# Quantitative Economics Workshop Paris

## Prelude to Dynamic Programming

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# Introduction

Summary of this lecture:

- Linear equations
- Fixed point theory
- Algorithms
- Job search

Operations on real numbers such as $|\cdot|$ and $\vee$ are applied to vectors element-by-element

Example.

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \Longrightarrow \quad |a| = \begin{pmatrix} |a_1| \\ \vdots \\ |a_n| \end{pmatrix}$$

$$a \vee b = \begin{pmatrix} a_1 \vee b_1 \\ \vdots \\ a_n \vee b_n \end{pmatrix} \quad \text{and} \quad a \wedge b = \begin{pmatrix} a_1 \wedge b_1 \\ \vdots \\ a_n \wedge b_n \end{pmatrix}$$

etc.

# Linear Equations

Given one-dimensional equation $x = ax + b$, we have

$$|a| < 1 \quad \Longrightarrow \quad x^* = \frac{b}{1-a} = \sum_{k \geqslant 0} a^k b$$

How can we extend this beyond one dimension?

- When does $x = Ax + b$ have a unique solution?
- How can we compute it?

Recall that $\lambda \in \mathbb{C}$ is an eigenvalue of $n \times n$ matrix $A$ if

$$Ae = \lambda e$$

for some nonzero $e \in \mathbb{C}^n$.

We define the **spectral radius** of square matrix $A$ as

$$r(A) := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$$

**Key idea**:

- $r(A) < 1$ is a generalization of $|a| < 1$

# Neumann Series Lemma

Suppose

- $b$ is $n \times 1$ and $A$ is $n \times n$
- $I$ is the $n \times n$ identity matrix

**Theorem.** If $r(A) < 1$, then

1. $I - A$ is nonsingular
2. $\sum_{k \geqslant 0} A^k$ converges
3. $(I - A)^{-1} = \sum_{k \geqslant 0} A^k$ and
4. $x = Ax + b$ has the unique solution

$$x^* := (I - A)^{-1} b$$

Intuitive idea: with $S := \sum_{k \geqslant 0} A^k$, we have

$$\begin{aligned} I + AS &= I + A(I + A + \cdots) \\ &= I + A + A^2 + \cdots \\ &= S \end{aligned}$$

Rearranging $I + AS = S$ gives $S = (I - A)^{-1}$

$$\therefore \quad x = Ax + b \iff (I - A)x = b \iff x^* = (I - A)^{-1} b$$

# Fixed Points

To solve nonlinear equations we use fixed point theory

Recall: if $S$ is any set then

- $T$ is a **self-map** on $S$ if $T$ maps $S$ into itself
- $x^* \in S$ is called a **fixed point** of $T$ in $S$ if $Tx^* = x^*$

Examples.

- Every $x$ in $S$ is fixed for the **identity map** $I\colon x \mapsto x$
- If $S = \mathbb{N}$ and $Tx = x + 1$, then $T$ has no fixed point

Examples.

- If $S = \mathbb{R}$ and $Tx = x^2$, then $T$ has fixed points at $0, 1$
- If $S \subset \mathbb{R}$, $Tx = x \iff T$ meets the 45 degree line

