This is a brief summary of the CDP model implemented in `suanPan`. For detailed theories, readers are recommended to check the original literature.

1. Yield Function

The yield function is defined as

$$F = \alpha I_1 + \sqrt{\frac{3}{2}}|s| + \beta \langle \sigma_1 \rangle - (1 - \alpha) c_c, \tag{1}$$

with $c_c = -\bar{f}_c$ and $\beta = \dfrac{\bar{f}_c}{\bar{f}_t}(\alpha - 1) - (\alpha + 1)$.

2. Flow Rule

The flow potential is chosen to be

$$G = \sqrt{2J_2} + \alpha_p I_1 = |s| + \alpha_p \mathrm{tr}\sigma. \tag{2}$$

The flow rule is accordingly defined as

$$\Delta\varepsilon^p = \Delta\lambda \frac{\partial G}{\partial \sigma} = \Delta\lambda \left(\frac{s}{|s|} + \alpha_p I \right), \tag{3}$$

so that

$$\Delta\varepsilon^p_d = \Delta\lambda \frac{s}{|s|}, \quad \Delta\varepsilon^p_v = 3\alpha_p \Delta\lambda. \tag{4}$$

Noting that $s = s^{tr} - 2G\Delta\varepsilon^p_d = s^{tr} - 2G\Delta\lambda \dfrac{s}{|s|}$ and $p = p^{tr} - K\Delta\varepsilon^p_v = p^{tr} - 3K\alpha_p\Delta\lambda$, equivalently,

$$|s| = |s^{tr}| - 2G\Delta\lambda, \quad I_1 = I_1^{tr} - 9K\alpha_p\Delta\lambda. \tag{5}$$

Furthermore,

$$\frac{s}{|s|} = \frac{s^{tr}}{|s^{tr}|} \equiv n, \tag{6}$$

so that

$$\Delta\varepsilon^p = \Delta\lambda \left(n + \alpha_p I \right). \tag{7}$$

The flow direction is fixed for each sub-step.

3. Damage Evolution

Damage parameters shall satisfy the following expression.

$$\kappa - \kappa_n - H\Delta\lambda = 0. \tag{8}$$

For tension and compression, two separate scalar equations can be written as

$$\kappa^t_n + r\frac{f_t}{g_t}(n_1 + \alpha_p)\Delta\lambda - \kappa^t = 0, \tag{9}$$

$$\kappa^c_n + (1 - r)\frac{f_c}{g_c}(n_3 + \alpha_p)\Delta\lambda - \kappa^c = 0. \tag{10}$$

The backbone curves are related to the damage parameter $\kappa_\aleph$.

$$f_\aleph = f_{\aleph,0}\sqrt{\phi_\aleph}\Phi_\aleph,$$

with

$$\phi_\aleph = 1 + a_\aleph\left(2 + a_\aleph\right)\kappa_\aleph, \quad \Phi_\aleph = \dfrac{1 + a_\aleph - \sqrt{\phi_\aleph}}{a_\aleph}.$$

The effective counterpart is defined as

$$\bar{f}_\aleph = \dfrac{f_\aleph}{1-d} = f_{\aleph,0}\sqrt{\phi_\aleph}\Phi_\aleph^{1-c_\aleph/b_\aleph},$$

with

$$d = 1 - \Phi_\aleph^{c_\aleph/b_\aleph}.$$

The corresponding derivatives are

$$\dfrac{\mathrm{d}d}{\mathrm{d}\kappa_\aleph} = \dfrac{c_\aleph}{b_\aleph}\dfrac{a_\aleph+2}{2\sqrt{\phi_\aleph}}\Phi_\aleph^{c_\aleph/b_\aleph-1}, \quad \dfrac{\mathrm{d}f_\aleph}{\mathrm{d}\kappa_\aleph} = f_{\aleph,0}\dfrac{a_\aleph+2}{2\sqrt{\phi_\aleph}}\left(a_\aleph - 2\sqrt{\phi_\aleph}+1\right),$$

$$\dfrac{\mathrm{d}\bar{f}_\aleph}{\mathrm{d}\kappa_\aleph} = f_{\aleph,0}\dfrac{a_\aleph+2}{2\sqrt{\phi_\aleph}}\dfrac{\left(a_\aleph + 1 + \left(\dfrac{c_\aleph}{b_\aleph}-2\right)\sqrt{\phi_\aleph}\right)}{\Phi_\aleph^{c_\aleph/b_\aleph}}.$$

4. Residual

The yield function and the damage evolutions are three local equations shall be satisfied.

$$R\begin{cases}\alpha I_1^{tr} - 9K\alpha\alpha_p\Delta\lambda + \sqrt{\dfrac{3}{2}}\left(|s^{tr}| - 2G\Delta\lambda\right) + \beta\left\langle\sigma_1^{tr} - \Delta\lambda\left(2Gn_1 + 3K\alpha_p\right)\right\rangle + (1-\alpha)\bar{f}_c = 0,\\ \kappa_n^t + r\dfrac{f_t}{g_t}(n_1+\alpha_p)\Delta\lambda - \kappa^t = 0,\\ \kappa_n^c + (1-r)\dfrac{f_c}{g_c}(n_3+\alpha_p)\Delta\lambda - \kappa^c = 0.\end{cases} \tag{11}$$

