

# CG assignment 4

Jeferson Morales Mariciano, Martin Lettry

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## Ex 1 [5 points]

Let us consider a 2D coordinate system with points  $p_1 = [1, 1]^T$ , and  $p = [1.5, 2.5]^T$ . Additionally, let's define a vector  $u = p - p_1$ . Perform the following tasks:

### Task 1 [1 points]

Construct two matrices,  $R_{90}, T \in \mathbb{R}^{3 \times 3}$ , such that the first one performs counter-clockwise rotation around the center of the coordinates system by angle  $90^\circ$ , and the second one performs translation by vector  $t = [1, -2]^T$ , in homogeneous coordinates.

$$p = \begin{bmatrix} 1.5 \\ 2.5 \end{bmatrix}, p_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$u = p - p_1 = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}$$

From generic counter clockwise rotation in 2D coord system:

$$R(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

Then, rotation matrix with homogeneous coordinates for  $\alpha = 90^\circ$  is:

$$R(90) = \begin{bmatrix} \cos(90) & -\sin(90) & 0 \\ \sin(90) & \cos(90) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Given rotation matrix  $R$  and translation vector  $t$  for 2D coordinates system, the affine transformation matrix  $T$  is:

$$T = \begin{bmatrix} R & t \\ 0 & \dots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$
$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

## Task 2 [1 points]

Represent the points  $p_1$  and  $p$  as well as the vector  $u$  using homogeneous coordinates and transform them first using the  $R(90)$  and then  $T$  matrices. Convert the obtained points and vector back into Cartesian coordinates and denote them by  $p'_1$ ,  $p'$ , and  $u'$ . Draw all the points and vectors before and after the transformation, and verify that  $u' = p' - p'_1$ . What influence did the matrix  $T$  have on the  $u'$ ?

Using homogeneous coordinates for position points:

$$p_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, p = \begin{bmatrix} 1.5 \\ 2.5 \\ 1 \end{bmatrix}$$

$u$  is displacement vector, so 0 in homogeneous coordinates last element:

$$u = p - p_1 = \begin{bmatrix} 0.5 \\ 1.5 \\ 0 \end{bmatrix}$$

Using homogeneous coordinates allow chaining sequence of transformations to be represented with a single matrix being the product of the transformation matrices. In this case, the transformation matrix is:

$$\begin{aligned} M &= T \cdot R(90) \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3 \times 3} \end{aligned}$$

Calculating with matrix  $M$ :

$$p' = M \cdot p = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1.5 \\ 2.5 \\ 1 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -0.5 \\ 1 \end{bmatrix} \Rightarrow p' = \begin{bmatrix} -1.5 \\ -0.5 \end{bmatrix}$$

$$p'_1 = M \cdot p_1 = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \Rightarrow p'_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$u' = M \cdot u = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.5 \\ 1.5 \\ 0 \end{bmatrix} = \begin{bmatrix} -1.5 \\ 0.5 \\ 0 \end{bmatrix} \Rightarrow u' = \begin{bmatrix} -1.5 \\ 0.5 \end{bmatrix} = p' - p'_1$$

We can see that  $T$  had no effect on  $u'$ , this is because  $u'$  is a displacement vector and so is not affected by translation.

**Task 3 [1 points]**

$$S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S.p = S * p = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -1.5 \\ -0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$$

$$p'' = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$$

$$S.p1 = S * p1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$p1'' = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$S.u = S * u = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -1.5 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

$$u'' = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

**Task 4 [2 points]**

$$S^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{90}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M = R_{90}^{-1} * T^{-1} * S^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 2 \\ -1/2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M * p'' = \begin{bmatrix} 0 & 1/2 & 2 \\ -1/2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 2.5 \\ 1 \end{bmatrix} == p$$

$$M * p1'' = \begin{bmatrix} 0 & 1/2 & 2 \\ -1/2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} == p1$$

$$M * u'' = \begin{bmatrix} 0 & 1/2 & 2 \\ -1/2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \\ 0 \end{bmatrix} == u$$

## Ex 2

$$p1 = \begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix}$$

$$p2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$p3 = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

$$p = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$$area\_of\_triangle = t = (p3 - p1) * (p2 - p1) = \begin{bmatrix} 16 \\ 16 \\ -16 \end{bmatrix}$$

$$normal\_of\_sub\_triangle\_1 = t1 = (p3 - p) * (p1 - p) = \begin{bmatrix} 4 \\ 4 \\ -4 \end{bmatrix}$$

$$normal\_of\_sub\_triangle\_2 = t2 = (p1 - p) * (p2 - p) = \begin{bmatrix} 4 \\ 4 \\ -4 \end{bmatrix}$$

$$normal\_of\_sub\_triangle\_3 = t3 = (p2 - p) * (p3 - p) = \begin{bmatrix} 8 \\ 8 \\ -8 \end{bmatrix}$$

We then compute the dot product between the sub triangles and t

$$\text{dot}(t1, t) = 192$$

$$\text{dot}(t2, t) = 192$$

$$\text{dot}(t3, t) = 384$$

Because all these dot products are positive, we can confirm that the point is indeed inside the triangle.

### Ex 3

To demonstrate that the centroid of a triangle divides its medians in a 2:1 ratio using barycentric coordinates, let's rephrase the explanation:

Let's begin by visualizing a triangle with vertices  $A$ ,  $B$ , and  $C$ , and we'll employ a barycentric coordinate system to analyze it. In this system, any point in the plane can be represented as  $(\alpha, \beta, \gamma)$  with the constraint that  $\alpha + \beta + \gamma$  equals 1.

