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Rate of Convergence of Shepard's Global Interpolation Formula

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Abstract. Given any data points $x_1, \ldots, x_n$ in $\mathbf{R}^s$ and values $f(x_1), \ldots, f(x_n)$ of a function f, Shepard's global interpolation formula reads as follows:

$$S_p^0 f(x) = \sum_i f(x_i) w_i(x), \qquad w_i(x) = |x - x_i|^{-p} / \sum_j |x - x_j|^{-p},$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbf{R}^s$. This interpolation scheme is stable, but if $p > 1$, the gradient of the interpolating function vanishes in all data points. The interpolation operator S_p^q is defined by replacing the values $f(x_i)$ in $S_p^0 f$ by Taylor polynomials of f of degree $q \in \mathbf{N}$. In this paper, we investigate the approximating power of S_p^q for all values of p, q and s.

1. Introduction. In [12] D. Shepard introduced an interpolation scheme which is easily programmable and whose interpolating function can be written down explicitly. Given any arbitrarily spaced points $x_1, \ldots, x_n \in \mathbf{R}^s$ and values $f(x_1), \ldots, f(x_n)$ of a function f, the first version of Shepard's interpolation formula is given by

$$S_p^0 f(x) = \sum_{i=1}^{n} f(x_i) w_i(x)$$

with basis functions

$$w_i(x) = \frac{|x - x_i|^{-p}}{\sum_j |x - x_j|^{-p}}.$$

Here $p > 0$, and $|\cdot|$ may denote any norm in $\mathbf{R}^s$, but for reasons of differentiability it is natural to use the Euclidean norm. The basis functions are not differentiable at the data points if $p \leqslant 1$; otherwise, at least the first derivatives vanish. Generally, the parameter p is chosen to equal 2 so that, in addition, the basis functions are rational and infinitely differentiable. For a qualitative discussion of the parameter p, see Barnhill, Dube and Little [3], Gordon and Wixom [5], Lancaster and Salkauskas [7] and Poeppelmeier [10]. Since

$$w_i(x_j) = \delta_{ij}, \qquad w_i(x) \geqslant 0 \quad \text{and} \quad \sum_i w_i(x) = 1,$$

the linear interpolation operator S_p^0 is stable in the sense that

$$\min_i f(x_i) \leqslant S_p^0 f(x) \leqslant \max_i f(x_i).$$

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D. J. Newman and T. J. Rivlin [9] show that, if $p = 2$, then S_p^0 is optimal in a certain sense among all 'universally' stable rational interpolation schemes. But the approximating power of a stable interpolation operator cannot exceed $n^{-1/s}$, as is shown in Section 3.

If $p > 1$, the interpolating function $S_p^0 f$ has flat spots in the neighborhood of all data points. This drawback can be avoided by using Taylor polynomials of f of degree q instead of the values $f(x_1), \ldots, f(x_n)$, if all partial derivatives $D^\nu f(x_i)$ of f up to degree q, or approximations to them, are known:

$$S_p^q f(x) = \sum_{i=1}^{n} \sum_{|\nu| \leqslant q} \frac{1}{\nu!} D^\nu f(x_i)(x - x_i)^\nu w_i(x),$$

where $\nu = (\nu_1, \ldots, \nu_s)$ denotes a multi-index and $|\nu| = \nu_1 + \cdots + \nu_s$. If $q \geqslant 1$, S_p^q is no longer stable, but $S_p^q f$ will approximate f (and its derivatives) much better than $S_p^0 f$ does. Generalizations of Shepard's global interpolation formula are described by Barnhill [2], McLain [6], Little [8], Schumaker [11] and in [3], [7] and [10].

In applications, the global character of Shepard's interpolation formula is often undesirable. Further, the evaluation of $S_p^q f(x)$ requires a considerable amount of work. These disadvantages are avoided by using local versions of Shepard's formula. There the basic functions have 'small' compact support which may even depend on the local distribution of data points (see [2], [11] and [12]). Bash-Ayan [1] and Franke [4] tested Shepard's global formula and several local versions and compared them to other multivariate interpolation schemes.

If the n data points are distributed in a homogeneous way throughout a given domain, and if $n \to \infty$, then $S_p^q f$ will converge to the given function f. In [9], the rate of convergence of S_2^0 is given in the univariate case. In this paper, we investigate the approximating power of S_p^q for all p and q and in every dimension s. Section 2 shows that $p - s = q + 1$ or $p - s > q + 1$ is a good choice of the parameter p if q is given. In Section 3, we prove (cf. [9], if $s = 1$, $q = 0$, $p = 2$) that the results of our main theorem, Theorem 2.3, cannot be improved for any set of data points.

2. Approximating Power of Shepard's Global Interpolation Formula. In the s-dimensional space $\mathbf{R}^s$ let $|\cdot|$ denote the Euclidean norm, $\|\cdot\|$ the maximum norm, and let $B_r(y)$ denote the closed cube $\{x \in \mathbf{R}^s;\ \|x - y\| \leqslant r\}$ with center y and radius r. We use standard multi-index notation. In particular, given any multi-index $\nu = (\nu_1, \ldots, \nu_s) \in \mathbf{N}^s$, $|\nu|$ denotes the sum $\nu_1 + \cdots + \nu_s$ and not the Euclidean norm of the vector ν.