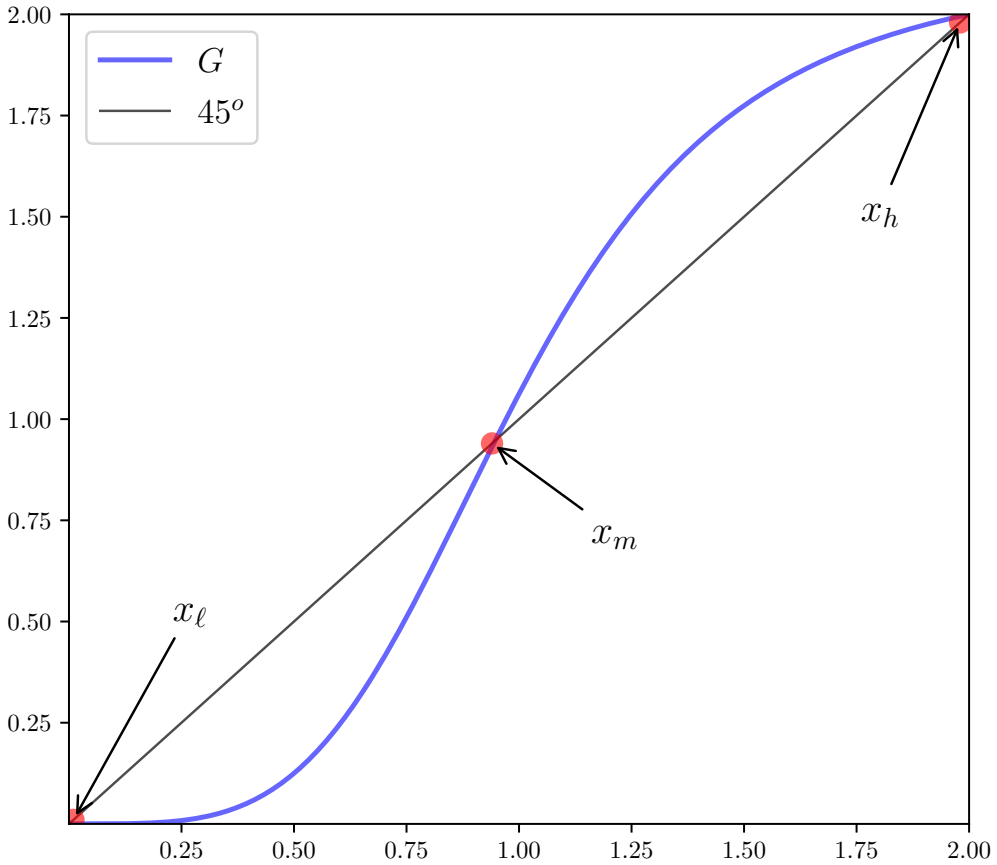



Figure: Graph and fixed points of $G\colon x \mapsto 2.125/(1+x^{-4})$

**Key idea**: Fixed point theory is for solving equations

1. solve $x = Tx \iff$ find fixed points of $T$
2. solve $Gx = 0 \iff$ find fixed points of $Tx := Gx + x$

Example. If $S = \mathbb{R}^n$ and $Tx = Ax + b$, then

$$x^* \text{ solves equation } x = Ax + b \iff x^* \text{ is a fixed point of } T$$

(But fixed point theory is mainly for nonlinear equations)

Point on notation:

- $T^2 = T \circ T$
- $T^3 = T \circ T \circ T$
- etc.

Example. $Tx = Ax + b$ implies $T^2x = A(Ax + b) + b$

Self-map $T$ is called **globally stable** on $S$ if

1. $T$ has a unique fixed point $x^*$ in $S$ and
2. $T^k x \to x^*$ as $k \to \infty$ for all $x \in S$

Example. Let $S = \mathbb{R}^n$ and $Tx = Ax + b$

**Ex.** Prove: $r(A) < 1$ implies $T$ is globally stable on $S$

Example. Consider Solow–Swan growth dynamics

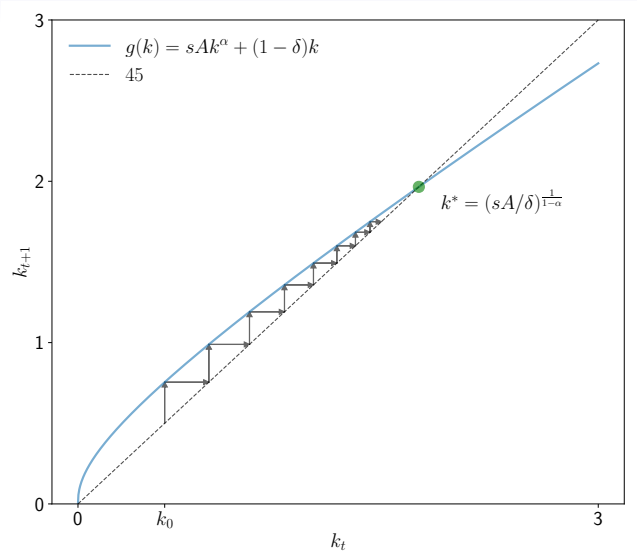
$$k_{t+1} = g(k_t) := sAk_t^\alpha + (1-\delta)k_t, \qquad t = 0, 1, \ldots,$$

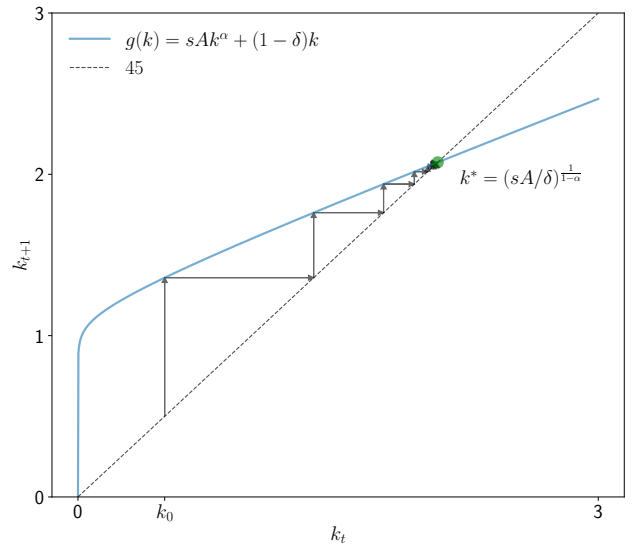
where

- $k_t$ is capital stock per worker,
- $A, \alpha > 0$ are production parameters, $\alpha < 1$
- $s > 0$ is a savings rate, and
- $\delta \in (0, 1)$ is a rate of depreciation

Iterating with $g$ from $k_0$ generates a time path for capital stock

The map $g$ is globally stable on $(0, \infty)$


$g(k) = sAk^{\alpha} + (1-\delta)k$
45
$k^* = (sA/\delta)^{\frac{1}{1-\alpha}}$
$k_{t+1}$
$k_t$
0
$k_0$
3
1
2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$k_t$
0
$k_0$
3
1
2

Note from last slide

- If $g$ is flat near $k^*$, then $g(k) \approx k^*$ for $k$ near $k^*$
- A flat function near the fixed point $\implies$ fast convergence

Conversely

- If $g$ is close to the 45 degree line near $k^*$, then $g(k) \approx k$
- Close to 45 degree line means high persistence, slow convergence

Given a self-map $T$ on $S$, we typically ask

- Does $T$ have at least one fixed point on $S$ (existence)?
- Does $T$ have at most one fixed point on $S$ (uniqueness)?
- How can we compute fixed points of $T$?

