Proof of Theorem 1

The corresponding Lagrangian form of Problem 1 is

$$\begin{aligned} L(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\mu}) &= -\sum_{j\in\mathcal{U}} log\left(\sum_{i\in M}\omega_{i,j,r}x_{i,j,r}\right) \\ &+ \sum_{i\in\mathcal{M}}\lambda_i\left(\sum_{j\in U}x_{i,j,r}-Q_{i,r}\right)+\sum_{j\in\mathcal{U}}\mu_j\left(\sum_{i\in M}p_{i,r}x_{i,j,r}-B_{j,r}\right)\end{aligned} \tag{1}$$

where$\boldsymbol{\lambda}=\{\lambda_i\}$ and $\boldsymbol{\mu}=\{\mu_j\}$ are the Lagrangian multipliers.

The Karush–Kuhn–Tucker (KKT) conditions are necessary and sufficient for optimality [1]. Inspired by refs [2] and [3], we also adopt the KKT conditions when solving the problem. The KKT conditions are given as follows.

$$\begin{cases} \frac{\partial L(x_{i,j,r},\lambda_i,\mu_j)}{(x_{i,j,r})}=0, & \forall i\in\mathcal{M},\forall j\in\mathcal{U} \\ \lambda_i(\sum_{j\in\mathcal{U}}x_{i,j,r}-Q_{i,r})=0, & \forall i\in\mathcal{M} \\ \mu_j(\sum_{i\in\mathcal{M}}p_{j,r}x_{i,j,r}-B_{j,r})=0, & \forall j\in\mathcal{U} \\ x_{i,j,r}\geq 0, & \forall i\in\mathcal{M},\forall j\in\mathcal{U} \\ \lambda_i\geq 0, & \forall i\in\mathcal{M} \\ \mu_j\geq 0, & \forall j\in\mathcal{U} \end{cases} \tag{2}$$

After eliminating λ_i, we have

$$\begin{cases} (\frac{\omega_{i,j,r}}{\sum_{i\in\mathcal{M}}\omega_{i,j,r}x_{i,j,r}}-\mu_j p_i)(\sum_{j\in\mathcal{U}}x_{i,j,r}-Q_{i,r})=0, & \forall i\in\mathcal{M} \\ \mu_j(\sum_{i\in\mathcal{M}}p_{j,r}x_{i,j,r}-B_{j,r})=0, & \forall i\in\mathcal{M} \\ x_{i,j,r}\geq 0, & \forall i,j \\ \frac{\omega_{i,j,r}}{\sum_{i\in\mathcal{M}}\omega_{i,j,r}x_{i,j,r}}-\mu_j p_i\geq 0 & \forall i\in M \\ \mu_j\geq 0 & \forall j\in U \end{cases} \tag{3}$$

When

$$\frac{\omega_{i,j,r}}{\sum_{i\in\mathcal{M}}\omega_{i,j,r}x_{i,j,r}}-\mu_j p_i=0 \tag{4}$$

we have

$$\mu_j=\frac{\omega_{i,j,r}}{p_i\sum_{i\in\mathcal{M}}\omega_{i,j,r}x_{i,j,r}}\geq 0 \tag{5}$$

Since

$$\mu_j(\sum_{i\in\mathcal{M}}p_{j,r}x_{i,j,r}-B_{j,r})=0 \tag{6}$$

we have

$$B_{j,r}=\sum_{i\in\mathcal{M}}p_{j,r}x_{i,j,r} \tag{7}$$

Namely, if $\sum_{j\in\mathcal{U}} x_{i,j,r} - Q_{i,r} \neq 0,\ \forall i \in \mathcal{M}$ then,we have $\sum_{i\in\mathcal{M}} p_{j,r}x_{i,j,r} - B_{j,r} = 0$. Similarly, if $\sum_{i\in\mathcal{M}} p_{j,r}x_{i,j,r} - B_{j,r} \neq 0$ then we have $\sum_{j\in\mathcal{U}} x_{i,j,r} - Q_{i,r} = 0$.

Therefore, Problem 1 can be decomposed into two sub-problems as follow.