Let f be a function sufficiently smooth in a domain A of $\mathbf{R}^s$ such that the function values $f(x_i)$ and all partial derivatives $D^\nu f(x_i)$ up to some order $q \in \mathbf{N}$ ($|\nu| \leqslant q$) are known in n pairwise distinct data points $x_1, \ldots, x_n$. Then, if $p > 0$, Shepard's interpolation operator S_p^q is defined as

$$S_p^q f(x) = \sum_{i=1}^{n} \sum_{|\nu| \leqslant q} \frac{1}{\nu!} D^\nu f(x_i)(x - x_i)^\nu w_i(x) \tag{2.1}$$

with the ith basis function

$$w_i(x) = \frac{|x - x_i|^{-p}}{\sum_{j=1}^{n} |x - x_j|^{-p}} = \frac{\prod_{k \neq i} |x - x_k|^p}{\sum_{j=1}^{n} \prod_{k \neq j} |x - x_k|^p}. \tag{2.2}$$

Note that

$$w_i(x_j) = \delta_{ij}, \quad w_i(x) \geqslant 0, \quad \sum_{i=1}^{n} w_i(x) = 1,$$

and that $w_i(x)$ is infinitely differentiable in $\mathbf{R}^s$ except in the data points where all partial derivatives up to order $p-1$ ($p \in \mathbf{N}$) or $[p]$ ($p \notin \mathbf{N}$) exist and vanish. Thus, if $q \leqslant p-1$ or $q \leqslant [p]$, the interpolating function $S_p^q f$ is at least q-times continuously differentiable and

$$D^\nu S_p^q f(x_i) = D^\nu f(x_i) \qquad \bigl(|\nu| \leqslant q,\ i = 1, \ldots, n\bigr).$$

Assume that the domain A is compact and satisfies two conditions of regularity:

(A1) There is a number $\gamma \geqslant 1$ such that any two points x and y in A are joined by a rectifiable curve Γ in A with length $|\Gamma| \leqslant \gamma|x-y|$.

(A2) There is a compact cone K such that for every $x \in A$ there is a cone $K(x) \subset A$ with vertex x and congruent with K.

Note that the cone K may be defined as $K = \{\lambda y;\ y \in B_{r_1}(z),\ \lambda \geqslant 0\} \cap B_{r_2}(0)$ ($r_1 > 0$, $r_2 > 0$, $0 \notin B_{r_1}(z)$) and that the number γ in (A1) may be assumed to be an integer. The condition (A1) implies connectedness and excludes domains with too thin cusps directed into the interior $\mathring{A}$ of A, while the cone property (A2) does not admit domains with too thin cusps directed outwards and guarantees that the closure of $\mathring{A}$ equals A.

Next, we discuss some consequences of the condition (A1). According to H. Whitney [13], a function $f\colon A \to \mathbf{R}$ is said to be of class C^q in A if and only if functions $D^\nu f(x)$ and $R_\nu(x;y)$ ($|\nu| \leqslant q$) exist in A such that Taylor's formula holds in the following sense ($x, y \in A$, $|\nu| \leqslant q$)

$$D^\nu f(x) = \sum_{|\mu| \leqslant q-|\nu|} \frac{1}{\mu!} D^{\nu+\mu} f(y)(x-y)^\mu + R_\nu(x;y). \tag{2.3}$$

The remainder terms $R_\nu(x;y)$ shall have the following property: Given any point $z \in A$ and any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|R_\nu(x;y)| \leqslant \varepsilon |x-y|^{q-|\nu|} \quad \text{for every } x,\ y \in A,\ |x-z| < \delta,\ |y-z| < \delta. \tag{2.4}$$

(2.3) and (2.4) imply that f is continuous in A and, if $q \geqslant 1$, that f has continuous partial derivatives up to order q in the interior of A satisfying

$$D^\nu f(x) = \frac{\partial^{|\nu|} f(x)}{\partial_{x_1}^{\nu_1} \cdots \partial_{x_s}^{\nu_s}}, \qquad |\nu| \leqslant q,\ x \in A.$$

A function f is said to be of class $C^{q,1}$ in A if and only if f is of class C^q in A and, additionally, the partial derivatives $D^\nu f$ of f of order q ($|\nu| = q$) are Lipschitz-continuous in A. In this case, the seminorm $|\cdot|_{q,1}$ is defined as

$$|f|_{q,1} = \sup\left\{ \frac{|D^\nu f(x_1) - D^\nu f(x_2)|}{|x_1 - x_2|};\ x_1, x_2 \in A,\ x_1 \neq x_2,\ |\nu| = q \right\}.$$

The results of H. Whitney [13] yield important estimates of the remainder terms $R_\nu(x;y)$.

LEMMA 2.1. *Let A satisfy* (A1) *and let f be of class $C^{q,1}$ in A. Then, for every x, $y \in A$,*

$$|R_\nu(x;y)| \leqslant c_{q-|\nu|}\gamma^{q-|\nu|}|x-y|^{q-|\nu|+1}|f|_{q,1},$$

where $c_p = s^p/(p-1)!$ $(p>0)$ or $c_p = 1$ $(p=0)$.

If f is assumed to be only continuous in A, we will use its modulus of continuity ω_f:

$$\omega_f(\delta) = \sup\{|f(x)-f(y)|;\ x,\ y \in A,\ |x-y| \leqslant \delta\}.$$

LEMMA 2.2. *Let A satisfy* (A1), *and let $f \in C^0(A)$. Then, for arbitrary positive numbers $0 < \varepsilon < \delta$,*

$$\omega_f(\delta) \leqslant 2\gamma\frac{\delta}{\varepsilon}\omega_f(\varepsilon).$$

Proof. Let m be the integer defined by $m\varepsilon < \delta \leqslant (m+1)\varepsilon$. Then, it is sufficient to prove that $\omega_f((m+1)\varepsilon) \leqslant \gamma(m+1)\omega_f(\varepsilon)$ because $m+1 \leqslant 2\delta/\varepsilon$. Let x, $y \in A$ with $|x-y| \leqslant (m+1)\varepsilon$ and let Γ be a rectifiable curve in A joining x and y with length $|\Gamma| \leqslant \gamma|x-y|$. Parameterizing Γ by $x(s) \in C^0[0, |\Gamma|]$, where s denotes the length of the part of Γ between x and $x(s)$, we partition the interval $[0, |\Gamma|]$ into $\gamma(m+1)$ consecutive parts $[s_{i-1}, s_i]$ of length $|s_i - s_{i-1}| \leqslant \delta$. Noting that $x = x(0)$ and $y = x(|\Gamma|)$, we get

$$|f(x)-f(y)| \leqslant \sum_{i=1}^{\gamma(m+1)} |f(x(s_i)) - f(x(s_{i-1}))| \leqslant \gamma(m+1)\omega_f(\delta). \quad \square$$

