This document introduces the implementation of the von Mises based model that employs a Voce (1955) type isotropic hardening rule, a Armstrong–Frederick (Armstrong and Frederick, 1966) type kinematic hardening rule that supports multiplicative back stresses as proposed by Chaboche and Rousselier (1983) and a Peric (1993) type viscosity rule.
References:

1. https://doi.org/10.1017/S0368393100118759
2. https://doi.org/10.1179/096034007X207589
3. https://doi.org/10.1016/0749-6419(89)90015-6
4. https://doi.org/10.1002/nme.1620360807

1 Yield Function

A von Mises yielding function is used.

$$F = \sqrt{\frac{3}{2}}\left|\eta\right| - k = q - k, \tag{1}$$

in which $\eta = s - \beta$ is the shifted stress, s is the stress deviator, β is the back stress and k is the isotropic hardening stress.

2 Flow Rule

The associated plasticity flow is adopted. The plastic strain rate is then

$$\mathrm{d}\varepsilon^p = \gamma\frac{\partial F}{\partial \sigma} = \sqrt{\frac{3}{2}}\gamma n, \tag{2}$$

where $n = \dfrac{\eta}{\left|\eta\right|}$. The corresponding accumulated plastic strain rate is

$$\mathrm{d}p = \sqrt{\frac{2}{3}\mathrm{d}\varepsilon^p : \mathrm{d}\varepsilon^p} = \gamma. \tag{3}$$

3 Plastic Multiplier

The rate of plastic multiplier is defined as

$$\frac{\gamma}{\Delta t} = \dot{\gamma} = \frac{1}{\mu}\left(\left(\frac{q}{k}\right)^{\dfrac{1}{\epsilon}} - 1\right), \tag{4}$$

in which μ and ϵ are two material constants. Equivalently, it is

$$q\left(\frac{\Delta t}{\Delta t + \mu\gamma}\right)^{\epsilon} - k = 0. \tag{5}$$

4 Hardening Rules

An exponential function with a linear component is used for isotropic hardening stress.

$$k = \sigma_y + k_l p + k_s - k_s e^{-mp}. \tag{6}$$

The corresponding derivative is

$$\frac{\mathrm{d}k}{\mathrm{d}\gamma} = k_l + k_s m e^{-mp}. \tag{7}$$

The rate form of back stress $\beta = \sum \beta^i$ is defined as

$$\mathrm{d}\beta^i = \sqrt{\frac{2}{3}} a^i \mathrm{d}\varepsilon^p - b^i \beta^i \mathrm{d}p.$$

In terms of γ, it is $\mathrm{d}\beta^i = a^i \gamma n - b^i \gamma \beta^i$. The incremental form is thus

$$\beta^i = \beta^i_n + a^i \gamma n - b^i \gamma \beta^i, \qquad \beta^i = \frac{\beta^i_n + a^i \gamma n}{1 + b^i \gamma}. \tag{8}$$

5 Incremental Form

The shifted stress can be computed as

$$\eta = s - \beta = 2GI_d \left(\varepsilon^{tr} - \varepsilon^p_n - \sqrt{\frac{3}{2}} \gamma n \right) - \beta = s^{tr} - \sqrt{6} G \gamma n - \sum \frac{\beta^i_n + a^i \gamma n}{1 + b^i \gamma} \tag{9}$$

with $s^{tr} = 2GI_d \left(\varepsilon^{tr} - \varepsilon^p_n \right)$. Hence,

$$\left|\eta\right| n + \sqrt{6} G \gamma n + \sum \frac{a^i \gamma}{1 + b^i \gamma} n = s^{tr} - \sum \frac{\beta^i_n}{1 + b^i \gamma}, \qquad \left|\eta\right| + \sqrt{6} G \gamma + \sum \frac{a^i \gamma}{1 + b^i \gamma} = \left| s^{tr} - \sum \frac{\beta^i_n}{1 + b^i \gamma} \right|.$$

Eventually,

$$\left|\eta\right| = \left| s^{tr} - \sum \frac{\beta^i_n}{1 + b^i \gamma} \right| - \sqrt{6} G \gamma - \sum \frac{a^i \gamma}{1 + b^i \gamma}. \tag{10}$$

Then η can be expressed as,

$$\eta = \frac{\left| s^{tr} - \sum \frac{\beta^i_n}{1 + b^i \gamma} \right| - \sqrt{6} G \gamma - \sum \frac{a^i \gamma}{1 + b^i \gamma}}{\left| s^{tr} - \sum \frac{\beta^i_n}{1 + b^i \gamma} \right|} \left(s^{tr} - \sum \frac{\beta^i_n}{1 + b^i \gamma} \right).$$