Sub-problem 1:

$$\begin{aligned} &\max_{\boldsymbol{x}_r} \sum_{j\in\mathcal{U}} \log\left(\sum_{i\in\mathcal{M}} \omega_{i,j,r}x_{i,j,r}\right) \\ &s.t. \sum_{j\in\mathcal{U}} x_{i,j,r} \leq Q_{i,r}, \forall i \in \mathcal{M} \\ &\quad\ \sum_{i\in M} p_{i,r}x_{i,j,r} = B_{j,r}, \forall j \in \mathcal{U} \end{aligned} \tag{8}$$

The KKT conditions of the corresponding Lagrangian form of sub-problem 1 are given as follows.

$$\begin{cases} \frac{\partial L(x_{i,j,r},\lambda_i,\mu_j)}{(x_{i,j,r})} = 0, & \forall i \in \mathcal{M}, \forall j \in \mathcal{U} \\ \lambda_i(\sum_{j\in\mathcal{U}} x_{i,j,r} - Q_{i,r}) = 0, & \forall i \in \mathcal{M} \\ x_{i,j,r} \geq 0, & \forall i \in \mathcal{M}, \forall j \in \mathcal{U} \\ \sum_{i\in M} p_{i,r}x_{i,j,r} = B_{j,r}, & \forall j \in \mathcal{U} \\ \lambda_i \geq 0, & \forall i \in \mathcal{M} \end{cases} \tag{9}$$

Eliminating λ_i, we have

$$\begin{cases} \frac{\omega_{i,j,r}}{\sum_{i\in\mathcal{M}} \omega_{i,j,r}x_{i,j,r}} - \mu_j p_i \geq 0, & \forall i \in \mathcal{M} \\ (\frac{\omega_{i,j,r}}{\sum_{i\in\mathcal{M}} \omega_{i,j,r}x_{i,j,r}} - \mu_j p_i)(\sum_{j\in\mathcal{U}} x_{i,j,r} - Q_{i,r}) = 0, & \forall i \in \mathcal{M} \\ x_{i,j,r} \geq 0, & \forall i, j \\ \sum_{i\in M} p_{i,r}x_{i,j,r} - B_{j,r} = 0, & \forall j \in \mathcal{U} \end{cases} \tag{10}$$

On one hand, the stationarity and complementary slackness in Eq.(9) imply that

$$\frac{\omega_{i,j,r}}{\sum_{i\in\mathcal{M}} \omega_{i,j,r}x_{i,j,r}} \geq \mu_j p_i \tag{11}$$

and

$$x^*_{i,j,r} = \begin{cases} \frac{\omega_{i,j,r}}{p_{i,r}\sum_{i\in\mathcal{M}} \omega_{i,j,r}\mu^*_j} & \mu^*_j \geq 0 \\ 0 & otherwise \end{cases} \tag{12}$$

For simplicity, Eq. (12) can be rewritten as

$$x^*_{i,j,r} = (\frac{\omega_{i,j,r}}{p_{i,r}\sum_{i\in\mathcal{M}} \omega_{i,j,r}\mu^*_j})^+ \tag{13}$$

with$(.)^+ = max(.,0)$

On the other hand, the primal feasibility in Eq.(9) implies that

$$\sum_{i\in M} p_{i,r}\frac{\omega_{i,j,r}}{p_{i,r}\sum_{i\in\mathcal{M}} \omega_{i,j,r}\mu^*_j} = B_{j,r} \tag{14}$$

Therefore,

$$x^*_{i,j,r} = \frac{B_{j,r}\omega_{i,j,r}}{p_{i,r}\sum_{i\in\mathcal{M}}\omega_{i,j,r}} \tag{15}$$

Sub-problem 2:

$$\begin{aligned} &\max_{\boldsymbol{x}_r} \sum_{j\in\mathcal{U}} \log\left(\sum_{i\in\mathcal{M}}\omega_{i,j,r}x_{i,j,r}\right) \\ &s.t. \sum_{j\in\mathcal{U}} x_{i,j,r} = Q_{i,r}, \forall i\in\mathcal{M} \\ &\quad\sum_{i\in M} p_{i,r}x_{i,j,r} \le B_{j,r}, \forall j\in\mathcal{U} \end{aligned} \tag{16}$$