Let X be a set of n pairwise distinct data points $x_1, \dots, x_n$ in A. We set

$$r = \inf\{\rho > 0;\ \text{for every } x \in A,\ B_\rho(x) \text{ contains at least one element of } X\}$$

and

$$M = \sup_y \operatorname{card}(B_r(y) \cap X).$$

Analogously to the half-open unit interval $(0,1]$, the half-open unit cube in $\mathbf{R}^s$ is defined as $(0,1]^s$. Multiplying by $2r$ and shifting, we get the half-open cube $Q_r(x)$ with center x and radius r (with respect to the maximum norm). The finiteness of X implies for every $x \in A$ that $B_r(x)$ and $Q_r(x)$, if additionally $Q_r(x) \subset \mathring{A}$, contain at least one element of X. Our main theorem, Theorem 2.3, yields an estimate of $\|S_p^q f - f\|_A$, the supremum of $S_p^q f - f$ in A.

THEOREM 2.3. *Let the compact domain A satisfy condition* (A1) *and, if $p = s$, also* (A2) *and let $X = \{x_1, \dots, x_n\}$ be a set of n data points. If f is of class $C^{q,1}$ in A, then*

$$\|S_p^q f - f\|_A \leqslant C\gamma^q M|f|_{q,1}\varepsilon_p^q(r), \tag{2.5}$$

where

$$\varepsilon_p^q(r) = \begin{cases} |\log r|^{-1}, & p = s, \\ r^{p-s}, & p-s < q+1, p > s, \\ r^{p-s}|\log r|, & p-s = q+1, \\ r^{q+1}, & p-s > q+1, \end{cases} \tag{2.6}$$

and C is a positive constant independent of f, γ and X. If $f \in C^0(A)$, then

$$\left\| S_p^0 f - f \right\|_A \leqslant C\gamma M \omega_f\left(\varepsilon_p^0(r)\right). \tag{2.7}$$

Proof. Define

$$s_p^q(x) = \frac{\sum_{i=1}^n \|x - x_i\|^{q+1-p}}{\sum_{i=1}^n \|x - x_i\|^{-p}}. \tag{2.8}$$

If $f \in C^0(A)$, then by Lemma 2.2 ($\varepsilon(r) = \varepsilon_p^0(r)$)

$$\begin{aligned} \left|\left(S_p^q f - f\right)(x)\right| &\leqslant \sum_{\substack{i \\ |x - x_i| \leqslant \varepsilon(r)}} \omega_f(\varepsilon(r)) w_i(x) + \sum_{\substack{i \\ |x - x_i| > \varepsilon(r)}} |f(x_i) - f(x)| w_i(x) \\ &\leqslant \omega_f(\varepsilon(r))\left(1 + \frac{2\gamma}{\varepsilon(r)} \sum_i |x - x_i| w_i(x)\right) \\ &\leqslant \omega_f(\varepsilon(r))\gamma\left(1 + \frac{C}{\varepsilon(r)} s_p^0(x)\right). \end{aligned}$$

If $f \in C^{q,1}(A)$, then by (2.3) and Lemma 2.1,

$$\left|\left(S_p^q f - f\right)(x)\right| \leqslant \sum_i |R_0(x; x_i)| w_i(x) \leqslant c_q \gamma^q |f|_{q,1} s_p^q(x).$$

Thus, it remains to be shown that for fixed $x \in A$, $x \neq x_i$ ($1 \leqslant i \leqslant n$),

$$s_p^q(x) \leqslant MC\varepsilon_p^q(r) \tag{2.9}$$

with a positive constant C independent of x and X.

In order to prove (2.9) we need a disjoint covering of A by half-open cubes. Let $T_j = T_{j,r}(x)$ be the half-open annulus with center x and radius $2rj$ defined by

$$T_j = \bigcup_{\substack{k \in \mathbf{Z}^s \\ \|k\| = j}} Q_r(x + 2rk).$$

Note that $T_0 = Q_r(x)$ and that

$$A \subset \bigcup_{j=0}^{N} T_j \quad \text{with} \quad N = \left[\frac{d(A)}{2r}\right] + 1,$$

where $d(A) > 0$ denotes the diameter $\sup\{\|y - z\|;\ y, z \in A\}$ of A. Evidently, the number of half-open cubes $Q_r(\cdot)$ contained in T_j lies between $2s(2j-1)^{s-1}$ and $2s(2j+1)^{s-1}$, thus

$$\operatorname{card}\left(X \cap T_{j,r}(x)\right) \leqslant 2sM(2j+1)^{s-1} \qquad (1 \leqslant j \leqslant N).$$

Further, note that

$$(2j-1)r \leqslant \|x - x_i\| \leqslant (2j+1)r \quad \text{for every } x_i \in X \cap T_{j,r}(x).$$

