Assignment 1

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Exercise 1

$\underline{x} = [\sqrt{2}, 1, 0]^T \quad \underline{y} = [1\ 1\ 1]^T \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ -1 & -3 & -3 \end{bmatrix}$

$\|\underline{x}\| = \sqrt{\langle \underline{x}, \underline{x} \rangle} = \sqrt{\begin{bmatrix} 2^{\frac{1}{2}} \\ 1 \\ 0 \end{bmatrix} \cdot [2^{\frac{1}{2}}\ 1\ 0]^T} = \sqrt{2+1+0} = \sqrt{3} \qquad \|\underline{y}\| = \sqrt{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot [1\ 1\ 1]^T} = \sqrt{3}$

$\langle x, y \rangle = \underline{x} \cdot \underline{y}^T = \begin{bmatrix} 2^{1/2} \\ 1 \\ 0 \end{bmatrix} [1\ 1\ 1]^T = 2^{1/2} + 1 = 1 + \sqrt{2}$

$\cos(\angle(x,y)) = \frac{\langle \underline{x}, \underline{y} \rangle}{\|\underline{x}\| \cdot \|\underline{y}\|} = \frac{1+\sqrt{2}}{\sqrt{3} \cdot \sqrt{3}} = \frac{1+\sqrt{2}}{3}$

$\boxed{\text{Task 1 } \checkmark}$

$\underline{z} = \underline{x} \times \underline{y} \ \wedge \ \underline{z} \perp \underline{x} \ \wedge \ \underline{z} \perp \underline{y} \ \wedge \ \|\underline{z}\| = 1 = \hat{\underline{z}}$

$z_1 = \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$

$z_2 = -\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} = -\begin{vmatrix} \sqrt{2} & 1 \\ 0 & 1 \end{vmatrix} = -\sqrt{2}$

$z_3 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \begin{vmatrix} \sqrt{2} & 1 \\ 1 & 1 \end{vmatrix} = \sqrt{2} - 1$

$\underline{z} = [1\ \ -\sqrt{2}\ \ \sqrt{2}-1]^T$

$\|\underline{z}\| = \sqrt{\langle \underline{z}, \underline{z} \rangle} = \begin{bmatrix} 1 \\ -\sqrt{2} \\ \sqrt{2}-1 \end{bmatrix} \cdot [1\ \ -\sqrt{2}\ \ \sqrt{2}-1]^T =$

$= \sqrt{1+2+2+1-2\sqrt{2}} = \sqrt{6-2\sqrt{2}}$

normalized $\hat{\underline{z}} = \frac{\underline{z}}{\|\underline{z}\|} = [1\ \ -\sqrt{2}\ \ \sqrt{2}-1]^T / \sqrt{6-2\sqrt{2}}$

$\boxed{\text{Task 2 } \checkmark}$

$\underline{u} = A\,\underline{z}$

$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ -1 & -3 & -3 \end{bmatrix}}_{\mathbb{R}^{3,3}} \cdot \underbrace{\begin{bmatrix} 1 \\ -\sqrt{2} \\ \sqrt{2}-1 \end{bmatrix}}_{\mathbb{R}^{3,1}} / \sqrt{6-2\sqrt{2}} \longrightarrow \begin{bmatrix} (1-\sqrt{2}+\sqrt{2}-1) \\ (2-2\sqrt{2}+\sqrt{2}-1) \\ (-1+3\sqrt{2}-3\sqrt{2}+3) \end{bmatrix} / \sqrt{6-2\sqrt{2}} \longrightarrow \frac{1}{\sqrt{6-2\sqrt{2}}} \cdot \begin{bmatrix} 0 \\ 1-\sqrt{2} \\ 2 \end{bmatrix} = \underline{u}$

$\boxed{\text{Task 3 } \checkmark}$

Exercise 2

Derive solution by using directly ray equation and implicit definition of sphere

ray $\gamma(t) = t\underline{d}$ $\quad \underline{d} \to$ direction

sphere $F(x, y, z) = (x - c_x)^2 + (y - c_y)^2 + (z - c_z)^2 - r^2 = 0$ $\quad r \to$ radius

$\underline{c} = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} \to$ center

Proof:

$$F(x, y, z) = (x - c_x)^2 + (y - c_y)^2 + (z - c_z)^2 - r^2 = 0$$

$$\Rightarrow \begin{bmatrix} x - c_x \\ y - c_y \\ z - c_z \end{bmatrix} \cdot \begin{bmatrix} x - c_x & y - c_y & z - c_z \end{bmatrix} - r^2 = 0$$

Let $\underline{p} = [x \; y \; z]^T$, then $\underline{p} - \underline{c} = [x - c_x \quad y - c_y \quad z - c_z]^T$

