

Proof of Theorem 1

1 Proof of Theorem 1

The corresponding Lagrangian form of Problem 1 is

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = - \sum_{j \in \mathcal{U}} \log \left(\sum_{i \in \mathcal{M}} \omega_{i,j,r} x_{i,j,r} \right) + \sum_{i \in \mathcal{M}} \lambda_i \left(\sum_{j \in \mathcal{U}} x_{i,j,r} - Q_{i,r} \right) + \sum_{j \in \mathcal{U}} \mu_j \left(\sum_{i \in \mathcal{M}} p_{i,r} x_{i,j,r} - B_{j,r} \right) \quad (1)$$

where $\boldsymbol{\lambda} = \{\lambda_i\}$ and $\boldsymbol{\mu} = \{\mu_j\}$ are the Lagrangian multipliers.

The Karush–Kuhn–Tucker (KKT) conditions are necessary and sufficient for optimality [1]. Inspired by refs [2] and [3], we also adopt the KKT conditions when solving the problem. The KKT conditions are given as follows.

$$\begin{cases} \frac{\partial L(x_{i,j,r}, \lambda_i, \mu_j)}{\partial (x_{i,j,r})} = 0, & \forall i \in \mathcal{M}, \forall j \in \mathcal{U} \\ \lambda_i (\sum_{j \in \mathcal{U}} x_{i,j,r} - Q_{i,r}) = 0, & \forall i \in \mathcal{M} \\ \mu_j (\sum_{i \in \mathcal{M}} p_{j,r} x_{i,j,r} - B_{j,r}) = 0, & \forall j \in \mathcal{U} \\ x_{i,j,r} \geq 0, & \forall i \in \mathcal{M}, \forall j \in \mathcal{U} \\ \lambda_i \geq 0, & \forall i \in \mathcal{M} \\ \mu_j \geq 0, & \forall j \in \mathcal{U} \end{cases} \quad (2)$$

After eliminating λ_i , we have

$$\begin{cases} (\frac{\omega_{i,j,r}}{\sum_{i \in \mathcal{M}} \omega_{i,j,r} x_{i,j,r}} - \mu_j p_i) (\sum_{j \in \mathcal{U}} x_{i,j,r} - Q_{i,r}) = 0, & \forall i \in \mathcal{M} \\ \mu_j (\sum_{i \in \mathcal{M}} p_{j,r} x_{i,j,r} - B_{j,r}) = 0, & \forall i \in \mathcal{M} \\ x_{i,j,r} \geq 0, & \forall i, j \\ \frac{\omega_{i,j,r}}{\sum_{i \in \mathcal{M}} \omega_{i,j,r} x_{i,j,r}} - \mu_j p_i \geq 0 & \forall i \in \mathcal{M} \\ \mu_j \geq 0 & \forall j \in \mathcal{U} \end{cases} \quad (3)$$

When

$$\frac{\omega_{i,j,r}}{\sum_{i \in \mathcal{M}} \omega_{i,j,r} x_{i,j,r}} - \mu_j p_i = 0 \quad (4)$$

we have

$$\mu_j = \frac{\omega_{i,j,r}}{p_i \sum_{i \in \mathcal{M}} \omega_{i,j,r} x_{i,j,r}} \geq 0 \quad (5)$$

Since

$$\mu_j (\sum_{i \in \mathcal{M}} p_{j,r} x_{i,j,r} - B_{j,r}) = 0 \quad (6)$$

we have

$$B_{j,r} = \sum_{i \in \mathcal{M}} p_{j,r} x_{i,j,r} \quad (7)$$

Namely, if $\sum_{j \in \mathcal{U}} x_{i,j,r} - Q_{i,r} \neq 0$, $\forall i \in \mathcal{M}$ then, we have $\sum_{i \in \mathcal{M}} p_{j,r} x_{i,j,r} - B_{j,r} = 0$. Similarly, if $\sum_{i \in \mathcal{M}} p_{j,r} x_{i,j,r} - B_{j,r} \neq 0$ then we have $\sum_{j \in \mathcal{U}} x_{i,j,r} - Q_{i,r} = 0$.

Therefore, Problem 1 can be decomposed into two sub-problems as follow.