For the last question, we seek an algorithm

Then we investigate its properties

# Contractions

Let

- $U$ be a nonempty subset of $\mathbb{R}^n$,
- $\|\cdot\|$ be a norm on $\mathbb{R}^n$, and
- $T$ be a self-map on $U$

$T$ is called a **contraction** on $U$ with respect to $\|\cdot\|$ if

$$\exists\, \lambda < 1 \text{ such that } \|Tu - Tv\| \leqslant \lambda \|u - v\| \quad \text{for all} \quad u, v \in U$$

Example. $Tx = ax + b$ is a contraction on $\mathbb{R}$ with respect to $|\cdot|$ if and only if $|a| < 1$

Indeed,

$$|Tx - Ty| = |ax + b - ay - b| = |a||x - y|$$

# Banach's Contraction Mapping Theorem

**Theorem** If

1. $U$ is closed in $\mathbb{R}^n$ and
2. $T$ is a contraction of modulus $\lambda$ on $U$ with respect to some norm $\|\cdot\|$ on $\mathbb{R}^n$,

then $T$ has a unique fixed point $u^*$ in $U$ and

$$\|T^n u - u^*\| \leqslant \lambda^n \|u - u^*\| \quad \text{for all } n \in \mathbb{N} \text{ and } u \in U$$

In particular, $T$ is globally stable on $U$

Proof: See the DP text

# Successive approximation

fix a guess $x_0$ and some error tolerance $\tau$

$\varepsilon \leftarrow \tau + 1$

$x \leftarrow x_0$

**while** $\varepsilon > \tau$ **do**

$\quad y \leftarrow Tx$

$\quad \varepsilon \leftarrow \|y - x\|$

$\quad x \leftarrow y$

**end**

**return** $x$

If $T$ is a contraction, say, then the output will be close to to $x^*$

```python
import numpy as np
def successive_approx(T,                    # Operator (callable)
                      x_0,                  # Initial condition
                      tolerance=1e-6,       # Error tolerance
                      max_iter=10_000,      # Max iteration bound
                      print_step=25,        # Print at multiples
                      verbose=False):
    x = x_0
    error = tolerance + 1
    k = 1
    while error > tolerance and k <= max_iter:
        x_new = T(x)
        error = np.max(np.abs(x_new - x))
        if verbose and k % print_step == 0:
            print(f"Completed iteration {k} with error {error}.")
        x = x_new
        k += 1
    if error > tolerance:
        print(f"Warning: Iteration hit upper bound {max_iter}.")
    elif verbose:
        print(f"Terminated successfully in {k} iterations.")
    return x
```

