BIRKBECK

(UNIVERSITY OF LONDON)

ENTRANCE EXAMINATION

Department of Economics, Mathematics and Statistics

Solutions

1 a Want to proof that $2n^3+3n^2+n$ is divisible by 6 for all positive integers n.

ie $\forall n \in \mathbb{Z}^+$, $\frac{2n^3+3n^2+n}{6} = j$, $j \in \mathbb{Z}$.

By induction,

Base: Let $n=1$, for $n \in \mathbb{Z}$

$\Rightarrow 2(1)^3+3(1)^2+1 = 2+3+1=6$

Now $6/6 = 1 \in \mathbb{Z}$, hence true for $n=1$.

Hypothesis: Let's assume true for $n=k$ for $k \in \mathbb{Z}$

ie $\frac{2k^3+3k^2+k}{6} = j$; $j \in \mathbb{Z}$ (divisible by 6).

Want to show that ~~the k~~ it is true for $n=k+1$.

ie $2(k+1)^3+3(k+1)^2+k+1$

$= 2(k^3+3k^2+3k+1)+3(k^2+2k+1)+k+1$

$\cancel{= 2k^3+6k^2+6k+2+3k^2+6k+2+k+1}$

$= 2k^3+6k^2+6k+2+3k^2+6k+3+k+1$

$= 2k^3+9k^2+13k+6$

Now since $2k^3+3k^2+k$ is divisible 6;

we have $(2k^3+3k^2+k)+6k^2+12k+6$

$= (2k^3+3k^2+k)+6(k^2+2k+1)$

$= (2k^3+3k^2+k)+6(k+1)^2$

Con't'd

① ⓐ for any $k \in \mathbb{Z}$, $(k+1)^2 \in \mathbb{Z}$

Hence $6(k+1)^2$ is divisible by 6.

$\Rightarrow$ Assumption holds for $k+1$.

$\therefore$ $2n^3+3n^2+n$ is divisible by 6.

① ⓑ Counterexamples

(i) $\forall n \in \mathbb{Z}^+$; $2n^2+5$ is prime.

for $n=5$, $2n^2+5 = 2(5)^2+5$

$= 2(25)+5 = 50+5 = 55$.

Since 55 has factors 11, 5 inclusive, 55 is not prime.

Hence the counterexample is $\underline{\underline{n=5}}$

(ii) $\forall n \in \mathbb{Z}^+$; ~~$3n^2+2$~~ $3n^2-22n-16$ is not prime.

$\Rightarrow$ $3n^2-22n-16 = 3n^2-24n+2n-16$

$= 3n(n-8)+2(n-8) = (3n+2)(n-8)$.

for $n=9$, $(3(9)+2)(9-8) = (29)(1) = 29$.

Since 29 is a prime number, $\underline{\underline{n=9}}$, is a counterexample.

① ⓒ $z = \sqrt{3} - i$

Polar form

ie $z = r\cos\theta + i\sin\theta$.

$\Rightarrow r^2 = (\sqrt{3})^2 + (-1)^2 = 3+1$

$r^2 = 4$

$\Rightarrow r = \sqrt{4} = \underline{2}$

To find θ, we can use $\tan\theta = \frac{1}{\sqrt{3}}$ or ~~cos θ~~

$\Rightarrow \theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \pi/6$

$\therefore$ Polar form of $z = 2\left(\cos \pi/6 + i \sin \pi/6\right)$.

Next, Cube Roots of ~~the~~ z. let q denote the cube root

ie $q = \sqrt[3]{2}\left[\cos\left(\frac{\theta + 2\pi k}{3}\right) + i\sin\left(\frac{\theta + 2\pi k}{3}\right)\right]$

$= \sqrt[3]{2}\left[\cos\left(\frac{\pi/6 + 2\pi k}{3}\right) + i\sin\left(\frac{\pi/6 + 2\pi k}{3}\right)\right]$.

for $k=0$; $q = \sqrt[3]{2}\left[\cos\left(\frac{\pi}{6}\times\frac{1}{3}\right) + i\sin\left(\pi/6 \times \frac{1}{3}\right)\right]$.

$= \sqrt[3]{2}\left(\cos \pi/18 + i\sin \pi/18\right)$

for $k=1$; $q = \sqrt[3]{2}\left[\cos\left(\frac{\pi/6 + 2\pi}{3}\right) + i\sin\left(\frac{\pi/6 + 2\pi}{3}\right)\right]$

$= \sqrt[3]{2}\left[\cos\left(\frac{13\pi}{18}\right) + i\sin\left(\frac{13}{18}\pi\right)\right]$.

for $k=2$ $q = \sqrt[3]{2}\left[\cos\left(\frac{\pi/6 + 4\pi}{3}\right) + i\sin\left(\frac{\pi/6 + 4\pi}{3}\right)\right]$.

