# A Generalized LQR Controller Design Problem

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**Problem Formulation**
Given a time-varying linear system:

$$x_{k+1} = A_k x_k + B_k u_k + g_k \tag{1}$$

It is required to find a control law to minimize the cost function in a finite time steps $N$:

$$L = \sum_{k=1}^{N-1} (u_k^T R_k u_k + 2l_k^T u_k + x_k^T Q_k x_k + 2h_k^T x_k + 2x_k^T M_k u_k) + x_N^T Q_N x_N + 2h_N^T x_N + c \tag{2}$$

**Solution**
1. The optimal cost-to-go function at each step is a quadratic function of the form:

$$v(x, k) = x^T P_k x + 2p_k^T x + c_k \tag{3}$$

First set $P_N = Q_N$, $p_N = h_N$ and $c_N = c$, then $P_k$, $p_k$ and $c_k$ can be computed backward from $k = N - 1$ to $k = 2$ as following:

$$P_k = A_k^T P_{k+1} A_k + Q_k - (B_k^T P_{k+1} A_k + M_k^T)^T (B_k^T P_{k+1} B_k + R_k)^{-1} (B_k^T P_{k+1} A_k + M_k^T) \tag{4}$$

$$p_k = A_k^T P_{k+1} g_k + A_k^T p_{k+1} + h_k - (B_k^T P_{k+1} A_k + M_k^T)^T (B_k^T P_{k+1} B_k + R_k)^{-1} (B_k^T P_{k+1} g_k + B_k^T p_{k+1} + l_k) \tag{5}$$

$$\begin{aligned} c_k = &-(B_k^T P_{k+1} g_k + B_k^T p_{k+1} + l_k)^T (B_k^T P_{k+1} B_k + R_k)^{-1} (B_k^T P_{k+1} g_k + B_k^T p_{k+1} + l_k) \\ &+ g_k^T P_{k+1} g_k + 2p_{k+1}^T g_k + c_{k+1} \end{aligned} \tag{6}$$

2. The optimal control law is an affine function of the state:

$$u_k^\star = K_k x_k + s_k \tag{7}$$

The optimal control gain and offset are given by:

$$K_k = -(R_k + B_k^T P_{k+1} B_k)^{-1} (B_k^T P_{k+1} A_k + M_k^T) \tag{8}$$

$$s_k = -(R_k + B_k^T P_{k+1} B_k)^{-1} (B_k^T P_{k+1} g_k + B_k^T p_{k+1} + l_k) \tag{9}$$

**Proof**
We already know the optimal cost-to-go function at time $k+1$ is a quadratic function:

$$v(x_{k+1}, k+1) = x_{k+1}^T P_{k+1} x_{k+1} + 2p_{k+1}^T x_{k+1} + c_{k+1} \tag{10}$$

Group terms with subscript $k$ in the cost function with $v(x_{k+1}, k+1)$:

$$x_{k+1}^T P_{k+1} x_{k+1} + 2p_{k+1}^T x_{k+1} + c_{k+1} + u_k^T R_k u_k + 2x_k^T M_k u_k + x_k^T Q_k x_k + 2h_k^T x_k + 2l_k^T u_k \tag{11}$$

Substitute the system dynamics $x_{k+1} = A_k x_k + B_k u_k + g_k$:

$$\begin{aligned}(A_k x_k + B_k u_k + g_k)^T P_{k+1}(A_k x_k + B_k u_k + g_k) + 2p_{k+1}^T(A_k x_k + B_k u_k + g_k) + c_{k+1} \\ + u_k^T R_k u_k + 2x_k^T M_k u_k + x_k^T Q_k x_k + 2h_k^T x_k + 2l_k^T u_k\end{aligned} \tag{12}$$

Regroup terms as a quadratic function of $u_k$, and the constant as a quadratic function of $x_k$:

$$\begin{aligned}u_k^T(B_k^T P_{k+1} B_k + R_k)u_k + [2B_k^T P_{k+1}(A_k x_k + g_k) + 2B_k^T p_{k+1} + 2M_k^T x_k + 2l_k]^T u_k \\ + x_k^T(A_k^T P_{k+1} A_k + Q_k)x_k + [2A_k^T P_{k+1} g_k + 2A_k^T p_{k+1} + 2h_k]^T x_k + g_k^T P_{k+1} g_k + 2p_{k+1}^T g_k + c_{k+1}\end{aligned} \tag{13}$$

