CG assignment 4

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Ex 1 [5 points]

Let us consider a 2D coordinate system with points $p_1 = [1, 1]^\intercal$, and $p = [1.5, 2.5]^\intercal$. Additionally, let's define a vector $u = p - p_1$. Perform the following tasks:

Task 1 [1 points]

Construct two matrices, R_{90}, $T \in \mathbb{R}^{3\times 3}$, such that the first one performs counter-clockwise rotation around the center of the coordinates system by angle $90°$, and the second one performs translation by vector $t = [1, -2]^\intercal$, in homogeneous coordinates.

$$p = \begin{bmatrix} 1.5 \\ 2.5 \end{bmatrix}, p_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$u = p - p_1 = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}$$

From generic counter clockwise rotation in 2D coord system:

$$R(\alpha) = \begin{bmatrix} cos(\alpha) & -sin(\alpha) \\ sin(\alpha) & cos(\alpha) \end{bmatrix}$$

Then, rotation matrix with homogeneous coordinates for $\alpha = 90°$ is:

$$R(90) = \begin{bmatrix} cos(90) & -sin(90) & 0 \\ sin(90) & cos(90) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Given rotation matrix R and translation vector t for 2D coordinates system, the affine transformation matrix T is:

$$T = \begin{bmatrix} & \text{R} & & \text{t} \\ & & & \\ 0 & \dots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Task 2 [1 points]

Represent the points p_1 and p as well as the vector u using homogeneous coordinates and transform them first using the $R(90)$ and then T matrices. Convert the obtained points and vector back into Cartesian coordinates and denote them by p_1', p', and u'. Draw all the points and vectors before and after the transformation, and verify that $u' = p' - p_1'$. What influence did the matrix T have on the u'?

Using homogeneous coordinates for position points:

$$p_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, p = \begin{bmatrix} 1.5 \\ 2.5 \\ 1 \end{bmatrix}$$

u is displacement vector, so 0 in homogeneous coordinates last element:

$$u = p - p_1 = \begin{bmatrix} 0.5 \\ 1.5 \\ 0 \end{bmatrix}$$

Using homogeneous coordinates allow chaining sequence of transformations to be represented with a single matrix being the product of the transformation matrices. In this case, the transformation matrix is:

$$\begin{aligned} M &= T \cdot R(90) \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3} \end{aligned}$$

Calculating with matrix M:

$$p' = M \cdot p = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1.5 \\ 2.5 \\ 1 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -0.5 \\ 1 \end{bmatrix} \Rightarrow p' = \begin{bmatrix} -1.5 \\ -0.5 \end{bmatrix}$$

$$p_1' = M \cdot p_1 = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \Rightarrow p_1' = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$u' = M \cdot u = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.5 \\ 1.5 \\ 0 \end{bmatrix} = \begin{bmatrix} -1.5 \\ 0.5 \\ 0 \end{bmatrix} \Rightarrow u' = \begin{bmatrix} -1.5 \\ 0.5 \end{bmatrix} = p' - p_1'$$

We can see that T had no effect on u', this is because u' is a displacement vector and so is not affected by translation.

Task 3 [1 points]

$$S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S\_p = S * p = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -1.5 \\ -0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$$

$$p'' = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$$

$$S\_p1 = S * p1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$p1'' = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$S\_u = S * u = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -1.5 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

$$u'' = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Task 4 [2 points]

$$S^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R\_90^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M = R\_90^{-1} * T^{-1} * S^{-1} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 2 \\ -1/2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M * p'' = \begin{bmatrix} 0 & 1/2 & 2 \\ -1/2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 2.5 \\ 1 \end{bmatrix} == p$$

$$M * p1'' = \begin{bmatrix} 0 & 1/2 & 2 \\ -1/2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} == p1$$

$$M * u'' = \begin{bmatrix} 0 & 1/2 & 2 \\ -1/2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \\ 0 \end{bmatrix} == u$$