Sub-problem 1:

$$\begin{aligned} \max_{\mathbf{x}_r} \quad & \sum_{j \in \mathcal{U}} \log \left(\sum_{i \in \mathcal{M}} \omega_{i,j,r} x_{i,j,r} \right) \\ \text{s.t.} \quad & \sum_{j \in \mathcal{U}} x_{i,j,r} \leq Q_{i,r}, \forall i \in \mathcal{M} \\ & \sum_{i \in \mathcal{M}} p_{i,r} x_{i,j,r} = B_{j,r}, \forall j \in \mathcal{U} \end{aligned} \quad (8)$$

The KKT conditions of the corresponding Lagrangian form of sub-problem 1 are given as follows.

$$\begin{cases} \frac{\partial L(x_{i,j,r}, \lambda_i, \mu_j)}{\partial (x_{i,j,r})} = 0, & \forall i \in \mathcal{M}, \forall j \in \mathcal{U} \\ \lambda_i (\sum_{j \in \mathcal{U}} x_{i,j,r} - Q_{i,r}) = 0, & \forall i \in \mathcal{M} \\ x_{i,j,r} \geq 0, & \forall i \in \mathcal{M}, \forall j \in \mathcal{U} \\ \sum_{i \in \mathcal{M}} p_{i,r} x_{i,j,r} = B_{j,r}, & \forall j \in \mathcal{U} \\ \lambda_i \geq 0, & \forall i \in \mathcal{M} \end{cases} \quad (9)$$

Eliminating λ_i , we have

$$\begin{cases} \frac{\omega_{i,j,r}}{\sum_{i \in \mathcal{M}} \omega_{i,j,r} x_{i,j,r}} - \mu_j p_i \geq 0, & \forall i \in \mathcal{M} \\ (\frac{\omega_{i,j,r}}{\sum_{i \in \mathcal{M}} \omega_{i,j,r} x_{i,j,r}} - \mu_j p_i) (\sum_{j \in \mathcal{U}} x_{i,j,r} - Q_{i,r}) = 0, & \forall i \in \mathcal{M} \\ x_{i,j,r} \geq 0, & \forall i, j \\ \sum_{i \in \mathcal{M}} p_{i,r} x_{i,j,r} - B_{j,r} = 0, & \forall j \in \mathcal{U} \end{cases} \quad (10)$$

On one hand, the stationarity and complementary slackness in Eq.(9) imply that

$$\frac{\omega_{i,j,r}}{\sum_{i \in \mathcal{M}} \omega_{i,j,r} x_{i,j,r}} \geq \mu_j p_i \quad (11)$$

and

$$x_{i,j,r}^* = \begin{cases} \frac{\omega_{i,j,r}}{p_{i,r} \sum_{i \in \mathcal{M}} \omega_{i,j,r} \mu_j^*} & \mu_j^* \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (12)$$

For simplicity, Eq. (12) can be rewritten as

$$x_{i,j,r}^* = \left(\frac{\omega_{i,j,r}}{p_{i,r} \sum_{i \in \mathcal{M}} \omega_{i,j,r} \mu_j^*} \right)^+ \quad (13)$$

with $(\cdot)^+ = \max(\cdot, 0)$

On the other hand, the primal feasibility in Eq.(9) implies that

$$\sum_{i \in \mathcal{M}} p_{i,r} \frac{\omega_{i,j,r}}{p_{i,r} \sum_{i \in \mathcal{M}} \omega_{i,j,r} \mu_j^*} = B_{j,r} \quad (14)$$

Therefore,

$$x_{i,j,r}^* = \frac{B_{j,r} \omega_{i,j,r}}{p_{i,r} \sum_{i \in \mathcal{M}} \omega_{i,j,r}} \quad (15)$$

Sub-problem 2:

$$\begin{aligned} \max_{\mathbf{x}_r} \quad & \sum_{j \in \mathcal{U}} \log \left(\sum_{i \in \mathcal{M}} \omega_{i,j,r} x_{i,j,r} \right) \\ \text{s.t.} \quad & \sum_{j \in \mathcal{U}} x_{i,j,r} = Q_{i,r}, \forall i \in \mathcal{M} \\ & \sum_{i \in \mathcal{M}} p_{i,r} x_{i,j,r} \leq B_{j,r}, \forall j \in \mathcal{U} \end{aligned} \quad (16)$$

The KKT conditions of the corresponding Lagrangian form of sub-problem 2 are given below.

