

# CDP Model

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This is a brief summary of the CDP model implemented in suanPan. For detailed theories, readers are recommended to check the original literature 10.1061/(ASCE)0733-9399(1998)124:8(892).

## 1 Yield Function

The yield function is defined as

$$F = \alpha I_1 + \sqrt{\frac{3}{2}} |s| + \beta \langle \sigma_1 \rangle - (1 - \alpha) c_c, \quad (1)$$

with  $c_c = -\bar{f}_c$  and  $\beta = \frac{\bar{f}_c}{\bar{f}_t}(\alpha - 1) - (\alpha + 1)$ .

## 2 Flow Rule

The flow potential is chosen to be

$$G = \sqrt{2J_2} + \alpha_p I_1 = |s| + \alpha_p \text{tr} \sigma. \quad (2)$$

The flow rule is accordingly defined as

$$\Delta \varepsilon^p = \Delta \lambda \frac{\partial G}{\partial \sigma} = \Delta \lambda \left( \frac{s}{|s|} + \alpha_p I \right), \quad (3)$$

so that

$$\Delta \varepsilon_d^p = \Delta \lambda \frac{s}{|s|}, \quad \Delta \varepsilon_v^p = 3\alpha_p \Delta \lambda. \quad (4)$$

Noting that  $s = s^{tr} - 2G\Delta \varepsilon_d^p = s^{tr} - 2G\Delta \lambda \frac{s}{|s|}$  and  $p = p^{tr} - K\Delta \varepsilon_v^p = p^{tr} - 3K\alpha_p \Delta \lambda$ , equivalently,

$$|s| = |s^{tr}| - 2G\Delta \lambda, \quad I_1 = I_1^{tr} - 9K\alpha_p \Delta \lambda. \quad (5)$$

Furthermore,

$$\frac{s}{|s|} = \frac{s^{tr}}{|s^{tr}|} \equiv n, \quad (6)$$

so that

$$\Delta \varepsilon^p = \Delta \lambda (n + \alpha_p I). \quad (7)$$

The flow direction is fixed for each sub-step.

### 3 Damage Evolution

Damage parameters shall satisfy the following expression.

$$\kappa - \kappa_n - H\Delta\lambda = 0. \quad (8)$$

For tension and compression, two separate scalar equations can be written as

$$\kappa_n^t + r \frac{f_t}{g_t} (n_1 + \alpha_p) \Delta\lambda - \kappa^t = 0, \quad (9)$$

$$\kappa_n^c + (1 - r) \frac{f_c}{g_c} (n_3 + \alpha_p) \Delta\lambda - \kappa^c = 0. \quad (10)$$

The backbone curves are related to the damage parameter  $\kappa_N$ .

$$f_N = f_{N,0} \sqrt{\phi_N} \Phi_N,$$

with

$$\phi_N = 1 + a_N (2 + a_N) \kappa_N, \quad \Phi_N = \frac{1 + a_N - \sqrt{\phi_N}}{a_N}.$$

The effective counterpart is defined as

$$\bar{f}_N = \frac{f_N}{1 - d} = f_{N,0} \sqrt{\phi_N} \Phi_N^{1 - c_N/b_N},$$

with

$$d = 1 - \Phi_N^{c_N/b_N}.$$

The corresponding derivatives are

$$\begin{aligned} \frac{dd}{d\kappa_N} &= \frac{c_N}{b_N} \frac{a_N + 2}{2\sqrt{\phi_N}} \Phi_N^{c_N/b_N - 1}, \quad \frac{df_N}{d\kappa_N} = f_{N,0} \frac{a_N + 2}{2\sqrt{\phi_N}} (a_N - 2\sqrt{\phi_N} + 1), \\ \frac{d\bar{f}_N}{d\kappa_N} &= f_{N,0} \frac{a_N + 2}{2\sqrt{\phi_N}} \frac{\left( a_N + 1 + \left( \frac{c_N}{b_N} - 2 \right) \sqrt{\phi_N} \right)}{\Phi_N^{c_N/b_N}}. \end{aligned}$$

### 4 Residual

The yield function and the damage evolutions are three local equations shall be satisfied.

$$R = \begin{cases} \alpha I_1^{tr} - 9K\alpha\alpha_p\Delta\lambda + \sqrt{\frac{3}{2}} (|s^{tr}| - 2G\Delta\lambda) + \beta \langle \sigma_1^{tr} - \Delta\lambda (2Gn_1 + 3K\alpha_p) \rangle + (1 - \alpha) \bar{f}_c, \\ \kappa_n^t + r \frac{f_t}{g_t} (n_1 + \alpha_p) \Delta\lambda - \kappa^t, \\ \kappa_n^c + (1 - r) \frac{f_c}{g_c} (n_3 + \alpha_p) \Delta\lambda - \kappa^c. \end{cases} \quad (11)$$