Ex 2

$$p1 = \begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix}$$

$$p2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$p3 = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

$$p = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$$area\_of\_triangle = t = (p3 - p1) * (p2 - p1) = \begin{bmatrix} 16 \\ 16 \\ -16 \end{bmatrix}$$

$$normal\_of\_sub\_triangle\_1 = t1 = (p3 - p) * (p1 - p) = \begin{bmatrix} 4 \\ 4 \\ -4 \end{bmatrix}$$

$$normal\_of\_sub\_triangle\_2 = t2 = (p1 - p) * (p2 - p) = \begin{bmatrix} 4 \\ 4 \\ -4 \end{bmatrix}$$

$$normal\_of\_sub\_triangle\_3 = t3 = (p2 - p) * (p3 - p) = \begin{bmatrix} 8 \\ 8 \\ -8 \end{bmatrix}$$

We then compute the dot product between the sub triangles and t
dot(t1, t) = 192
dot(t2, t) = 192
dot(t3, t) = 384
Because all these dot products are positive, we can confirm that the point is indeed inside the triangle.

Ex 3

To demonstrate that the centroid of a triangle divides its medians in a 2:1 ratio using barycentric coordinates, let's rephrase the explanation:

Let's begin by visualizing a triangle with vertices A, B, and C, and we'll employ a barycentric coordinate system to analyze it. In this system, any point in the plane can be represented as (α, β, γ) with the constraint that $\alpha + \beta + \gamma$ equals 1.

Now, let's focus on the three medians of the triangle, denoted as AM_a, BM_b, and CM_c, where M denotes the midpoint of the respective side. The barycentric coordinates of these medians can be expressed as follows:
- For AM_a, we have (α, β, γ) with the condition that $\beta = \gamma$.
- Similarly, for BM_b, the barycentric coordinates are (α, β, γ) with $\alpha = \gamma$.
- And for CM_c, they are (α, β, γ) with $\alpha = \beta$.

Now, the centroid of the triangle, often represented as G, is the point where these medians intersect. We can summarize the conditions for G as follows:
- $\alpha + \beta + \gamma = 1$ (This is the general constraint for barycentric coordinates).
- $\beta = \gamma$ (from the AM_a median).
- $\alpha = \gamma$ (from the BM_b median).
- $\alpha = \beta$ (from the CM_c median).

Solving this system of equations, we find that $\alpha = \beta = \gamma = 1/3$.

In other words, the barycentric coordinates for the centroid G are $(1/3, 1/3, 1/3)$. This means that the centroid divides each of the medians in a 1:2 ratio (1 part to the centroid, 2 parts to the vertex). This can also be expressed as a 2:1 ratio (2 parts to the vertex, 1 part to the centroid), and it holds for any triangle, illustrating the desired result.

Ex 4

Consider a transformation that projects all the points $p_i \in \mathbb{R}^2$ onto the line $x = 1$. For each point, the projection is performed along the line that passes through the center of the coordinate system and the point. Each point p'_i is a result of applying this transformation to point p_i. Construct a matrix $M \in \mathbb{R}^{3\times 3}$ which realizes this transformation using homogeneous coordinates. More specifically, after converting a point p_i into the homogeneous coordinates, multiplying it with the matrix M, and transforming it back to the Cartesian coordinates, you should obtain point p'_i. Note that this is not an affine transformation. Comment on how the matrix M transforms the points lying on the y-axis. Interpret the results both in the homogeneous and Cartesian coordinates.

Projection of points onto line $x = 1$ is a line passing always through the origin $O = (0, 0)$ of the Cartesian coordinate.
Such line from point to origin can be represented as $l : y = ax+b$ where a is the slope of the line and $b = 0$ is the y-intercept which passes through the origin O.

$$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{p_y - O_y}{p_x - O_x} = \frac{p_y}{p_x}$$

Hence, line $l : y = \frac{p_y}{p_x}x$ is the line passing through the origin O and any point p.
Since points need to be projected on $x = 1$, $l : y = \frac{p_y}{p_x}$.
Thus, $p' = (1, \frac{p_y}{p_x})$ is the projection of point p on line $x = 1$.