If (A2) is satisfied, we get a lower bound on card$(X \cap T_j)$. For there are a positive number r_0 and positive integers j_0 and $j_r \geqslant \max(r_0/r, j_0)$ (each of them independent of x) such that the number of half-open cubes $Q_r(x + 2rk)$ with $\|k\| = j$ contained in $\mathring{A} \cap K(x)$ is bounded below by $[j/j_0]^{s-1}$ for every $j_0 \leqslant j \leqslant j_r$, $r \leqslant r_0$. Thus,

$$\operatorname{card}\left(X \cap T_{j,r}(x)\right) \geqslant \left[\frac{j}{j_0}\right]^{s-1} \quad \text{for every } j_0 \leqslant j \leqslant j_r,\ r \leqslant r_0$$

$$\text{where } j_r \geqslant \max\left(\frac{r_0}{r}, j_0\right).$$

Note that, if $\mathring{A}$ is not empty and if M is bounded as $n \to \infty$, $r \to 0$, then $n = \operatorname{card} X$ is of exact order r^{-s}. Defining the point $x_{i_0} \in X$ by

$$\|x - x_{i_0}\| = \min_i \|x - x_i\|,$$

we get

$$\tag{2.10} \begin{aligned} s_p^q(x) &\leqslant \|x - x_{i_0}\|^p \left(\sum_{x_i \in T_0} \|x - x_i\|^{q+1-p} + \sum_{j=1}^{N} \sum_{x_i \in T_j} \|x - x_i\|^{q+1-p} \right) \\ &\leqslant M r^{q+1} \left(1 + C \sum_{j=1}^{N} j^{s-p+q} \right). \end{aligned}$$

Case 1 ($p > s$). If either $p - s > q + 1$ or $p - s = q + 1$ or $p - s < q + 1$, then $\sum j^{s-p+q}$ is either bounded or diverges with $O(\log N) = O(|\log r|)$ or diverges with $O(N^{s-p+q+1}) = O(r^{p-s-q-1})$. Thus (2.10) yields (2.9).

Case 2 ($p = s$). In this case we must examine the denominator of $s_p^q(x)$ more carefully. Since $\sum_{j_0}^{j_r} 1/j = O(\log j_r) = O(|\log r|)$,

$$\begin{aligned} \sum_{j=0}^{N} \sum_{x_i \in T_0} \|x - x_i\|^{-p} &\geqslant \sum_{x_i \in T_0} \|x - x_i\|^{-p} + \sum_{j=j_0}^{j_r} \left[\frac{j}{j_0} \right]^{s-1} ((2j+1)r)^{-p} \\ &\geqslant \sum_{x_i \in T_0} \|x - x_i\|^{-p} + \frac{C}{r^p} |\log r|. \end{aligned}$$

Using the inequality

$$\frac{\sum a_i}{\sum b_i} \leqslant \sum \frac{a_i}{b_i} \qquad (a_i \geqslant 0, b_i > 0),$$

we get

$$\begin{aligned} s_p^q(x) &\leqslant \sum_{x_i \in T_0} \|x - x_i\|^{q+1} + C_1 \frac{r^p}{|\log r|} \sum_{j=1}^{N} \sum_{x_i \in T_j} \|x - x_i\|^{q+1-p} \\ &\leqslant M r^{q+1} \left(1 + C_2 \frac{1}{|\log r|} \sum_{j=1}^{N} j^q \right). \end{aligned}$$

Finally, since

$$\sum_{j=1}^{N} j^q = O(N^{q+1}) = O(r^{-q-1}),$$

this last estimate yields (2.9). $\square$

THEOREM 2.4. *Assume* (A1) *and let f be of class $C^{q,1}$ in A, $q \geqslant 1$ and $p > s + 1$. Then, the first partial derivatives of $S_p^q f$ converge to f uniformly in A as $r \to 0$. More precisely, if μ denotes a multi-index with $|\mu| = 1$, then*

$$\left\| D^\mu (S_p^q f - f) \right\|_A \leqslant C \gamma^q M^2 |f|_{q,1} \frac{\varepsilon_p^q(r)}{r}.$$

Proof. Given any $x \in A$,

$$D^\mu S_p^q f(x) - D^\mu f(x) = \sum_i \left(\sum_{|\nu| \leqslant q-1} \frac{1}{\nu!} D^{\nu+\mu} f(x_i)(x - x_i)^\nu - D^\mu f(x) \right) w_i(x)$$
$$- \sum_i R_0(x; x_i) D^\mu w_i(x)$$
$$= -\sum_i R_\mu(x; x_i) w_i(x) - \sum_i R_0(x; x_i) D^\mu w_i(x).$$

Since

$$D^\mu w_i(x) = -p \frac{(x - x_i)^\mu |x - x_i|^{-p-2}}{\Sigma_j |x - x_j|^{-p}} + p \frac{1}{|x - x_i|^p} \frac{\Sigma_j (x - x_j)^\mu |x - x_j|^{-p-2}}{\left(\Sigma_j |x - x_j|^{-p}\right)^2},$$

Lemma 2.1 yields the estimate

$$\left| D^\mu (S_p^q f - f)(x) \right| \leqslant C |f|_{q,1} \gamma^q \left(s_p^{q-1}(x) + s_p^q(x) s_p^{-2}(x) \right).$$

Here $s_p^{-2}(x)$ is defined by (2.8) with $q = -2$, and $x \neq x_i$ $(1 \leqslant i \leqslant n)$. We also have

$$s_p^{-2}(x) = \frac{\Sigma_i \|x - x_i\|^{-p-1}}{\Sigma_i \|x - x_i\|^{-p}}$$
$$\leqslant \frac{1}{\|x - x_{i_0}\|} M \left(1 + C_1 \sum_{j=1}^N j^{s-p-2} \right) \leqslant \frac{MC_2}{\|x - x_{i_0}\|}.$$

Looking at the proof of the estimate (2.9) (see (2.10)), we have

$$s_p^q(x) s_p^{-2}(x) \leqslant CM^2 \frac{\varepsilon_p^q(r)}{r}.$$

Since for arbitrary values of p, s and q

$$s_p^{q-1}(x) \leqslant C_1 M \varepsilon_p^{q-1}(r) \leqslant C_2 M \frac{\varepsilon_p^q(r)}{r},$$

the theorem is proved. □

Remark 2.5. *A similar result holds for higher derivatives: Given any multi-index* μ *with* $|\mu| \leqslant q$ *and* $p > s + |\mu|$,

$$\left\| D^\mu (S_p^q f - f) \right\|_A \leqslant C \gamma^q M^{|\mu|} |f|_{q,1} \frac{\varepsilon_p^q(r)}{r^{|\mu|}}.$$

Generally, the derivatives of f in the data points are not known. The next theorem guarantees the same rate of convergence if all derivatives needed are approximated to the right order.