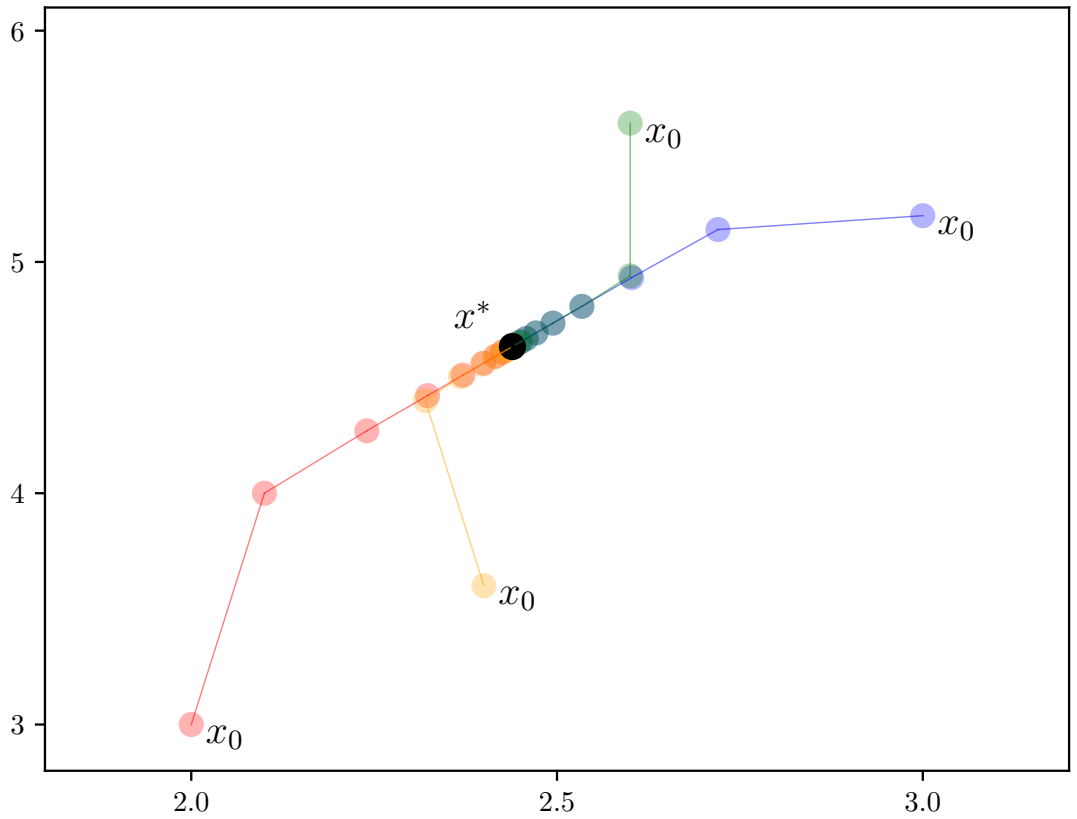


Figure: Successive approximation from different initial conditions

# Newton's Method

Let $g$ be a smooth self-map on $S := (a, b)$

We start with guess $x_0$ of the fixed point and update it using

$$g_a(x) \approx g(x_0) + g'(x_0)(x - x_0)$$

Set $g_a(x_1) = x_1$ and solve for $x_1$ to get

$$x_1 = \frac{x_0 - g'(x_0)x_0}{1 - g'(x_0)}$$

Now repeat logic to get $x_2, x_3, \ldots$

This gives **Newton's method**

We are iterating on the map

$$x_{k+1} = q(x_k) \quad \text{where} \quad q(x) := \frac{x - g'(x)x}{1 - g'(x)}$$

Note that we are applying successive approximation to $q$

Hence we can use the same code

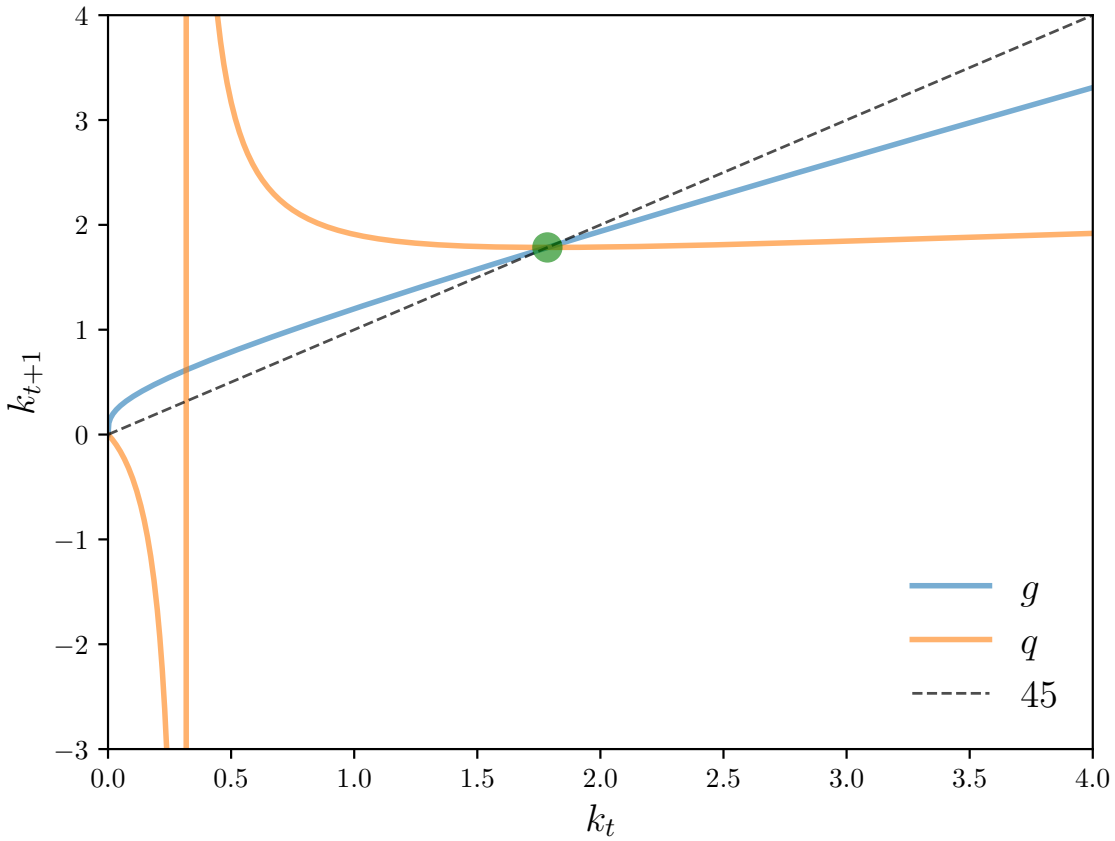


Figure: Successive approximation vs Newton's method

Comments:

- The map $q$ is flat close to the fixed point $k^*$
- Hence Newton's method converges quickly near $k^*$
- But Newton's method is not globally convergent
- Successive approximation is slower but more robust

## Key ideas

- There is almost always a trade-off between robustness and speed
- Speed requires assumptions, and assumptions can fail