$$\begin{cases} \frac{\partial L(x_{i,j,r}, \lambda_i, \mu_j)}{\partial x_{i,j,r}} = 0, & \forall i \in \mathcal{M}, \forall j \in \mathcal{U} \\ \mu_j (\sum_{i \in \mathcal{M}} p_{i,r} x_{i,j,r} - B_{j,r}) = 0, & \forall j \in \mathcal{U} \\ x_{i,j,r} \geq 0, & \forall i \in \mathcal{M}, \forall j \in \mathcal{U} \\ \sum_{j \in \mathcal{U}} x_{i,j,r} - Q_{i,r} = 0, & \forall i \in \mathcal{M} \\ \mu_j \geq 0, & \forall j \in \mathcal{U} \end{cases} \quad (17)$$

Eliminating μ_j , we have

$$\begin{cases} \frac{\omega_{i,j,r}}{p_{i,r} \sum_{i \in \mathcal{M}} \omega_{i,j,r} x_{i,j,r}} - \frac{\lambda_i}{p_{i,r}} \geq 0, & \forall j \in \mathcal{U} \\ (\frac{\omega_{i,j,r}}{p_{i,r} \sum_{i \in \mathcal{M}} \omega_{i,j,r} x_{i,j,r}}) (\sum_{i \in \mathcal{M}} p_{i,r} x_{i,j,r} - B_{j,r}) = 0, & \forall j \in \mathcal{U} \\ x_{i,j,r} \geq 0, & \forall i, j \\ \sum_{j \in \mathcal{U}} x_{i,j,r} - Q_{i,r} = 0, & \forall i \in \mathcal{M} \end{cases} \quad (18)$$

On one hand, the stationarity and complementary slackness in Eq.(17) imply that

$$\frac{\omega_{i,j,r}}{p_{i,r} \sum_{i \in \mathcal{M}} \omega_{i,j,r} x_{i,j,r}} \geq \frac{\lambda_i}{p_{i,r}} \quad (19)$$

and

$$x_{i,j,r}^* = \begin{cases} \frac{\omega_{i,j,r}}{\lambda_i^* \sum_{i \in \mathcal{M}} \omega_{i,j,r}} & \lambda_i^* \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (20)$$

Equation (20) can be rewritten as

$$x_{i,j,r}^* = \left(\frac{\omega_{i,j,r}}{\lambda_i^* \sum_{i \in \mathcal{M}} \omega_{i,j,r}} \right)^+ \quad (21)$$

On the other hand, the primal feasibility in Eq.(17) implies that

$$\sum_{j \in \mathcal{U}} x_{i,j,r} = Q_{i,r} \quad (22)$$

Therefore,

$$x_{i,j,r}^* = \frac{Q_{i,r} \omega_{i,j,r}}{\sum_{i \in \mathcal{M}} \omega_{i,j,r}} \quad (23)$$

In summary, the optimal solution to Problem 1 is given as

$$x_{i,j,r}^* = \begin{cases} \frac{B_{j,r} \omega_{i,j,r}}{p_{i,r} \sum_{i \in \mathcal{M}} \omega_{i,j,r}}, & \text{if } I \geq 0 \\ \frac{Q_{i,r} \omega_{i,j,r}}{\sum_{j \in \mathcal{M}} \omega_{i,j,r}}, & \text{otherwise} \end{cases} \quad \forall i \in \mathcal{M}, \forall j \in \mathcal{U} \quad (24)$$

where $I = \sum_{j \in \mathcal{U}} \log(\sum_{i \in \mathcal{M}} \frac{B_{j,r} \omega_{i,j,r}^2}{p_{j,r} \sum_{i \in \mathcal{M}} \omega_{i,j,r}}) - \sum_{j \in \mathcal{U}} \log(\sum_{i \in \mathcal{M}} \frac{Q_{j,r} \omega_{i,j,r}^2}{\sum_{j \in \mathcal{U}} \omega_{i,j,r}})$.

References

1. Boyd, S., Vandenberghe, L.: Convex Optimization, Cambridge university press (2004).
2. Shi, W., Cao, J., Zhang, Q., Li, Y., Xu, L.: Edge Computing: Vision and Challenges. IEEE Internet of Things 3 (5), 637-646 (2018).
3. Kang, X., Zhang, R., Motani, M.: Price-based Resource Allocation for Spectrum sharing Femto-cell networks: A stackelberg game approach, IEEE Journal on Selected Areas in Communications. 30 (3), 538-549 (2012).