COROLLARY 2.6. *Assume* (A1), *let* f *be of class* $C^{q,1}$ *in* A *and let* $d^\nu f(x_i)$ *be approximations to* $D^\nu f(x_i)$, $|\nu| \leqslant q$, $x_i \in X$, *such that*

$$d^\nu f(x_i) = D^\nu f(x_i) + O(r^{q+1-|\nu|}), \qquad |\nu| \leqslant q, \; r \to 0.$$

If $\tilde{S}_p^q$ *denotes the interpolation operator*

$$\tilde{S}_p^q f(x) = \sum_i \sum_{|\nu| \leqslant q} \frac{1}{\nu!} d^\nu f(x_i)(x - x_i)^\nu w_i(x),$$

then

$$\|\tilde{S}_p^q f - f\|_A = M \cdot O\big(\varepsilon_p^q(r)\big), \qquad r \to 0.$$

Proof. It suffices to consider $S_p^q f - \tilde{S}_p^q f$. Now

$$\begin{aligned}\left|\left(S_p^q f - \tilde{S}_p^q f\right)(x)\right| &\leqslant C_0 \sum_{|\nu| \leqslant q} r^{q+1-|\nu|} \sum_i \|x - x_i\|^{|\nu|} w_i(x)\\ &\leqslant C_1 \sum_{|\nu| \leqslant q} r^{q+1-|\nu|} s_p^{|\nu|-1}(x).\end{aligned}$$

Since by the definition (2.8) $s_p^{-1}(x) \equiv 1$, let $\varepsilon_p^{-1}(r) \equiv 1$, apply (2.9) and note that for all $|\nu| \leqslant q$, $r^{q+1-|\nu|}\varepsilon_p^{|\nu|-1}(r) = O(\varepsilon_p^q(r))$. □

Remark 2.7. *Analogously to Remark* 2.5,

$$\left\|D^\mu\big(\tilde{S}_p^q f - f\big)\right\|_A = M^{|\mu|} \cdot O\big(\varepsilon_p^q(r)/r^{|\mu|}\big), \qquad r \to 0.$$

3. Negative Results. Theorem 2.3 gives no information if $p < s$. In this case Shepard's formula fails to converge in the sense that there is no function $\varepsilon_p^q(r)$ such that $\varepsilon_p^q(r) \to 0$ as $r \to 0$ and such that (2.5) holds. Theorem 3.2 deals with this case and, further, shows that Theorem 2.3 cannot be improved. In the following we will use the test functions

$$g_y^q(x) = |x - y|^{q+1} \qquad \big(g_y = g_y^0\big),$$

some properties of which are listed in the next lemma.

LEMMA 3.1. *The function g_y^q is of class $C^{q,1}$ in $\mathbf{R}^s$. If $q = 0$, then $\omega_{g_y}(\delta) = \delta$ ($\delta > 0$). If $q \geqslant 1$, then all partial derivatives of g_y^q exist at every point $x \neq y$, and up to order $q + 1$ they are continuous and bounded functions in $\mathbf{R}^s \setminus \{y\}$. Further*,

$$\sum_{|\nu| \leqslant q} \frac{1}{\nu!} D^\nu g_y^q(x)(y - x)^\nu = (-1)^q |x - y|^{q+1}.$$

Proof. Let us only prove the last identity for the case $y = 0$. Using polar coordinates with $r = r(x) = |x|$, it is easily seen that for every $j = 1, \ldots, q$,

$$\begin{aligned}\frac{1}{j!}\frac{\partial^j}{\partial r^j} g_0^q(x) &= \frac{1}{j!} \sum_{i_1, \ldots, i_j = 1}^{s} \frac{\partial^j g_0^q(x)}{\partial x_{i_1} \cdots \partial x_{i_j}} \frac{\partial x_{i_1}}{\partial r} \cdots \frac{\partial x_{i_j}}{\partial r}\\ &= \frac{1}{r^j} \sum_{|\nu| = j} \frac{1}{\nu!} D^\nu g_0^q(x) x^\nu.\end{aligned}$$

Now insert this identity into Taylor's formula for $p(r) = r^{q+1} = g_0^q(x)$:

$$0 = p(0) = \sum_{j=0}^{q} \frac{1}{j!} p^{(j)}(r)(-r)^j + (-r)^{q+1}. \quad □$$

THEOREM 3.2. *Let the compact domain $A \subset [0,1]^s$ satisfy the cone property* (A2), *let $X(n)$ denote an arbitrary set of n pairwise distinct data points $\{x_1, \ldots, x_n\}$ in A and let $r = n^{-1/s}$. Then, there is a positive constant c dependent only on p, s, q and A with the following property: given any set $X(n)$ (n sufficiently large) there is a function g of class $C^{q,1}$ in A such that*

$$\|S_p^q g - g\|_A \geqslant c|g|_{q,1}\varepsilon_p^q(r). \tag{3.1}$$

Here $\varepsilon_p^q(r)$ is defined by (2.6) if $p \geqslant s$ and $\varepsilon_p^q(r) = 1$ if $p < s$. Analogously, there is a function $g \in C^0(A)$ such that

$$\left\| S_p^0 g - g \right\|_A \geqslant c\omega_g\left(\varepsilon_p^0(r)\right). \tag{3.2}$$