**Newton's method** extends naturally to **multiple dimensions**

When $h$ is a map from $S \subset \mathbb{R}^n$ to itself, we use

$$x_{k+1} = x_k - [J(x_k)]^{-1} h(x_k)$$

Here $J_h(x_k) :=$ the Jacobian of $h$ evaluated at $x_k$

Comments

- Typically faster but less robust
- Matrix operations can be parallelized
- Automatic differentiation can be helpful

# Job Search

A model of job search created by **John J. McCall**

We model the decision problem of an unemployed worker

Job search depends on

- current and likely future wage offers
- impatience, and
- the availability of unemployment compensation

We begin with a very simple version of the McCall model

(Later we consider extensions)

# Set Up

An agent begins working life at time $t = 0$ without employment

Receives a new job offer paying wage $w_t$ at each date $t$

She has two choices:

1. **accept** the offer and work permanently at $w_t$ or
2. **reject** the offer, receive unemployment compensation $c$, and reconsider next period

Assume $\{w_t\}$ is $\overset{\text{IID}}{\sim}$ $\varphi$, where

- $\mathsf{W} \subset \mathbb{R}_+$ is a finite set of wage outcomes and
- $\varphi \in \mathcal{D}(\mathsf{W})$

The agent cares about the future but is **impatient**

Impatience is parameterized by a **time discount factor** $\beta \in (0,1)$

- Present value of a next-period payoff of $y$ dollars is $\beta y$

Trade off:

- $\beta < 1$ indicating some impatience
- hence the agent will be tempted to accept reasonable offers, rather than always waiting for a better one
- The key question is how long to wait

The worker who aims to maximize

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t Y_t, \quad Y_t \in \{c, W_t\} \text{ is earnings at time } t \tag{1}$$

- $\{W_t\} \overset{\text{IID}}{\sim} \varphi$ for $\varphi \in \mathcal{D}(\mathsf{W})$
- $\mathsf{W} \subset \mathbb{R}_+$ with $|\mathsf{W}| < \infty$
- $c$ and $\beta$ are positive and $\beta < 1$
- jobs are <u>permanent</u>

What is max EPV of each option when lifetime is infinite?

What if we **accept** $w \in \mathrm{W}$ now?

$$\text{EPV} = \text{stopping value} = w + \beta w + \beta^2 w + \cdots = \frac{w}{1-\beta}$$

What if we **reject**?

$$\text{EPV} = \text{continuation value}$$

$$= \text{EPV of } \underline{\text{optimal}} \text{ choice in each subsequent period}$$

But what are optimal choices?!

Calculating optimal choice requires knowing optimal choice!

# The Value Function

Let $v^*(w) :=$ max lifetime EPV given wage offer $w$

We call $v^*$ the **value function**

**Suppose** that we know $v^*$

Then the (maximum) **continuation value** is

$$h^* := c + \beta \sum_{w' \in \mathsf{W}} v^*(w')\varphi(w')$$

$$= \text{ max EPV conditional on decision to continue}$$

The optimal choice is then

$$\mathbb{1}\{\text{stopping value} \geqslant \text{continuation value}\} = \mathbb{1}\left\{\frac{w}{1-\beta}, \; h^*\right\}$$

But how can we calculate $v^*$?

**Key idea**: We can use the Bellman equation to solve for $v^*$

**Theorem.** The value function $v^*$ satisfies the **Bellman equation**

$$v^*(w) = \max\left\{\frac{w}{1-\beta}, c + \beta \sum_{w' \in \mathsf{W}} v^*(w')\varphi(w')\right\} \qquad (w \in \mathsf{W})$$

Intuition:

- If accept, get $w/(1-\beta)$
- If reject and then choose optimally, get max continuation value
- Max value today is max of these alternatives

So how can we use the Bellman equation

$$v^*(w) = \max\left\{\frac{w}{1-\beta}, c + \beta \sum_{w' \in \mathsf{W}} v^*(w')\varphi(w')\right\} \qquad (w \in \mathsf{W})$$

to solve for $v^*$?

Let's now return to the job search problem

Recall that that the value function $v^*$ solves the Bellman equation

That is,

$$v^*(w) = \max\left\{\frac{w}{1-\beta}, c + \beta \sum_{w' \in \mathsf{W}} v^*(w')\varphi(w')\right\} \qquad (w \in \mathsf{W})$$

The infinite-horizon **continuation value** is defined as

$$h^* := c + \beta \sum_{w'} v^*(w')\varphi(w')$$

**Key question**: how to solve for $v^*$?

We introduce the **Bellman operator**, defined at $v \in \mathbb{R}^{\mathsf{W}}$ by

$$(Tv)(w) = \max\left\{\frac{w}{1-\beta}, c + \beta \sum_{w' \in \mathsf{W}} v(w')\varphi(w')\right\} \qquad (w \in \mathsf{W})$$

By construction, $Tv = v \iff v$ solves the Bellman equation

Let $\mathcal{V} := \mathbb{R}_+^{\mathsf{W}}$

**Proposition.** $T$ is a contraction $\mathcal{V}$ with respect to $\|\cdot\|_\infty$

In the proof, we use the elementary bound

$$|\alpha \vee x - \alpha \vee y| \leqslant |x - y| \qquad (\alpha, x, y \in \mathbb{R})$$