The KKT conditions of the corresponding Lagrangian form of sub-problem 2 are given below.

$$\begin{cases} \frac{\partial L(x_{i,j,r},\lambda_i,\mu_j)}{(x_{i,j,r})} = 0, & \forall i\in\mathcal{M},\forall j\in\mathcal{U} \\ \mu_j(\sum_{i\in\mathcal{M}} p_{i.r}x_{i,j,r} - B_{i,r}) = 0, & \forall j\in\mathcal{U} \\ x_{i,j,r}\ge 0, & \forall i\in\mathcal{M},\forall j\in\mathcal{U} \\ \sum_{j\in U} x_{i,j,r} - Q_{i,r} = 0, & \forall i\in\mathcal{M} \\ \mu_j \ge 0, & \forall j\in\mathcal{U} \end{cases} \tag{17}$$

Eliminating μ_j, we have

$$\begin{cases} \frac{\omega_{i,j,r}}{p_{i,r}\sum_{i\in\mathcal{M}}\omega_{i,j,r}x_{i,j,r}} - \frac{\lambda_i}{p_{i,r}} \ge 0, & \forall j\in\mathcal{U} \\ (\frac{\omega_{i,j,r}}{p_{i,r}\sum_{i\in\mathcal{M}}\omega_{i,j,r}x_{i,j,r}})(\sum_{i\in\mathcal{M}} p_{i,r}x_{i,j,r} - B_{j,r}) = 0, & \forall j\in\mathcal{U} \\ x_{i,j,r}\ge 0, & \forall i,j \\ \sum_{j\in U} x_{i,j,r} - Q_{i,r} = 0, & \forall i\in\mathcal{M} \end{cases} \tag{18}$$

On one hand, the stationarity and complementary slackness in Eq.(17) imply that

$$\frac{\omega_{i,j,r}}{p_{i,r}\sum_{i\in\mathcal{M}}\omega_{i,j,r}x_{i,j,r}} \ge \frac{\lambda_i}{p_{i,r}} \tag{19}$$

and

$$x^*_{i,j,r} = \begin{cases} \frac{\omega_{i,j,r}}{\lambda^*_i\sum_{i\in\mathcal{M}}\omega_{i,j,r}} & \lambda^*_i \ge 0 \\ 0 & otherwise \end{cases} \tag{20}$$

Equation (20) can be rewritten as

$$x^*_{i,j,r} = (\frac{\omega_{i,j,r}}{\lambda^*_i\sum_{i\in\mathcal{M}}\omega_{i,j,r}})^+ \tag{21}$$

On the other hand, the primal feasibility in Eq.(17) implies that

$$\sum_{j\in U} x_{i,j,r} = Q_{i,r} \tag{22}$$

Therefore,

$$x^*_{i,j,r} = \frac{Q_{i,r}\omega_{i,j,r}}{\sum_{i\in\mathcal{M}} \omega_{i,j,r}} \tag{23}$$

In summary, the optimal solution to Problem 1 is given as

$$x^*_{i,j,r} = \begin{cases} \dfrac{B_{j,r}\omega_{i,j,r}}{p_{i,r}\sum_{i\in M} \omega_{i,j,r}}, \; if \quad I \geq 0 \\ \dfrac{Q_{i,r}\omega_{i,j,r}}{\sum_{j\in M} \omega_{i,j,r}}, \quad otherwise \end{cases} \forall i \in \mathcal{M}, \forall j \in \mathcal{U} \tag{24}$$

where $I = \sum_{j\in\mathcal{U}} log(\sum_{i\in\mathcal{M}} \frac{B_{j,r}\omega^2_{i,j,r}}{p_{j,r}\sum_{i\in\mathcal{M}}\omega_{i,j,r}}) - \sum_{j\in\mathcal{U}} log(\sum_{i\in\mathcal{M}} \frac{Q_{j,r}\omega^2_{i,j,r}}{\sum_{j\in\mathcal{U}}\omega_{i,j,r}})$.

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