsilon}^{tr}}. \tag{45}$$

Then,

$$\dfrac{\partial\boldsymbol{x}}{\partial\boldsymbol{\varepsilon}^{tr}}=-\boldsymbol{J}^{-1}\dfrac{\partial\boldsymbol{R}}{\partial\boldsymbol{\varepsilon}^{tr}}. \tag{46}$$

Due to $\boldsymbol{\sigma} = \boldsymbol{s} + pI$, the stiffness can be assembled accordingly.