By choosing  $x = [\Delta\lambda \quad \kappa_t \quad \kappa_c]^T$  as independent variables and assuming  $n = \frac{s^{tr}}{|s^{tr}|}$  that is a function of  $\varepsilon^{tr}$  only thus does not contain  $\Delta\lambda$ , the Jacobian can be computed as

$$J = \begin{bmatrix} -9K\alpha\alpha_p - \sqrt{6}G - \beta(2Gn_1 + 3K\alpha_p)H(\sigma_1) & \langle\sigma_1\rangle \frac{\partial\beta}{\partial\kappa_t} & (1-\alpha)\bar{f}'_c + \langle\sigma_1\rangle \frac{\partial\beta}{\partial\kappa_c} \\ f_t \frac{n_1 + \alpha_p}{g_t} (r + \Delta\lambda \frac{\partial r}{\partial\Delta\lambda}) & r\Delta\lambda \frac{n_1 + \alpha_p}{g_t} f'_t - 1 & \cdot \\ f_c \frac{n_3 + \alpha_p}{g_c} (1 - r - \Delta\lambda \frac{\partial r}{\partial\Delta\lambda}) & \cdot & (1-r)\Delta\lambda \frac{n_3 + \alpha_p}{g_c} f'_c - 1 \end{bmatrix},$$

where

$$\frac{\partial\beta}{\partial\kappa_t} = (1-\alpha) \frac{\bar{f}'_c}{\bar{f}_t^2} f'_t, \quad \frac{\partial\beta}{\partial\kappa_c} = (\alpha-1) \frac{1}{\bar{f}_t} \bar{f}'_c.$$

In explicit form, if  $\sigma_1 > 0$ ,

$$J = \begin{bmatrix} -9K\alpha\alpha_p - \sqrt{6}G - \beta(2Gn_1 + 3K\alpha_p) & (1-\alpha) \frac{\bar{f}_c \sigma_1}{\bar{f}_t^2} \bar{f}'_t & (1-\alpha)(1 - \frac{\sigma_1}{\bar{f}_t}) \bar{f}'_c \\ f_t \frac{n_1 + \alpha_p}{g_t} (r + \Delta\lambda \frac{\partial r}{\partial\Delta\lambda}) & r\Delta\lambda \frac{n_1 + \alpha_p}{g_t} f'_t - 1 & \cdot \\ f_c \frac{n_3 + \alpha_p}{g_c} (1 - r - \Delta\lambda \frac{\partial r}{\partial\Delta\lambda}) & \cdot & (1-r)\Delta\lambda \frac{n_3 + \alpha_p}{g_c} f'_c - 1 \end{bmatrix}, \quad (12)$$

otherwise,

$$J = \begin{bmatrix} -9K\alpha\alpha_p - \sqrt{6}G & \cdot & (1-\alpha)\bar{f}'_c \\ f_t \frac{n_1 + \alpha_p}{g_t} (r + \Delta\lambda \frac{\partial r}{\partial\Delta\lambda}) & r\Delta\lambda \frac{n_1 + \alpha_p}{g_t} f'_t - 1 & \cdot \\ f_c \frac{n_3 + \alpha_p}{g_c} (1 - r - \Delta\lambda \frac{\partial r}{\partial\Delta\lambda}) & \cdot & (1-r)\Delta\lambda \frac{n_3 + \alpha_p}{g_c} f'_c - 1 \end{bmatrix}. \quad (13)$$

## 5 Consistent Tangent Stiffness

Taking derivatives with regard to trial strain of the residual equations gives

$$\frac{\partial R}{\partial \varepsilon^{tr}} = \begin{cases} 3K\alpha I + \sqrt{6}Gn + \beta \frac{d\hat{\sigma}}{d\bar{\sigma}_t} \frac{\partial \bar{\sigma}}{\partial \varepsilon^{tr}} H(\sigma), \\ \frac{f_t}{g_t} \Delta\lambda \left( r \frac{d\hat{\sigma}}{d\bar{\sigma}_t} \frac{dn}{de} + (n_1 + \alpha_p) r' \frac{d\hat{\sigma}}{d\bar{\sigma}} \frac{\partial \bar{\sigma}}{\partial \varepsilon^{tr}} \right), \\ \frac{f_c}{g_c} \Delta\lambda \left( (1-r) \frac{d\hat{\sigma}}{d\bar{\sigma}_c} \frac{dn}{de} - (n_3 + \alpha_p) r' \frac{d\hat{\sigma}}{d\bar{\sigma}} \frac{\partial \bar{\sigma}}{\partial \varepsilon^{tr}} \right). \end{cases} \quad (14)$$

so that

$$\frac{dx}{d\varepsilon^{tr}} = \begin{bmatrix} \frac{d\Delta\lambda}{d\varepsilon^{tr}} \\ \frac{d\kappa_t}{d\varepsilon^{tr}} \\ \frac{d\kappa_c}{d\varepsilon^{tr}} \end{bmatrix} = -\frac{\partial R}{\partial x}^{-1} \frac{\partial R}{\partial \varepsilon^{tr}}. \quad (15)$$