It is equivalent to

$$\eta = \left(\left| s^{tr} - \sum \frac{\beta^i_n}{1 + b^i \gamma} \right| - \sqrt{6} G \gamma - \sum \frac{a^i \gamma}{1 + b^i \gamma} \right) u.$$

It is easy to see that $n = u$ with $u = \dfrac{s^{tr} - \sum \frac{\beta^i_n}{1 + b^i \gamma}}{\left| s^{tr} - \sum \frac{\beta^i_n}{1 + b^i \gamma} \right|}$. The derivatives of u are

$$\frac{\partial u}{\partial \gamma} = \frac{1}{\left| s^{tr} - \sum \frac{\beta^i_n}{1 + b^i \gamma} \right|} \sum \frac{b^i}{(1 + b^i \gamma)^2} \left(\beta^i_n - u : \beta^i_n u \right), \qquad \frac{\partial u}{\partial \varepsilon^{tr}} = \frac{1}{\left| s^{tr} - \sum \frac{\beta^i_n}{1 + b^i \gamma} \right|} 2G \left(I_d - u \otimes u \right). \tag{11}$$

Furthermore,

$$\frac{\partial q}{\partial \gamma} = \sqrt{\frac{3}{2}} \sum \frac{b^i u : \beta^i_n - a^i}{(1 + b^i \gamma)^2} - 3G. \tag{12}$$

6 Scalar Equation Iteration

For viscoplasticity, the yield function F is not necessarily zero. The rate form of plastic multiplier is used here for iteration.

$$R = q \left(\frac{\Delta t}{\Delta t + \mu \gamma} \right)^\epsilon - k = 0.$$

The corresponding derivatives are then

$$\frac{\partial R}{\partial \gamma}=\left(\frac{\Delta t}{\Delta t+\mu\gamma}\right)^{\epsilon}\left(\frac{\partial q}{\partial \gamma}-\frac{q\epsilon\mu}{\Delta t+\mu\gamma}\right)-\frac{\mathrm{d}k}{\mathrm{d}\gamma}, \tag{13}$$

$$\frac{\partial R}{\partial \varepsilon^{tr}}=\left(\frac{\Delta t}{\Delta t+\mu\gamma}\right)^{\epsilon}\sqrt{6}Gu:I_d=\left(\frac{\Delta t}{\Delta t+\mu\gamma}\right)^{\epsilon}\sqrt{6}Gu, \tag{14}$$

with

$$k=\sigma_y+k_l\left(p_n+\gamma\right)+k_s\left(1-e^{-m(p_n+\gamma)}\right), \tag{15}$$

$$q=\sqrt{\frac{3}{2}}\left(\left|s^{tr}-\sum\frac{\beta_n^i}{1+b^i\gamma}\right|-\sqrt{6}G\gamma-\sum\frac{a^i\gamma}{1+b^i\gamma}\right). \tag{16}$$

7 Consistent Tangent Stiffness

For stiffness, ε^{tr} is now varying, then

$$\frac{\partial R}{\partial \varepsilon^{tr}}+\frac{\partial R}{\partial \gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}\varepsilon^{tr}}=0,\qquad \frac{\mathrm{d}\gamma}{\mathrm{d}\varepsilon^{tr}}=-\left(\frac{\partial R}{\partial \gamma}\right)^{-1}\frac{\partial R}{\partial \varepsilon^{tr}}. \tag{17}$$

Since the stress can be written as

$$\sigma=E(\varepsilon^{tr}-\varepsilon^p)=E(\varepsilon^{tr}-\varepsilon^p_n-\Delta\varepsilon^p)=E(\varepsilon^{tr}-\varepsilon^p_n)-\sqrt{6}G\gamma u. \tag{18}$$

The derivative is

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon^{tr}}&=E-\sqrt{6}G\left(\gamma\frac{\partial u}{\partial \varepsilon^{tr}}+\left(u+\gamma\frac{\partial u}{\partial \gamma}\right)\frac{\mathrm{d}\gamma}{\mathrm{d}\varepsilon^{tr}}\right)\\&=E+\sqrt{6}G\left(\left(u+\gamma\frac{\partial u}{\partial \gamma}\right)\left(\frac{\partial R}{\partial \gamma}\right)^{-1}\frac{\partial R}{\partial \varepsilon^{tr}}-\gamma\frac{\partial u}{\partial \varepsilon^{tr}}\right).\end{aligned} \tag{19}$$