By choosing $x = \begin{bmatrix}\Delta\lambda & \kappa_t & \kappa_c\end{bmatrix}^\mathrm{T}$ as independent variables and assuming $n = \dfrac{s^{tr}}{|s^{tr}|}$ that is a function of ε^{tr} only thus does not contain $\Delta\lambda$, the Jacobian can be computed as

$$J = \begin{bmatrix} -9K\alpha\alpha_p - \sqrt{6}G - \beta\left(2Gn_1 + 3K\alpha_p\right)H(\sigma_1) & \langle\sigma_1\rangle\dfrac{\partial\beta}{\partial\kappa_t} & (1-\alpha)\bar{f}_c' + \langle\sigma_1\rangle\dfrac{\partial\beta}{\partial\kappa_c}\\ f_t\dfrac{n_1+\alpha_p}{g_t}(r + \Delta\lambda\dfrac{\partial r}{\partial\Delta\lambda}) & r\Delta\lambda\dfrac{n_1+\alpha_p}{g_t}f_t' - 1 & \cdot\\ f_c\dfrac{n_3+\alpha_p}{g_c}(1 - r - \Delta\lambda\dfrac{\partial r}{\partial\Delta\lambda}) & \cdot & (1-r)\Delta\lambda\dfrac{n_3+\alpha_p}{g_c}f_c' - 1\end{bmatrix},$$

where

$$\dfrac{\partial\beta}{\partial\kappa_t} = (1-\alpha)\dfrac{\bar{f}_c}{\bar{f}_t^2}\bar{f}_t', \qquad \dfrac{\partial\beta}{\partial\kappa_c} = (\alpha-1)\dfrac{1}{\bar{f}_t}\bar{f}_c'.$$

In explicit form, if $\sigma_1 > 0$,

$$J = \begin{bmatrix} -9K\alpha\alpha_p - \sqrt{6}G - \beta\left(2Gn_1 + 3K\alpha_p\right) & (1-\alpha)\dfrac{\bar{f}_c\sigma_1}{\bar{f}_t^2}\bar{f}_t' & (1-\alpha)(1-\dfrac{\sigma_1}{\bar{f}_t})\bar{f}_c' \\ f_t\dfrac{n_1+\alpha_p}{g_t}(r+\Delta\lambda\dfrac{\partial r}{\partial\Delta\lambda}) & r\Delta\lambda\dfrac{n_1+\alpha_p}{g_t}f_t' - 1 & \cdot \\ f_c\dfrac{n_3+\alpha_p}{g_c}(1-r-\Delta\lambda\dfrac{\partial r}{\partial\Delta\lambda}) & \cdot & (1-r)\Delta\lambda\dfrac{n_3+\alpha_p}{g_c}f_c' - 1 \end{bmatrix}, \tag{12}$$

otherwise,

$$J = \begin{bmatrix} -9K\alpha\alpha_p - \sqrt{6}G & \cdot & (1-\alpha)\bar{f}_c' \\ f_t\dfrac{n_1+\alpha_p}{g_t}(r+\Delta\lambda\dfrac{\partial r}{\partial\Delta\lambda}) & r\Delta\lambda\dfrac{n_1+\alpha_p}{g_t}f_t' - 1 & \cdot \\ f_c\dfrac{n_3+\alpha_p}{g_c}(1-r-\Delta\lambda\dfrac{\partial r}{\partial\Delta\lambda}) & \cdot & (1-r)\Delta\lambda\dfrac{n_3+\alpha_p}{g_c}f_c' - 1 \end{bmatrix}. \tag{13}$$

5. Consistent Tangent Stiffness

Taking derivatives with regard to trial strain of the residual equations gives

$$\frac{\partial R}{\partial\varepsilon^{tr}} = \begin{cases} 3K\alpha I + \sqrt{6}Gn + \beta\dfrac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\bar{\sigma}}_t\dfrac{\partial\bar{\sigma}}{\partial\varepsilon^{tr}}H(\sigma), \\ \dfrac{f_t}{g_t}\Delta\lambda\left(r\dfrac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\bar{\sigma}}_t\dfrac{\mathrm{d}n}{\mathrm{d}e} + \left(n_1+\alpha_p\right)r'\dfrac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\bar{\sigma}}\dfrac{\partial\bar{\sigma}}{\partial\varepsilon^{tr}}\right), \\ \dfrac{f_c}{g_c}\Delta\lambda\left((1-r)\dfrac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\bar{\sigma}}_c\dfrac{\mathrm{d}n}{\mathrm{d}e} - \left(n_3+\alpha_p\right)r'\dfrac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\bar{\sigma}}\dfrac{\partial\bar{\sigma}}{\partial\varepsilon^{tr}}\right). \end{cases} \tag{14}$$

so that

$$\frac{\mathrm{d}x}{\mathrm{d}\varepsilon^{tr}} = \begin{bmatrix} \dfrac{\mathrm{d}\Delta\lambda}{\mathrm{d}\varepsilon^{tr}} \\ \dfrac{\mathrm{d}\kappa_t}{\mathrm{d}\varepsilon^{tr}} \\ \dfrac{\mathrm{d}\kappa_c}{\mathrm{d}\varepsilon^{tr}} \end{bmatrix} = -\frac{\partial R}{\partial x}^{-1}\frac{\partial R}{\partial\varepsilon^{tr}}. \tag{15}$$

In above equations, $\dfrac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\bar{\sigma}}$ is the transformation matrix between principal stress and nominal stress, which is a function of eigen vectors and does not change with any other variables. So it can be treated as a constant transform matrix.