Fixing $f, g$ in $\mathcal{V}$ fix any $w \in \mathrm{W}$, we have

$$|(Tf)(w) - (Tg)(w)| \leqslant \left| \beta \sum_{w'} f(w')\varphi(w') - \beta \sum_{w'} g(w')\varphi(w') \right|$$

$$= \beta \left| \sum_{w'} [f(w') - g(w')]\varphi(w') \right|$$

Applying the triangle inequality,

$$|(Tf)(w) - (Tg)(w)| \leqslant \beta \sum_{w'} |f(w') - g(w')|\varphi(w') \leqslant \beta \|f - g\|_\infty$$

$$\therefore \quad \|Tf - Tg\|_\infty \leqslant \beta \|f - g\|_\infty$$

Recall: The optimal decision at any given time, facing current wage draw $w \in \mathsf{W}$, is

$$\mathbb{1}\left\{\frac{w}{1-\beta} \geqslant h^*\right\}$$

Let's try to write this in the language of dynamic programming

Dynamic programming centers around the problem of finding optimal **policies**

# Optimal Policies

In general, for a dynamic program, choices consist of a sequence $(A_t)_{t \geqslant 0}$

- specifies how the agent acts at each $t$

Since agents are not clairvoyant, so we assume that $A_t$ cannot depend on future events

In other words, for some function $\sigma_t$,

$$A_t = \sigma_t(X_t, A_{t-1}, X_{t-1}, A_{t-2}, X_{t-2}, \ldots A_0, X_0)$$

In dynamic programming, $\sigma_t$ is called a **policy function**

**Key idea** Design the state such that $X_t$ is

- sufficient to determine the optimal current action
- but not so large as to be unmanagable

- Finding the state is an art!

Example. Recall retailer who chooses stock orders and prices in each period

What to include in the current state?

- level of current inventories
- interest rates and inflation?
- the rate at which inventories have changed?
- competitors prices?

So suppose state $X_t$ determines the current action $A_t$

Then we can write $A_t = \sigma(X_t)$ for some function $\sigma$

Note that we dropped the time subscript on $\sigma$

No loss of generality: can include time in the current state

- i.e., expand $X_t$ to $\hat{X}_t = (t, X_t)$

Depends on the problem at hand

- For the job search model with finite horizon, the date matters
- For the infinite horizon version of the problem, however, the agent always looks forward toward an infinite horizon

For job search model,

- state $=$ current wage offer and
- possible actions are accept (1) or reject (0)

A policy is a map $\sigma$ from W to $\{0, 1\}$

Let $\Sigma$ be the set of all such maps

For each $v \in \mathcal{V}$, let us define a **$v$-greedy policy** to be a $\sigma \in \Sigma$ satisfying

$$\sigma(w) = \mathbb{1}\left\{\frac{w}{1-\beta} \geqslant c + \beta \sum_{w' \in \mathsf{W}} v(w')\varphi(w')\right\} \quad \text{for all } w \in \mathsf{W}$$

Accepts iff $w/(1-\beta) \geqslant$ continuation value computed using $v$

Optimal choice:

- agent should adopt a $v^*$-greedy policy
- Sometimes called **Bellman's principle of optimality**

We can also express a $v^*$-greedy policy via

$$\sigma^*(w) = \mathbb{1}\{w \geqslant w^*\} \quad \text{where} \quad w^* := (1-\beta)h^* \tag{2}$$

The term $w^*$ in (2) is called the **reservation wage**

- Same ideas as before, different language
- We prove optimality more carefully later

# Computation

Since $T$ is globally stable on $\mathcal{V}$, we can compute an approximate optimal policy by

1. applying successive approximation on $T$ to compute $v^*$
2. calculate a $v^*$-greedy policy

In dynamic programming, this approach is called **value function iteration**

input $v_0 \in \mathcal{V}$, an initial guess of $v^*$
input $\tau$, a tolerance level for error
$\varepsilon \leftarrow \tau + 1$
$k \leftarrow 0$
**while** $\varepsilon > \tau$ **do**
    **for** $w \in \mathsf{W}$ **do**
        $v_{k+1}(w) \leftarrow (Tv_k)(w)$
    **end**
    $\varepsilon \leftarrow \|v_k - v_{k+1}\|_\infty$
    $k \leftarrow k + 1$
**end**
Compute a $v_k$-greedy policy $\sigma$
**return** $\sigma$

