SimpleSand Model

tlc

July 13, 2021

This sand model is a hybrid of some existing models with significant simplifications. Users should not expect this model to give accurate results.

Compared to other models for geomaterials, this model is more like a preliminary model just like the bilinear hardening J2 model compared to other models for metals.

Since the elastic part is assumed to be linear, the return mapping implicit algorithm is used to obtain consistent tangent operator.

The state parameter is taken from 10.1016/j.cma.2005.08.020. Other parts are mostly taken from 10.1061/(ASCE)0733-9399(2004)130:6(622). The Lode angle dependency is removed. The fabric effect is removed.

1 Elasticity

The strain increment is decomposed into elastic and plastic parts,

$$\Delta \varepsilon = \Delta \varepsilon^e + \Delta \varepsilon^p.$$

The stress increment is then

$$\Delta \sigma = E \left(\Delta \varepsilon - \Delta \varepsilon^p \right).$$

2 Yield Function

A wedge-like function is chosen to be the yield surface.

$$F = \left| s + p\alpha \right| + mp = \left| \eta \right| + mp,$$

where $s = \operatorname{dev} \sigma$ is the deviatoric stress, $p = \dfrac{1}{3}\operatorname{tr} \sigma$ is the hydrostatic stress, α is the so called back stress ratio and m characterises the size of the wedge. For simplicity, m is assumed to be a constant in this model.

3 Flow Rule

A non-associated plastic flow is used, the corresponding flow rule is defined as follows.

$$\Delta \varepsilon^p = \gamma \left(\boldsymbol{n} + \frac{1}{3} A \left(\alpha^d - m - \boldsymbol{\alpha} : \boldsymbol{n} \right) \boldsymbol{I} \right),$$

where $D = A \left(\alpha^d - m - \boldsymbol{\alpha} : \boldsymbol{n} \right)$ is the dilatancy parameter, A is a model constant, it can also be defined as a function of current state. Note since the sign convention is changed, a positive A leads to dilatant behaviour. Often it needs to be negative.

Again for simplicity, the linear elasticity is applied, hence the following expressions hold.

$$\boldsymbol{s} = \boldsymbol{s}^{tr} - 2G\gamma\boldsymbol{n}, \qquad p = p^{tr} - KA\gamma \left(\alpha^d - m - \boldsymbol{\alpha} : \boldsymbol{n} \right).$$

A hyper-elastic response can also be applied. However, in that case, the G and K would be zero for zero hydrostatic pressure.

4 Hardening Rule

The evolution rate of the back stress ratio $\boldsymbol{\alpha}$ is defined in terms of a proper distance measure from the bounding surface, Here, such a distance measure is chosen to be $h \left(\left(\alpha^b - m \right) \boldsymbol{n} - \boldsymbol{\alpha} \right)$, where h is a model constant. Thus,

$$\Delta \boldsymbol{\alpha} = \gamma h \left(\left(\alpha^b - m \right) \boldsymbol{n} - \boldsymbol{\alpha} \right).$$

Essentially, this is similar to the Armstrong–Frederick type kinematic hardening rule.

5 Critical State

The state parameter is defined to be

$$\psi = v_0 \left(1 + \mathrm{tr}\, \boldsymbol{\varepsilon}^{tr} \right) - v_c + \lambda_c \ln \left(\frac{p}{p_c} \right),$$

where v_0 is the initial specific volume, v_c is the corresponding specific volume on the critical line at p_c and λ_c is absolute value of the slope of the critical line in v–$\ln(-p)$ space.

The Lode angle dependence is removed for simple derivations of the corresponding terms. Hence both the dilatancy surface and bounding surface will be circular cones in the principal stress space.

The dilatancy surface is defined as

$$\alpha^d = \alpha^c \exp \left(n^d \psi \right).$$

The corresponding derivatives are

$$\frac{\partial \alpha^d}{\partial \varepsilon^{tr}} = \alpha^d n^d v_0 \boldsymbol{I}, \qquad \frac{\partial \alpha^d}{\partial p} = \alpha^d n^d \frac{\lambda_c}{p}.$$

The bounding surface is defined as

$$\alpha^b = \alpha^c \exp\left(-n^b \psi\right).$$

The corresponding derivatives are

$$\frac{\partial \alpha^b}{\partial \varepsilon^{tr}} = -\alpha^b n^b v_0 \boldsymbol{I}, \qquad \frac{\partial \alpha^b}{\partial p} = -\alpha^b n^b \frac{\lambda_c}{p}.$$

The symbol $\boldsymbol{I}$ denotes the unit second order tensor.