Proof. The function g will be the test function $g_y^q(x) = |x-y|^{q+1}$, where the choice of the point $y \in A$ depends on the set $X(n)$. Since $|g_y^q|_{q,1}$ is bounded independently of $y \in A$, Lemma 3.1 implies that

$$\left\| S_p^q g_y^q - g_y^q \right\|_A \geqslant \left| S_p^q g_y^q(y) \right| = \sum_i |y - x_i|^{q+1} w_i(y)$$
$$\geqslant c_1 s_p^q(y) \geqslant c_0 |g_y^q|_{q,1} s_p^q(y)$$

with positive constants c_0 and c_1. Analogously, for $g_y \in C^0(A)$,

$$\left\| S_p^0 g_y - g_y \right\|_A \geqslant c s_p^0(y) = c\omega_{g_y}\left(s_p^0(y)\right).$$

Thus, it suffices to show the existence of a positive constant c such that for every set $X(n)$ (n sufficiently large) there is a point $y \in A \setminus X(n)$ satisfying the inequality

$$s_p^q(y) \geqslant c\varepsilon_p^q(r).$$

Let $N(n)$ be an increasing function

$$N\colon \mathbf{N} \to \mathbf{N} \quad \text{with } N(n) \to \infty \text{ as } n \to \infty,$$

which will be defined later. By virtue of the transformation $x \mapsto N(n)x$ it remains to show that

$$s_p^q(y) \geqslant cN^{q+1}\varepsilon_p^q(r), \tag{3.3}$$

where now y and the data points $x_1, \ldots, x_n$ range through the set $A_N = N(n)A \subset [0, N(n)]^s$. If $k \in \mathbf{N}^s$, let $Q(k)$ denote the half-open cube of radius $\frac{1}{2}$ and center k and note that

$$A_N \subset \bigcup_{\substack{k \in \mathbf{N}^s \\ 0 \leqslant \|k\| \leqslant N}} Q(k).$$

Following the ideas of D. J. Newman and T. J. Rivlin [9] for the univariate case, we construct a point $y \in A_N \setminus X(n)$ suitable for our purpose. Let

$$m = \operatorname{card}(Q(l) \cap X(n)) = \min_{Q(k) \subset A_N} \operatorname{card}(Q(k) \cap X(n)).$$

Since the number of half-open cubes $Q(k) \subset A_N$ increases asymptotically like $\mu(A_N) = \mu(A)N^s$, as $N \to \infty$ ($\mu(A)$ denotes the Lebesgue measure of A and is positive because of (A2)),

$$m \leqslant \frac{n}{\frac{1}{2}\mu(A)N^s} \quad (N \text{ sufficiently large}). \tag{3.4}$$

Assume $m \geqslant 1$, the trivial case $m = 0$ being considered later. Let

$$T = \left\{ x \in Q(l);\ \|x - l\| \leqslant \tfrac{1}{4},\ \|x - x_i\| \geqslant \tfrac{1}{4}(2m)^{-1/s} \text{ for every } x_i \in X(n) \right\}$$

and note that

$$\mu(T) \geqslant \left(\tfrac{1}{2}\right)^s - m\left(\tfrac{2}{4}(2m)^{-1/s}\right)^s = 1/2^{s+1}. \tag{3.5}$$

In order to simplify the notation, define

$$\sigma_1(y) = \sum_{x_i \in Q(l)} \|y - x_i\|^{-p} \quad \text{and} \quad \sigma_2(y) = \sum_{x_i \notin Q(l)} \|y - x_i\|^{q+1-p}.$$

Case 1 ($p > s$). Since

$$\int_T d\mu(x) \sum_{x_i \in Q(l)} |x - x_i|^{-p} \leqslant m \int_{|x| \geqslant (2m)^{-1/s}/4} d\mu(x) |x|^{-p}$$
$$= cm \int_{(2m)^{-1/s}/4}^{\infty} dr\, r^{s-p-1} = \frac{cm}{p-s}\left(4(2m)^{1/s}\right)^{p-s},$$

c being a positive constant arising from the introduction of polar coordinates, (3.5) guarantees the existence of a point $y \in T$ such that

$$\sigma_1(y) \leqslant cm^{p/s} \qquad (p > s). \tag{3.6}$$

Case 2 ($p = s$). Since $|x - x_i| \leqslant \sqrt{s}$ for every $x \in T$ and $x_i \in Q(l)$,

$$\int_T d\mu(x) \sum_{x_i \in Q(l)} |x - x_i|^{-p} \leqslant m \int_{(2m)^{-1/s}/4 \leqslant |x| \leqslant \sqrt{s}} d\mu(x) |x|^{-p}$$
$$= c_1 m \int_{(2m)^{-1/s}/4}^{\sqrt{s}} \frac{dr}{r} \leqslant c_2 m (1 + \log m).$$

Thus, (3.5) yields the existence of a point $y \in T$ such that

$$\sigma_1(y) \leqslant cm(1 + \log m) \qquad (p = s). \tag{3.7}$$

Case 3 ($p < s$).

$$\int_T d\mu(x) \sum_{x_i \in Q(l)} |x - x_i|^{-p} \leqslant mc_1 \int_0^{\sqrt{s}} dr\, r^{s-p-1} = mc_2,$$

thus there is a point $y \in T$ such that

$$\sigma_1(y) \leqslant cm \qquad (p < s). \tag{3.8}$$

If $m = 0$, the inequalities (3.6), (3.7) and (3.8) obviously hold for every $y \in T$: $\sigma_1(y) = 0$.