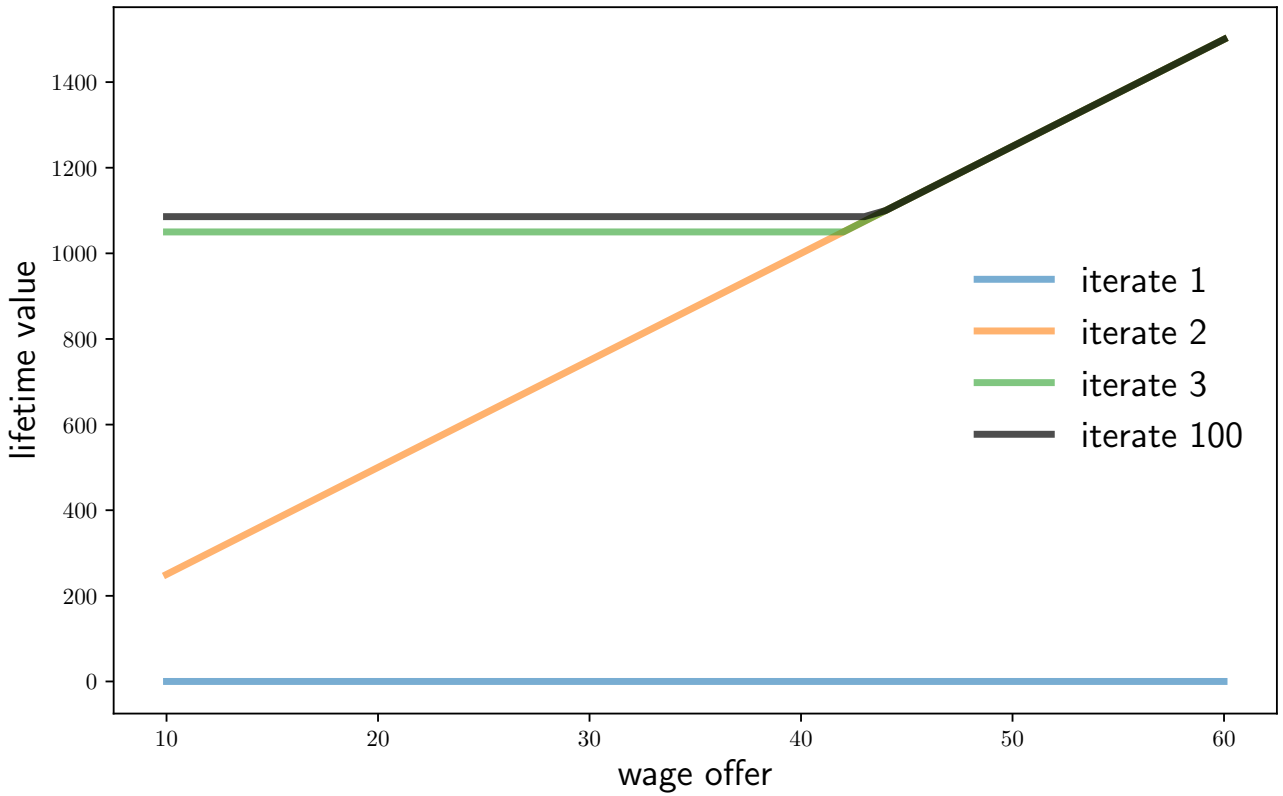


Figure: A sequence of iterates of the Bellman operator

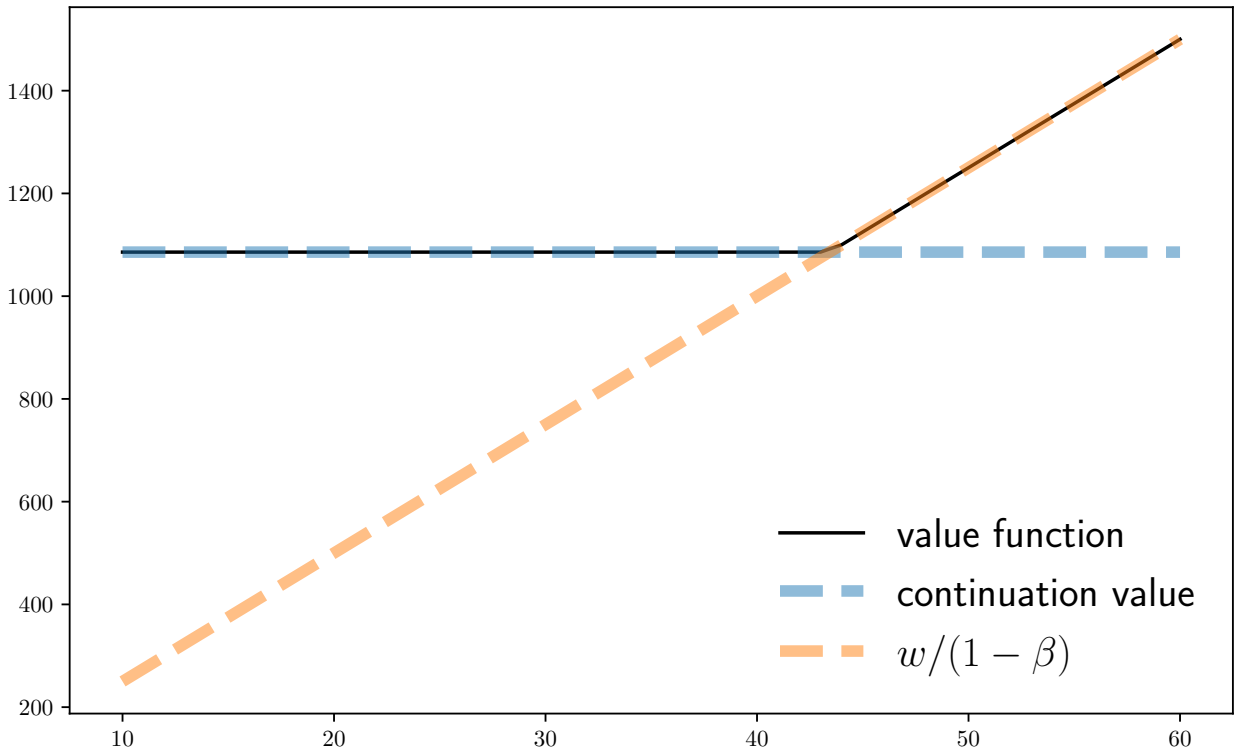


Figure: The approximate value function for job search

# Computing the Continuation Value Directly

We used a standard dynamic programming approach to solve this problem

Sometimes we can find more efficient ways to solve particular problems

For the infinite horizon job search problem, a more efficient way exists

The idea is to compute the continuation value directly

This shifts the problem from $n$-dimensional to one-dimensional

Method: Recall that

$$v^*(w) = \max\left\{\frac{w}{1-\beta}, c + \beta \sum_{w'} v^*(w')\varphi(w')\right\} \qquad (w \in \mathsf{W})$$

Using the definition of $h^*$, we can write

$$v^*(w') = \max\left\{w'/(1-\beta), h^*\right\} \qquad (w' \in \mathsf{W})$$

Take expectations, multiply by $\beta$ and add $c$ to obtain