6   Local Iteration

According to tensor algebra, the following expressions can be derived.

$$\frac{\partial |\boldsymbol{\eta}|}{\partial} = \boldsymbol{n} : \frac{\partial \boldsymbol{\eta}}{\partial}, \qquad \frac{\partial \boldsymbol{n}}{\partial} = \frac{1}{|\boldsymbol{\eta}|} \left(\frac{\partial \boldsymbol{\eta}}{\partial} - \boldsymbol{n} \otimes \left(\boldsymbol{n} : \frac{\partial \boldsymbol{\eta}}{\partial} \right) \right).$$

Hence,

$$\frac{\partial |\boldsymbol{\eta}|}{\partial p} = \boldsymbol{n} : \boldsymbol{\alpha}, \qquad \frac{\partial \boldsymbol{n}}{\partial p} = \frac{1}{|\boldsymbol{\eta}|} \left(\boldsymbol{\alpha} - (\boldsymbol{n} : \boldsymbol{\alpha}) \, \boldsymbol{n} \right),$$

$$\frac{\partial |\boldsymbol{\eta}|}{\partial s} = \boldsymbol{n} : \boldsymbol{I}, \qquad \frac{\partial \boldsymbol{n}}{\partial s} = \frac{1}{|\boldsymbol{\eta}|} \left(\boldsymbol{I} - \boldsymbol{n} \otimes (\boldsymbol{n} : \boldsymbol{I}) \right),$$

$$\frac{\partial |\boldsymbol{\eta}|}{\partial \boldsymbol{\alpha}} = p \boldsymbol{n} : \boldsymbol{I}, \qquad \frac{\partial \boldsymbol{n}}{\partial \boldsymbol{\alpha}} = \frac{p}{|\boldsymbol{\eta}|} \left(\boldsymbol{I} - \boldsymbol{n} \otimes (\boldsymbol{n} : \boldsymbol{I}) \right).$$

In summary, there are four residual equations.

$$\boldsymbol{R} = \begin{cases} |\boldsymbol{\eta}| + mp, \\[2mm] p - p^{tr} + KA\gamma \left(\alpha^d - m - \boldsymbol{\alpha} : \boldsymbol{n} \right), \\[2mm] \boldsymbol{s} - \boldsymbol{s}^{tr} + 2G\gamma \boldsymbol{n}, \\[2mm] \boldsymbol{\alpha}_n + \gamma h \left(\alpha^b - m \right) \boldsymbol{n} - (\gamma h + 1) \, \boldsymbol{\alpha}. \end{cases}$$

By defined $\boldsymbol{x} = \begin{bmatrix} \gamma & p & s & \alpha \end{bmatrix}^{\mathrm{T}}$, the local Jacobian can be derived as

$$\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{x}} = \begin{bmatrix} 0 & \boldsymbol{\alpha} : \boldsymbol{n} + m & \boldsymbol{n} : \boldsymbol{I} & p\boldsymbol{n} : \boldsymbol{I} \\[3mm] KA \left(\alpha^d - m - \boldsymbol{\alpha} : \boldsymbol{n} \right) & 1 + KA\gamma \left(\dfrac{\partial \alpha^d}{\partial p} - \boldsymbol{\alpha} : \dfrac{\partial \boldsymbol{n}}{\partial p} \right) & -KA\gamma \boldsymbol{\alpha} : \dfrac{\partial \boldsymbol{n}}{\partial s} & -KA\gamma \left(\boldsymbol{I} : \boldsymbol{n} + \boldsymbol{\alpha} : \dfrac{\partial \boldsymbol{n}}{\partial \boldsymbol{\alpha}} \right) \\[3mm] 2G\boldsymbol{n} & 2G\gamma \dfrac{\partial \boldsymbol{n}}{\partial p} & \boldsymbol{I} + 2G\gamma \dfrac{\partial \boldsymbol{n}}{\partial s} & 2G\gamma \dfrac{\partial \boldsymbol{n}}{\partial \boldsymbol{\alpha}} \\[3mm] h \left(\alpha^b - m \right) \boldsymbol{n} - h\boldsymbol{\alpha} & \gamma h \left(\dfrac{\partial \alpha^b}{\partial p} \boldsymbol{n} + \left(\alpha^b - m \right) \dfrac{\partial \boldsymbol{n}}{\partial p} \right) & \gamma h \left(\alpha^b - m \right) \dfrac{\partial \boldsymbol{n}}{\partial s} & \gamma h \left(\alpha^b - m \right) \dfrac{\partial \boldsymbol{n}}{\partial \boldsymbol{\alpha}} - (\gamma h + 1) \boldsymbol{I} \end{bmatrix}.$$