Simplify a bit:

$$\begin{aligned}u_k^T(B_k^T P_{k+1} B_k + R_k)u_k + 2[(B_k^T P_{k+1} A_k + M_k^T)x_k + B_k^T P_{k+1} g_k + B_k^T p_{k+1} + l_k]^T u_k \\ + x_k^T(A_k^T P_{k+1} A_k + Q_k)x_k + 2(A_k^T P_{k+1} g_k + A_k^T p_{k+1} + h_k)^T x_k + g_k^T P_{k+1} g_k + 2p_{k+1}^T g_k + c_{k+1}\end{aligned} \tag{14}$$

The optimal input minimizing the quadratic terms of $u_k$ is:

$$u_k^* = -(B_k^T P_{k+1} B_k + R_k)^{-1}[(B_k^T P_{k+1} A_k + M_k^T)x_k + B_k^T P_{k+1} g_k + B_k^T p_{k+1} + l_k] \tag{15}$$

The optimal feedback control law is then:

$$u_k^\star = K_k x_k + s_k \tag{16}$$

$$K_k = -(R_k + B_k^T P_{k+1} B_k)^{-1}(B_k^T P_{k+1} A_k + M_k^T) \tag{17}$$

$$s_k = -(R_k + B_k^T P_{k+1} B_k)^{-1}(B_k^T P_{k+1} g_k + B_k^T p_{k+1} + l_k) \tag{18}$$

Substituting the optimal input into the cost terms, we can get the optimal cost-to-go function at time step $k$:

$$\begin{aligned}v(x_k, k) = -[(B_k^T P_{k+1} A_k + M_k^T)x_k + B_k^T P_{k+1} g_k + B_k^T p_{k+1} + l_k]^T(B_k^T P_{k+1} B_k + R_k)^{-1} \\ [(B_k^T P_{k+1} A_k + M_k^T)x_k + B_k^T P_{k+1} g_k + B_k^T p_{k+1} + l_k] \\ + x_k^T(A_k^T P_{k+1} A_k + Q_k)x_k + 2(A_k^T P_{k+1} g_k + A_k^T p_{k+1} + h_k)^T x_k + g_k^T P_{k+1} g_k + 2p_{k+1}^T g_k + c_{k+1}\end{aligned} \tag{19}$$

Regroup terms into a quadratic form of $x_k$:

$$\begin{aligned}x_k^T[A_k^T P_{k+1} A_k + Q_k - (B_k^T P_{k+1} A_k + M_k^T)^T(B_k^T P_{k+1} B_k + R_k)^{-1}(B_k^T P_{k+1} A_k + M_k^T)]x_k \\ + 2[A_k^T P_{k+1} g_k + A_k^T p_{k+1} + h_k - (B_k^T P_{k+1} A_k + M_k^T)^T(B_k^T P_{k+1} B_k + R_k)^{-1}(B_k^T P_{k+1} g_k + B_k^T p_{k+1} + l_k)]^T x_k \\ -(B_k^T P_{k+1} g_k + B_k^T p_{k+1} + l_k)^T(B_k^T P_{k+1} B_k + R_k)^{-1}(B_k^T P_{k+1} g_k + B_k^T p_{k+1} + l_k) + g_k^T P_{k+1} g_k + 2p_{k+1}^T g_k + c_{k+1}\end{aligned} \tag{20}$$

From here we can have the back propagation of optimal cost-to-go function:

$$P_k = A_k^T P_{k+1} A_k + Q_k - (B_k^T P_{k+1} A_k + M_k^T)^T(B_k^T P_{k+1} B_k + R_k)^{-1}(B_k^T P_{k+1} A_k + M_k^T) \tag{21}$$

$$p_k = A_k^T P_{k+1} g_k + A_k^T p_{k+1} + h_k - (B_k^T P_{k+1} A_k + M_k^T)^T(B_k^T P_{k+1} B_k + R_k)^{-1}(B_k^T P_{k+1} g_k + B_k^T p_{k+1} + l_k) \tag{22}$$

$$\begin{aligned}c_k = -(B_k^T P_{k+1} g_k + B_k^T p_{k+1} + l_k)^T(B_k^T P_{k+1} B_k + R_k)^{-1}(B_k^T P_{k+1} g_k + B_k^T p_{k+1} + l_k) \\ + g_k^T P_{k+1} g_k + 2p_{k+1}^T g_k + c_{k+1}\end{aligned} \tag{23}$$