In above equations,  $\frac{d\hat{\sigma}}{d\bar{\sigma}}$  is the transformation matrix between principal stress and nominal stress, which is a function of eigen vectors and does not change with any other variables. So it can be treated as a constant transform matrix.

The stress update is computed as follows.

$$\sigma = (1 - d_c)(1 - sd_t)\bar{\sigma}. \quad (16)$$

The tangent stiffness of which is

$$\frac{d\sigma}{d\varepsilon^{tr}} = (1 - d_c)(1 - sd_t) \frac{d\bar{\sigma}}{d\varepsilon^{tr}} + \bar{\sigma} \otimes \frac{d(1 - d_c)(1 - sd_t)}{d\varepsilon^{tr}}. \quad (17)$$

The effective stress  $\bar{\sigma}$  only depends on  $\varepsilon^{tr}$  and  $\Delta\lambda$ .

$$\begin{aligned} \frac{d\bar{\sigma}}{d\varepsilon^{tr}} &= \frac{d \left( s^{tr} - 2G\Delta\lambda \frac{s^{tr}}{|s^{tr}|} + (p^{tr} - 3K\alpha_p\Delta\lambda)I \right)}{d\varepsilon^{tr}} \\ &= \frac{d \left( 2G\varepsilon_d^{tr} - 2G\Delta\lambda \frac{\varepsilon_d^{tr}}{|\varepsilon_d^{tr}|} + (K\varepsilon_v^{tr} - 3K\alpha_p\Delta\lambda)I \right)}{d\varepsilon^{tr}}, \\ &= 2GI_d - \frac{4G^2\Delta\lambda}{|s^{tr}|} (I_d - n \otimes n) + KI \otimes I - (2Gn + 3K\alpha_p I) \otimes \frac{d\Delta\lambda}{d\varepsilon^{tr}} \\ &= D^e - \frac{4G^2\Delta\lambda}{|s^{tr}|} (I_d - n \otimes n) - (2Gn + 3K\alpha_p I) \otimes \frac{d\Delta\lambda}{d\varepsilon^{tr}} \end{aligned} \quad (18)$$

in which

$$\frac{\partial \bar{\sigma}}{\partial \varepsilon^{tr}} = D^e - \frac{4G^2\Delta\lambda}{|s^{tr}|} (I_d - n \otimes n). \quad (19)$$

The damage factor can be expressed as

$$\begin{aligned} \frac{d(1 - d_c)(1 - sd_t)}{d\varepsilon^{tr}} &= \frac{d(1 - sd_t - d_c + sd_t d_c)}{d\varepsilon^{tr}} = \frac{d(sd_t d_c)}{d\varepsilon^{tr}} - \frac{d(sd_t)}{d\varepsilon^{tr}} - \frac{dd_c}{d\varepsilon^{tr}} \\ &= d_t d_c \frac{ds}{d\varepsilon^{tr}} + sd_c \frac{dd_t}{d\varepsilon^{tr}} + sd_t \frac{dd_c}{d\varepsilon^{tr}} - d_t \frac{ds}{d\varepsilon^{tr}} - s \frac{dd_t}{d\varepsilon^{tr}} - \frac{dd_c}{d\varepsilon^{tr}}, \\ &= d_t(d_c - 1) \frac{ds}{d\varepsilon^{tr}} + s(d_c - 1) \frac{dd_t}{d\varepsilon^{tr}} + (sd_t - 1) \frac{dd_c}{d\varepsilon^{tr}} \\ &= d_t(d_c - 1)(1 - s_0) \frac{dr}{d\varepsilon^{tr}} + s(d_c - 1)d'_t \frac{d\kappa_t}{d\varepsilon^{tr}} + (sd_t - 1)d'_c \frac{d\kappa_c}{d\varepsilon^{tr}} \end{aligned} \quad (20)$$

where

$$\frac{dr}{d\varepsilon^{tr}} = \frac{dr}{d\hat{\sigma}} \frac{d\hat{\sigma}}{d\bar{\sigma}} \frac{d\bar{\sigma}}{d\varepsilon^{tr}}. \quad (21)$$