The stress update is computed as follows.

$$\sigma = (1-d_c)(1-sd_t)\bar{\sigma}. \tag{16}$$

The tangent stiffness of which is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon^{tr}} = (1-d_c)(1-sd_t)\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\varepsilon^{tr}} + \bar{\sigma}\otimes\frac{\mathrm{d}(1-d_c)(1-sd_t)}{\mathrm{d}\varepsilon^{tr}}. \tag{17}$$

The effective stress $\bar{\sigma}$ only depends on ε^{tr} and $\Delta\lambda$.

$$\begin{aligned}
\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\varepsilon^{tr}} &= \frac{\mathrm{d}\left(s^{tr} - 2G\Delta\lambda\dfrac{s^{tr}}{|s^{tr}|} + (p^{tr} - 3K\alpha_p\Delta\lambda)I\right)}{\mathrm{d}\varepsilon^{tr}} \\
&= \frac{\mathrm{d}\left(2G\varepsilon_d^{tr} - 2G\Delta\lambda\dfrac{\varepsilon_d^{tr}}{|\varepsilon_d^{tr}|} + \left(K\varepsilon_v^{tr} - 3K\alpha_p\Delta\lambda\right)I\right)}{\mathrm{d}\varepsilon^{tr}} \\
&= 2GI_d - \frac{4G^2\Delta\lambda}{|s^{tr}|}\left(I_d - n\otimes n\right) + KI\otimes I - \left(2Gn + 3K\alpha_p I\right)\otimes\frac{\mathrm{d}\Delta\lambda}{\mathrm{d}\varepsilon^{tr}} \\
&= D^e - \frac{4G^2\Delta\lambda}{|s^{tr}|}\left(I_d - n\otimes n\right) - \left(2Gn + 3K\alpha_p I\right)\otimes\frac{\mathrm{d}\Delta\lambda}{\mathrm{d}\varepsilon^{tr}}
\end{aligned}, \tag{18}$$

in which

$$\frac{\partial\bar{\sigma}}{\partial\varepsilon^{tr}} = D^e - \frac{4G^2\Delta\lambda}{|s^{tr}|}\left(I_d - n\otimes n\right). \tag{19}$$

The damage factor can be expressed as

$$\begin{aligned}
\frac{\mathrm{d}(1-d_c)(1-sd_t)}{\mathrm{d}\varepsilon^{tr}} = \frac{\mathrm{d}\left(1 - sd_t - d_c + sd_td_c\right)}{\mathrm{d}\varepsilon^{tr}} &= \frac{\mathrm{d}\left(sd_td_c\right)}{\mathrm{d}\varepsilon^{tr}} - \frac{\mathrm{d}\left(sd_t\right)}{\mathrm{d}\varepsilon^{tr}} - \frac{\mathrm{d}d_c}{\mathrm{d}\varepsilon^{tr}} \\
&= d_td_c\frac{\mathrm{d}s}{\mathrm{d}\varepsilon^{tr}} + sd_c\frac{\mathrm{d}d_t}{\mathrm{d}\varepsilon^{tr}} + sd_t\frac{\mathrm{d}d_c}{\mathrm{d}\varepsilon^{tr}} - d_t\frac{\mathrm{d}s}{\mathrm{d}\varepsilon^{tr}} - s\frac{\mathrm{d}d_t}{\mathrm{d}\varepsilon^{tr}} - \frac{\mathrm{d}d_c}{\mathrm{d}\varepsilon^{tr}} \\
&= d_t(d_c-1)\frac{\mathrm{d}s}{\mathrm{d}\varepsilon^{tr}} + s(d_c-1)\frac{\mathrm{d}d_t}{\mathrm{d}\varepsilon^{tr}} + (sd_t-1)\frac{\mathrm{d}d_c}{\mathrm{d}\varepsilon^{tr}} \\
&= d_t(d_c-1)(1-s_0)\frac{\mathrm{d}r}{\mathrm{d}\varepsilon^{tr}} + s(d_c-1)d_t'\frac{\mathrm{d}\kappa_t}{\mathrm{d}\varepsilon^{tr}} + (sd_t-1)d_c'\frac{\mathrm{d}\kappa_c}{\mathrm{d}\varepsilon^{tr}}
\end{aligned}, \tag{20}$$

where

$$\frac{\mathrm{d}r}{\mathrm{d}\varepsilon^{tr}} = \frac{\mathrm{d}r}{\mathrm{d}\hat{\sigma}}\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\bar{\sigma}}\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\varepsilon^{tr}}. \tag{21}$$