Hence $\Rightarrow (\underline{p} - \underline{c}) \cdot (\underline{p} - \underline{c})^T - r^2 = 0$

$$\Rightarrow \langle \underline{p} - \underline{c}, \underline{p} - \underline{c} \rangle - r^2 = 0$$

Since $\|\underline{v}\| = \sqrt{\langle \underline{v}, \underline{v} \rangle}$, then $\|\underline{v}\|^2 = \left(\sqrt{\langle \underline{v}, \underline{v} \rangle}\right)^2 = \langle \underline{v}, \underline{v} \rangle$

Finally $F(\underline{p}) \Rightarrow \|\underline{p} - \underline{c}\|^2 - r^2 = 0$ ■ $\boxed{\text{Task 2 } \checkmark}$

Show that to find intersections of the ray with sphere, you need t s.t.

$$\|t\underline{d} - \underline{c}\|^2 - r^2 = 0$$

$F(\underline{p}) = \|\underline{p} - \underline{c}\|^2 - r^2 = 0$ $\qquad p = [x \; y \; z]^T$

$\underline{p}$ ray $\to$ $\underline{p} = \gamma(t) = t\underline{d} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\qquad \underline{d} \to$ direction, $t \to$ scalar

Hence $F(\underline{p}) = F(\gamma(t)) = \|t\underline{d} - \underline{c}\|^2 - r^2 = 0$ ■ $\boxed{\text{Task 3 } \checkmark}$

Use properties of dot product to transform equation

$$\|t\underline{d}-\underline{c}\|^2 - r^2 = 0 \quad \text{to} \quad t^2 - 2\langle \underline{d}, \underline{c}\rangle t + \|\underline{c}\|^2 - r^2 = 0$$

$$\|t\underline{d} - \underline{c}\|^2 - r^2 = \langle t\underline{d} - \underline{c}, t\underline{d} - \underline{c}\rangle - r^2 = (t\underline{d} - \underline{c}) \cdot (t\underline{d} - \underline{c})^T - r^2 =$$

$$= t^2 \underline{d} \cdot \underline{d}^T - 2t\underline{d} \cdot \underline{c}^T + \underline{c} \cdot \underline{c}^T - r^2 = t^2 \|\underline{d}\|^2 - 2t\langle \underline{d}, \underline{c}\rangle + \|\underline{c}\|^2 - r^2$$

Since d is unit vector, $\hat{d} = d \wedge \|\underline{d}\|^2 = \left(\sqrt{\langle \underline{d}, \underline{d}\rangle}\right)^2 = \langle \underline{d}, \underline{d}\rangle = 1$

then $t^2 \|\underline{d}\|^2 - 2t\langle \underline{d}, \underline{c}\rangle + \|\underline{c}\|^2 - r^2 =$

$$= t^2 - 2\langle \underline{d}, \underline{c}\rangle t + \|\underline{c}\|^2 - r^2 \quad \blacksquare$$

Task 4

Solve equation to find t, write formulas for intersection points and verify results

$$ax^2 + bx + c = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t^2 - 2\langle \underline{d}, \underline{c}\rangle t + \|\underline{c}\|^2 - r^2 = 0$$

let $a = \langle \underline{d}, \underline{c}\rangle = \langle \underline{c}, \underline{d}\rangle$

$$t^2 - 2at + \|\underline{c}\|^2 - r^2 = 0$$

$$t_{1,2} = \frac{2a \pm \sqrt{4a^2 - 4\|\underline{c}\|^2 + 4r^2}}{2} = \frac{2a \pm \sqrt{4(a^2 - \|\underline{c}\|^2 + r^2)}}{2} = a \pm \sqrt{a^2 - \|\underline{c}\|^2 + r^2} =$$

$$= a \pm \sqrt{r^2 + a^2 - \|\underline{c}\|^2}$$

let $b = \sqrt{r^2 - D^2} = \sqrt{r^2 + a^2 - \|\underline{c}\|^2}$

$$D = \sqrt{\|\underline{c}\|^2 - a^2}$$

$$D^2 = \|\underline{c}\|^2 - a^2$$

if $D < r$ $\to$ 2 solutions : ray secant to sphere
else if $D = r$ $\to$ 1 unique solutions : ray tangent to sphere
else if $D > r$ $\to$ no solutions : ray external to sphere

then $t_{1,2} = a \pm b$

Intersection points using t_1, t_2 $\begin{cases} P_1 = \underline{O} + t_1 \cdot \underline{d} \\ P_2 = \underline{O} + t_2 \cdot \underline{d} \end{cases}$

Task 5 ✓