Following the proof of Theorem 2.3, let T_j be the half-open annulus with center l and radius j defined by

$$T_j = \bigcup_{\substack{k \in \mathbf{Z}^s \\ \|k - l\| = j}} Q(k).$$

Note that $\frac{1}{4}j \leqslant \|y - x_i\| \leqslant 2j$ for every $x_i \in X(n) \cap T_j$, while $\|y - x_i\| \geqslant \frac{1}{4}$ for every $x_i \in X(n)$. By virtue of (A2), we get a lower bound on card$(X(n) \cap T_j)$. For there are positive integers N_0, j_0 and $j(N) \geqslant \max(N/N_0, j_0)$ (each of them independent of y) such that the number of half-open cubes $Q(k)$ with $\|k - l\| = j$ contained in $N \cdot K(y) \subset A_N$ is bounded below by $[j/j_0]^{s-1}$ for every $j_0 \leqslant j \leqslant j(N)$, $N \geqslant N_0$. Thus, by the definition of m,

$$\operatorname{card}\left(X(n) \cap T_j\right) \geqslant m[j/j_0]^{s-1} \quad \text{for } j_0 \leqslant j \leqslant j(N),\ N \geqslant N_0$$
$$\text{where } j(N) \geqslant \max(N/N_0, j_0).$$

Hence

$$\sigma_2(y) \geqslant cm \sum_{j=j_0}^{j(N)} j^{s-p+q}, \tag{3.9}$$

and, obviously,

$$s_p^q(y) \geqslant \frac{\sigma_2(y)}{\sigma_1(y) + 4^{q+1}\sigma_2(y)}. \tag{3.10}$$

Note that $t/(a+bt)$ is strictly increasing in $t > 0$ $(a, b > 0)$.

Case 1 $(p > s)$. If $m > 0$, insert (3.6) and (3.9) in (3.10) and cancel m. Then (3.4) implies that

$$s_p^q(y) \geqslant c\frac{\sum_{j=j_0}^{j(N)} j^{s-p+q}}{\left(n^{1/s}/N\right)^{p-s} + \sum_{j=j_0}^{j(N)} j^{s-p+q}}. \tag{3.11}$$

Case 1.1 $(p - s > q + 1)$. Define $N(n) = [n^{1/s}]$. Since $\sum j^{s-p+q}$ is bounded, by (3.11),

$$s_p^q(y) \geqslant c \geqslant c\frac{N^{q+1}}{\left(n^{1/s}\right)^{q+1}} = cN^{q+1}\varepsilon_p^q(r).$$

Hence, (3.3) is proved if $m \neq 0$. Otherwise, (3.10) implies that $s_p^q(y) \geqslant 4^{-q-1} \geqslant cN^{q+1}\varepsilon_p^q(r)$.

Case 1.2 $(p - s = q + 1)$. Define $N(n) = [n^{1/s}/\log n]$ and note that asymptotically

$$\log N \sim \frac{1}{s}\log n, \qquad (\log n)^{-q} \sim N^{q+1}\frac{\log n}{\left(n^{1/s}\right)^{q+1}}$$

and that

$$\sum_{j=j_0}^{j(N)} j^{s-p+q} \geqslant c_1 \log j(N) \geqslant c_2 \log N.$$

Thus (3.11) implies that

$$s_p^q(y) \geqslant c_1\frac{1}{(\log n)^{p-s-1} + 1} \geqslant c_2(\log n)^{-q} \geqslant c_3N^{q+1}\varepsilon_p^q(r)$$

if $m \neq 0$. Otherwise, by (3.10),

$$s_p^q(y) \geqslant 4^{-q-1} \geqslant c_2(\log n)^{-q} \geqslant c_3N^{q+1}\varepsilon_p^q(r).$$

Case 1.3 $(p - s < q + 1)$. Note that

$$\sum_{j=j_0}^{j(N)} j^{s-p+q} \geqslant cN^{s-p+q+1},$$

hence, by (3.11),

$$s_p^q(y) \geqslant c\frac{N^{q+1}}{\left(n^{1/s}\right)^{p-s} + N^{q+1}}.$$

Now define $N(n) = [(n^{1/s})^{(p-s)/(q+1)}]$ in order to get

$$s_p^q(y) \geqslant cN^{q+1}\frac{1}{\left(n^{1/s}\right)^{p-s}} = cN^{q+1}\varepsilon_p^q(r)$$

if $m \neq 0$. Otherwise (3.10) yields the same result.

Case 2 ($p = s$). This time,

$$\sum_{j=j_0}^{j(N)} j^{s-p+q} \geqslant cN^{q+1}.$$

If $m > 0$ insert (3.7) and (3.9) in (3.10) and cancel m. Then (3.4) implies that

$$s_p^q(y) \geqslant c \frac{N^{q+1}}{1 + \log \dfrac{n^{1/s}}{N} + N^{q+1}}.$$

Define $N(n) = [(\log n)^{1/(q+1)}]$ such that

$$\log \frac{n^{1/s}}{N} \sim \frac{1}{s} \log n.$$

Hence,

$$s_p^q(y) \geqslant c \geqslant c \frac{N^{q+1}}{\log n} = cN^{q+1} \varepsilon_p^q(r),$$

the same result being true if $m = 0$.