We can see that, when dealing with a given homogeneous coordinate $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, the corresponding Cartesian coordinate f can be determined by dividing each component by the value of the third dimension and omitting that dimension. This operation yields $f = \begin{bmatrix} a/c \\ b/c \end{bmatrix}$.

We can see that for any point p, its transformed version can be represented as $p'' = \begin{bmatrix} 1 \\ y/x \end{bmatrix}$. This point transformation can be shown using the following matrix multiplication:

$$p'' = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} 1 \\ p_y/p_x \\ 1 \end{bmatrix}$$

To achieve $p''_x = 1$, we need to simplify p_x by dividing it by itself. Converting homogeneous coordinates to Cartesian requires dividing all elements by the third element, we can conveniently set p_z equal to p_x to facilitate this simplification:

$$p'' = \begin{bmatrix} 1 & b & c \\ d & e & f \\ 1 & h & i \end{bmatrix} \cdot \begin{bmatrix} p_x \\ p_y \\ p_x \end{bmatrix} = \begin{bmatrix} p_x/p_x \\ 1/p_x \\ p_x/p_x \end{bmatrix} = \begin{bmatrix} 1 \\ 1/p_x \\ 1 \end{bmatrix}$$

To preserve the value of y, we set e = 1, as shown below:

$$p'' = \begin{bmatrix} 1 & b & c \\ d & 1 & f \\ 1 & h & i \end{bmatrix} \cdot \begin{bmatrix} p_x \\ p_y \\ p_x \end{bmatrix} = \begin{bmatrix} 1 \\ p_y/p_x \\ 1 \end{bmatrix}$$

The transformation matrix can be represented as $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ for all $p_x \in \mathbb{R}$, $p_x \neq 0$.

When a point p lies on the y-axis (i.e., $p_x = 0$), there is no valid solution, as p''_y would involve division by zero.

Bonus exercise [2 points]

Consider the same task as in Exercise 4, but the line onto which the points are projected can be now arbitrary, and it is defined by a line equation $y = ax + b$, where $a, b \in \mathbb{R}$ are constants. Derive the matrix M for this more general case.

Represent the line equation as $l_1 : y = ax + b$ and put it in system with the previous exercise 4 answer.

$$\begin{cases} l_1 : y = \frac{p_y}{p_x} x \\ y = ax + b \end{cases} \rightarrow \begin{cases} l_1 : y = \frac{p_y}{p_x} x \\ \frac{p_y}{p_x} x = ax + b \end{cases} \rightarrow \begin{cases} l_1 : y = \frac{p_y}{p_x} x \\ x(\frac{p_y - ap_x}{p_x}) = b \end{cases} \rightarrow \begin{cases} x = \frac{bp_x}{p_y - ap_x} \\ y = \frac{bp_y}{p_y - ap_x} \end{cases}$$

Then, the matrix M is given by:

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

where

$$p'' = \begin{cases} p''_x = \frac{M_{11}p_x + M_{12}p_y + M_{13}p_x}{M_{31}p_x + M_{32}p_y + M_{33}p_x} = \frac{bp_x}{p_y - ap_x} \\ \\ p''_y = \frac{M_{21}p_x + M_{22}p_y + M_{23}p_x}{M_{31}p_x + M_{32}p_y + M_{33}p_x} = \frac{bp_y}{p_y - ap_x} \end{cases}$$

so the only defined values in M are:

$$M_{11} = b, M_{22} = b, M_{31} = -a, M_{32} = 1$$

Finally:

$$M = \begin{bmatrix} b & 0 & 0 \\ 0 & b & 0 \\ -a & 1 & 0 \end{bmatrix}$$