In the explicit form, it is

$$
\begin{bmatrix}
0 & \boldsymbol{\alpha} : \boldsymbol{n} + m & \boldsymbol{n} : \mathbf{I} & p\boldsymbol{n} : \mathbf{I} \\[2ex]
KA\left(\alpha^d - m - \boldsymbol{\alpha} : \boldsymbol{n}\right) & 1 + KA\gamma\left(\dfrac{\partial \alpha^d}{\partial p} - \dfrac{\boldsymbol{\alpha} : \boldsymbol{\alpha} - (\boldsymbol{n} : \boldsymbol{\alpha})^2}{|\boldsymbol{\eta}|}\right) & \dfrac{KA\gamma}{|\boldsymbol{\eta}|}\left((\boldsymbol{\alpha} : \boldsymbol{n}) \cdot \boldsymbol{n} - \boldsymbol{\alpha}\right) : \mathbf{I} & KA\gamma\left(\boldsymbol{n} + \dfrac{p}{|\boldsymbol{\eta}|}\left((\boldsymbol{\alpha} : \boldsymbol{n}) \cdot \boldsymbol{n} - \boldsymbol{\alpha}\right)\right) : \mathbf{I} \\[2ex]
2G\boldsymbol{n} & \dfrac{2G\gamma}{|\boldsymbol{\eta}|}\left(\boldsymbol{\alpha} - (\boldsymbol{n} : \boldsymbol{\alpha}) \cdot \boldsymbol{n}\right) & \mathbf{I} + \dfrac{2G\gamma}{|\boldsymbol{\eta}|}\left(\mathbf{I} - \boldsymbol{n} \otimes (\boldsymbol{n} : \mathbf{I})\right) & \dfrac{2G\gamma p}{|\boldsymbol{\eta}|}\left(\mathbf{I} - \boldsymbol{n} \otimes (\boldsymbol{n} : \mathbf{I})\right) \\[2ex]
h\left(\alpha^b - m\right)\boldsymbol{n} - h\boldsymbol{\alpha} & \gamma h\left(\dfrac{\partial \alpha^b}{\partial p}\boldsymbol{n} + \dfrac{\alpha^b - m}{|\boldsymbol{\eta}|}\left(\boldsymbol{\alpha} - (\boldsymbol{n} : \boldsymbol{\alpha}) \cdot \boldsymbol{n}\right)\right) & \dfrac{\gamma h\left(\alpha^b - m\right)}{|\boldsymbol{\eta}|}\left(\mathbf{I} - \boldsymbol{n} \otimes (\boldsymbol{n} : \mathbf{I})\right) & \dfrac{\gamma h\left(\alpha^b - m\right)p}{|\boldsymbol{\eta}|}\left(\mathbf{I} - \boldsymbol{n} \otimes (\boldsymbol{n} : \mathbf{I})\right) - (\gamma h + 1)\mathbf{I}
\end{bmatrix}.
$$

7 Consistent Tangent Stiffness

Accordingly,

$$
\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{\varepsilon}^{tr}} =
\begin{bmatrix}
0 \\[1ex]
\left(KA\gamma\alpha^d n^d v_0 - K\right)\boldsymbol{I} \\[1ex]
-2G\mathbf{I}_d \\[1ex]
-\alpha^b n^b v_0 \gamma h \boldsymbol{n} \otimes \boldsymbol{I}
\end{bmatrix}.
$$

Thus,

$$
\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\boldsymbol{\varepsilon}^{tr}} = -\left(\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{x}}\right)^{-1}\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{\varepsilon}^{tr}}.
$$

The stress update is

$$
\boldsymbol{\sigma} = \boldsymbol{s} + p\boldsymbol{I}.
$$

The consistent tangent stiffness is

$$
\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}\boldsymbol{\varepsilon}^{tr}} = \frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}\boldsymbol{\varepsilon}^{tr}} + \boldsymbol{I} \otimes \frac{\mathrm{d}p}{\mathrm{d}\boldsymbol{\varepsilon}^{tr}}.
$$