$$h^* = c + \beta \sum_{w'} \max\left\{\frac{w'}{1-\beta}, h^*\right\} \varphi(w')$$

How to find $h^*$ from the equation

$$h^* = c + \beta \sum_{w'} \max \left\{ \frac{w'}{1-\beta}, h^* \right\} \varphi(w') \tag{3}$$

We introduce the map $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$g(h) = c + \beta \sum_{w'} \max \left\{ \frac{w'}{1-\beta}, h \right\} \varphi(w')$$

By construction, $h^*$ solves (3) if and only if $h^*$ is a fixed point of $g$

**Ex.** Show that $g$ is a contraction map on $\mathbb{R}_+$

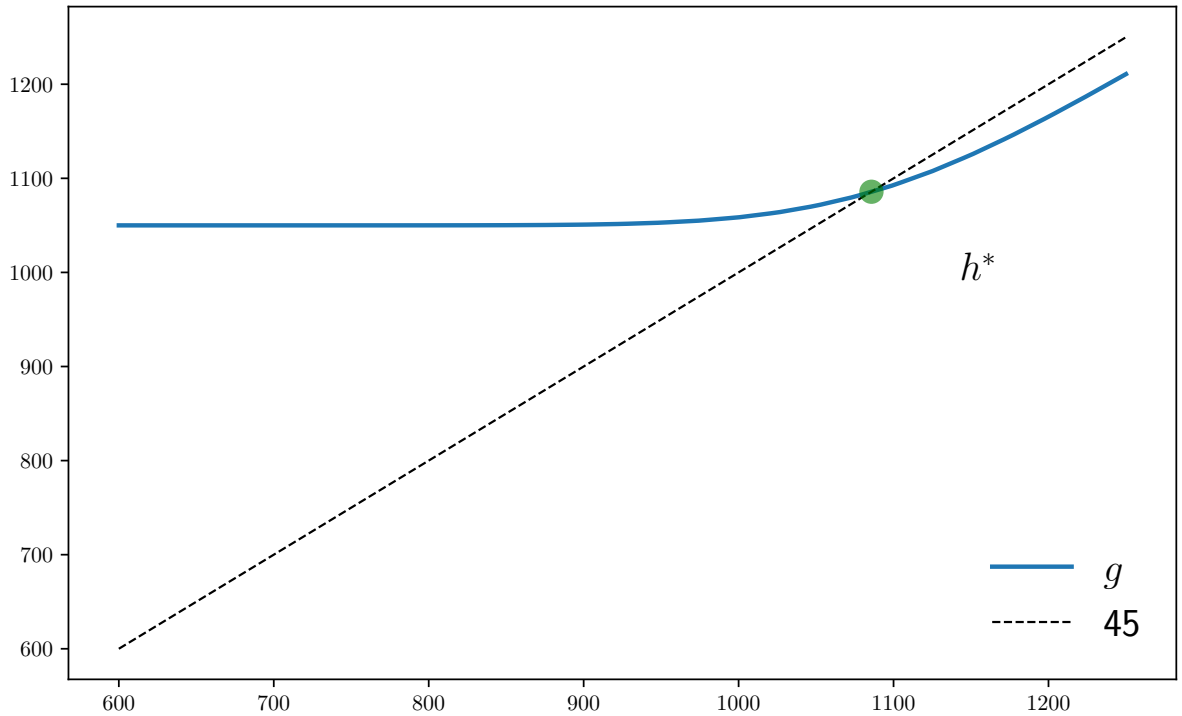


Figure: Computing the continuation value as the fixed point of $g$

New algorithm:

1. Compute $h^*$ via successive approximation on $g$
   - Iteration in $\mathbb{R}$, not $\mathbb{R}^n$
2. Optimal policy is

$$\sigma^*(w) = \mathbb{1}\left\{\frac{w}{1-\beta} \geqslant h^*\right\}$$