Now, let's focus on the three medians of the triangle, denoted as  $AM_a$ ,  $BM_b$ , and  $CM_c$ , where  $M$  denotes the midpoint of the respective side. The barycentric coordinates of these medians can be expressed as follows:

- For  $AM_a$ , we have  $(\alpha, \beta, \gamma)$  with the condition that  $\beta = \gamma$ .
- Similarly, for  $BM_b$ , the barycentric coordinates are  $(\alpha, \beta, \gamma)$  with  $\alpha = \gamma$ .
- And for  $CM_c$ , they are  $(\alpha, \beta, \gamma)$  with  $\alpha = \beta$ .

Now, the centroid of the triangle, often represented as  $G$ , is the point where these medians intersect. We can summarize the conditions for  $G$  as follows:

- $\alpha + \beta + \gamma = 1$  (This is the general constraint for barycentric coordinates).
- $\beta = \gamma$  (from the  $AM_a$  median).
- $\alpha = \gamma$  (from the  $BM_b$  median).
- $\alpha = \beta$  (from the  $CM_c$  median).

Solving this system of equations, we find that  $\alpha = \beta = \gamma = 1/3$ .

In other words, the barycentric coordinates for the centroid  $G$  are  $(1/3, 1/3, 1/3)$ . This means that the centroid divides each of the medians in a 1:2 ratio (1 part to the centroid, 2 parts to the vertex). This can also be expressed as a 2:1 ratio (2 parts to the vertex, 1 part to the centroid), and it holds for any triangle, illustrating the desired result.

## Ex 4

We can see that, when dealing with a given homogeneous coordinate  $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ , the corresponding Cartesian coordinate  $f$  can be determined by dividing each component by the value of the third dimension and omitting that dimension. This operation yields  $f = \begin{bmatrix} a/c \\ b/c \end{bmatrix}$ .

We can see that for any point  $p$ , its transformed version can be represented as  $p'' = \begin{bmatrix} 1 \\ y/x \end{bmatrix}$ . This point transformation can be shown using the following matrix multiplication:

$$p'' = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} 1 \\ p_y/p_x \\ 1 \end{bmatrix}$$

To achieve  $p''_x = 1$ , we need to simplify  $p_x$  by dividing it by itself. Converting homogeneous coordinates to Cartesian requires dividing all elements by the third element, we can conveniently set  $p_z$  equal to  $p_x$  to facilitate this simplification:

$$p'' = \begin{bmatrix} 1 & b & c \\ d & e & f \\ 1 & h & i \end{bmatrix} \cdot \begin{bmatrix} p_x \\ p_y \\ p_x \end{bmatrix} = \begin{bmatrix} p_x/p_x \\ 1/p_x \\ p_x/p_x \end{bmatrix} = \begin{bmatrix} 1 \\ 1/p_x \\ 1 \end{bmatrix}$$

To preserve the value of  $y$ , we set  $e = 1$ , as shown below:

$$p'' = \begin{bmatrix} 1 & b & c \\ d & 1 & f \\ 1 & h & i \end{bmatrix} \cdot \begin{bmatrix} p_x \\ p_y \\ p_x \end{bmatrix} = \begin{bmatrix} 1 \\ p_y/p_x \\ 1 \end{bmatrix}$$

The transformation matrix can be represented as  $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$  for all

$p_x \in \mathbb{R}, p_x \neq 0$ .

When a point  $p$  lies on the y-axis (i.e.,  $p_x = 0$ ), there is no valid solution, as  $p''_y$  would involve division by zero.

## Bonus exercise [2 points]

Consider the same task as in Exercise 4, but the line onto which the points are projected can be now arbitrary, and it is defined by a line equation  $y = ax + b$ , where  $a, b \in \mathbb{R}$  are constants. Derive the matrix  $M$  for this more general case.

Represent the line equation as  $l_1 : y = ax + b$  and put it in system with the previous exercise 4 answer.

$$\begin{cases} l_1 : y = \frac{p_y}{p_x}x \\ y = ax + b \end{cases} \rightarrow \begin{cases} l_1 : y = \frac{p_y}{p_x}x \\ \frac{p_y}{p_x}x = ax + b \end{cases} \rightarrow \begin{cases} l_1 : y = \frac{p_y}{p_x}x \\ x(\frac{p_y - ap_x}{p_x}) = b \end{cases} \rightarrow \begin{cases} x = \frac{bp_x}{p_y - ap_x} \\ y = \frac{bp_y}{p_y - ap_x} \end{cases}$$

Then, the matrix  $M$  is given by:

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

where

$$p'' = \begin{cases} p''_x = \frac{M_{11}p_x + M_{12}p_y + M_{13}p_x}{M_{31}p_x + M_{32}p_y + M_{33}p_x} = \frac{bp_x}{p_y - ap_x} \\ p''_y = \frac{M_{21}p_x + M_{22}p_y + M_{23}p_x}{M_{31}p_x + M_{32}p_y + M_{33}p_x} = \frac{bp_y}{p_y - ap_x} \end{cases}$$

so the only defined values in  $M$  are:

$$M_{11} = b, M_{22} = b, M_{31} = -a, M_{32} = 1$$

Finally:

$$M = \begin{bmatrix} b & 0 & 0 \\ 0 & b & 0 \\ -a & 1 & 0 \end{bmatrix}$$