Case 3 ($p < s$). Insert (3.8) and (3.9) in (3.10) to get

$$s_p^q(y) \geqslant 2c \frac{\sum_{j=j_0}^{j(N)} j^{s-p+q}}{1 + \sum_{j=j_0}^{j(N)} j^{s-p+q}} \geqslant c$$

if $m \neq 0$ and N is sufficiently large. If $m = 0$, $s_p^q(y) \geqslant c$ as well. Thus, in this case, the function $N(n)$ is defined as a constant N_1 sufficiently large, and we get (3.1) and (3.2) ($c > 0$):

$$\|S_p^q g_y^q - g_y^q\|_A \geqslant c|g_y^q|_{q,1}, \qquad \|S_p^q g_y - g_y\|_A \geqslant c\omega_{g_y}(1). \quad \square$$

Remark 3.3. The estimate (3.1) is trivial if $p - s > q + 1$. Since $\mu(A) > 0$, there is a positive constant c such that for all sets $X(n)$ of n data points of A there is a point $y \in A$ with

$$\|y - x_i\| \geqslant \frac{c}{n^{1/s}} \quad \text{for all } x_i \in X(n).$$

If $v_i(x) \geqslant 0$, $\sum v_i \equiv 1$ and $v_i(x_j) = \delta_{ij}$, then the approximating power of the interpolating operator

$$S^q f(x) = \sum_{i=1}^{n} \sum_{|\nu| \leqslant q} \frac{1}{\nu!} D^\nu f(x_i)(x - x_i)^\nu v_i(x)$$

cannot exceed r^{q+1}. For, if $q \geqslant 1$,

$$\|S^q g_y^q - g_y^q\|_A \geqslant \sum_i |y - x_i|^{q+1} v_i(x) \geqslant cr^{q+1} \geqslant c_1 |g_y^q|_{q,1} r^{q+1}.$$

Analogously, the estimate $\|S^0 g_y - g_y\|_A \geqslant c\omega_{g_y}(r)$ shows that there exists no stable interpolation operator S^0 in $C^0(A)$ whose approximating power is better than $O(n^{-1/s})$.

Example 3.4. Let A be the following two-dimensional compact domain which fails to satisfy the cone property (A2):

$$A = \{(x, y) \in [0,1]^s;\ x \leqslant y^2\}.$$

For every $N \in \mathbf{N}$, consider the set X of data points

$$X = \left\{ (x_i, y_j) = \left(\frac{i}{N}, \frac{j}{N} \right) \in A;\ i, j = 0, 1, \ldots, N \right\},$$

thus $r = 1/2N$ and $M = 4$. We note that if $j < \sqrt{N}$, no data point except for $(0, y_j)$ lies on the straight line $y = y_j$, and if $j \geqslant \sqrt{N}$, exactly $[j^2/N] + 1$ data points will do. Thus card X is asymptotic to $\frac{1}{3}N^2$. Let $f(x, y) = g^q_{(0,r)}(x, y)$ and apply Lemma (3.1) to get

$$\left|\left(S^q_p f - f\right)(0, r)\right| \geqslant c_1 r^{q+1} \frac{\sum_{1 \leqslant j < \sqrt{N}} j^{q+1-p} + r \sum_{\sqrt{N} \leqslant j \leqslant N} j^{q+3-p}}{\sum_{1 \leqslant j < \sqrt{N}} j^{-p} + r \sum_{\sqrt{N} \leqslant j \leqslant N} j^{2-p}}.$$

The preceding inequality holds with the direction reversed, if c_1 is replaced by a positive constant c_2. If $p = 2$ and $r \to 0$, the denominator is bounded, while the numerator behaves like r^{-q-1}. Thus Shepard's formula fails to converge like $\varepsilon^q_2(r) = |\log r|^{-1}$:

$$\| S^q_2 f - f \|_A \geqslant c \qquad (r \to 0). \quad \square$$

Example 3.5. Let A be the unit interval $[0, 1] \subset \mathbf{R}^1$, and let (j_n) be a sequence in $\mathbf{N}$ converging to $+\infty$ $(n \to \infty)$ such that $1 < j_n < n$ and $M_n = [j_n^p] = O(n)$. The set X will be defined as follows: $x_i = i/n$ for $1 \leqslant i \leqslant n$, while all the data points x_i for $n < i \leqslant n + M_n$ are arbitrarily spaced in the open interval (x_{j_n}, x_{j_n+1}), thus $r = 1/2n$ and $M = M_n + 2$. Then Lemma 3.1 implies that $(f = g^q_r)$

$$\left|\left(S^q_p f - f\right)(y)\right| \geqslant \frac{\sum_{i=3}^{n+M_n} |x - x_i|^{q+1-p}}{2(1/r)^p + \sum_{i=3}^{n+M_n} |x - x_i|^{-p}}$$

$$\geqslant c r^{q+1} \frac{\sum_{i=3}^{n} i^{q+1-p} + M_n j_n^{q+1-p}}{\sum_{i=3}^{n} i^{-p} + M_n j_n^{-p}}.$$

Case 1.1 $(p - 1 > q + 1)$. Since $\sum i^{-p}$ and $\sum i^{q+1-p}$ are bounded, we get

$$\| S^q_p f - f \|_A \geqslant c |f|_{q,1} \varepsilon^q_p(r) M_n^{(q+1)/p} \qquad (r \to 0, n \to \infty).$$

Case 1.2 $(p - 1 = q + 1)$. $\sum i^{-p}$ is bounded, while $\sum i^{q+1-p}$ diverges like $\log n$. Choosing (j_n) such that

$$\frac{M_n^{(q+1)/p}}{\log n} \to +\infty \quad \text{if } n \to \infty,$$

we get

$$\| S^q_p f - f \|_A \geqslant c |f|_{q,1} \varepsilon^q_p(r) \frac{M_n^{(q+1)/p}}{\log n} \qquad (r \to 0, n \to \infty).$$

In both cases Shepard's formula fails to converge like $\varepsilon^q_p(r)$, thus showing that the factor M in (2.5) and (2.7) is inevitable. $\square$

Remark 3.6. If $f(x, y) = y^{q+1}$ in Example 3.3 and $f(x) = x^{q+1}$ in Example 3.4, we get the same estimates as before. Essentially, this works because each term except the first in the numerator of $S_p^q f - f$, evaluated in $(0, r)$ (Example 3.3) or in r (Example 3.4), has the same sign. This argument shows that the results of Theorem 2.3 cannot be improved for functions of higher differentiability, not even for polynomials.

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