$= \sqrt[3]{2}\left[\cos\left(\frac{25}{18}\pi\right) + i\sin\left(\frac{25}{18}\pi\right)\right]$

① ⓓ

Ⓐ $A = \{2\sin x : x \in \mathbb{R}\}$.

for $x \in \mathbb{R}$, we know $-1 \leq \sin x \leq 1$

$\Rightarrow -2 \leq 2\sin x \leq 2$

Hence A is bounded above. The least upper bound is $\underline{\underline{2}}$

$B = \left\{\frac{2x+7}{x^2} : x \in \mathbb{R}, x \geq 1\right\}$.

for $x \geq 1$, $\Rightarrow 2x \geq 2$

$\Rightarrow 2x+7 \geq 9$

$\Rightarrow \frac{2x+7}{x^2} \geq \frac{9}{x^2}$

Hence B is not bounded above and has no least upper bound.

$C = \left\{\frac{2x+7}{x^2} \quad x \in \mathbb{R}; x > 0\right.$

for ~~$x>$~~ $x \in \mathbb{R}$; $x > 0 \Rightarrow 2x > 0$

$\Rightarrow 2x+7 > 7 \Rightarrow \frac{2x+7}{x^2} > \frac{7}{x^2}$

Hence C ~~has~~ is not bounded above

and there is no least upper bound

① ⓔ $\dfrac{2n^2-4n+3}{5n^2+n+2}$; Divide by highest power of n.(n^2)

ie $\dfrac{\frac{2n^2}{n^2}-\frac{4n}{n^2}+\frac{3}{n^2}}{\frac{5n^2}{n^2}+\frac{n}{n^2}+\frac{2}{n^2}} = \dfrac{2-\frac{4}{n}+\frac{3}{n^2}}{5+\frac{1}{n}+\frac{2}{n^2}}$

Now finding the limit as $n\to\infty$ and keep in mind $\lim\limits_{n\to\infty}\frac{1}{n}\approx 0$

Hence $\lim\limits_{n\to\infty}\left(\dfrac{2-\frac{4}{n}+\frac{3}{n^2}}{5+\frac{1}{n}+\frac{2}{n^2}}\right) = \dfrac{\lim\limits_{n\to\infty}\left(2-\frac{4}{n}+\frac{3}{n^2}\right)}{\lim\limits_{n\to\infty}\left(5+\frac{1}{n}+\frac{2}{n^2}\right)}$

$= \dfrac{\lim\limits_{n\to\infty}2-\lim\limits_{n\to\infty}\left(\frac{4}{n}\right)+\lim\limits_{n\to\infty}\left(\frac{3}{n^2}\right)}{\lim\limits_{n\to\infty}(5)+\lim\limits_{n\to\infty}\frac{1}{n}+\lim\limits_{n\to\infty}\left(\frac{2}{n^2}\right)}$ for $\lim\limits_{n\to\infty}\left(\frac{1}{n}\right)\approx 0$

So, we have $\underline{2/5}$ as the limit

Hence the Sequence converges at $\underline{2/5}$

② ⓐ (i) $8-3x > 14-5x$

i.e. $5x-3x > 14-8$

$2x > 6$

$x > 3$

$\therefore \{x : x > 3;\ x \in \mathbb{R}\}$.

(ii) $|x+3| \geq |2x-5|$

For absolute inequalities, square both sides.

$\Rightarrow (x+3)^2 \geq (2x-5)^2$

$\Rightarrow x^2+6x+9 \geq 4x^2-20x+25$

$\Rightarrow 4x^2-x^2-20x-6x+25-9 \leq 0$

$\Rightarrow 3x^2-26x+16 \leq 0$

$\Rightarrow \cancel{3x(x} 3x^2-24x-2x+16 \leq 0$

$\Rightarrow 3x(x-8)-2(x-8) \leq 0$

$\Rightarrow (3x-2)(x-8) \leq 0$

Now we test the various points, $0, \frac{2}{3}, 8$.

$\therefore \frac{2}{3} \leq x \leq 8$

2 ⓑ (i) $2\log x - 3\log(x+1) + \frac{1}{2}\log(x-1)$

$\Rightarrow \log x^2 - \log(x+1)^3 + \log(x-1)^{1/2}$

$\Rightarrow \log\left[\frac{x^2}{(x+1)^3}\cdot(x-1)^{1/2}\right]$

$\Rightarrow \log\left(\frac{x^2\sqrt{x-1}}{(x+1)^3}\right)$

2. (b) (ii) $(2+3i)(4-7i)$

$= 2(4-7i) + 3i(4-7i)$

$= 8 - 14i + 12i + 21$

$= 29 - 2i$

$\frac{2+3i}{4-7i} \times \frac{4+7i}{4+7i} = \frac{8+12i+14i-21}{16+49}$

$= \frac{8-21+26i}{65} = \frac{-13+26i}{65}$

$= \frac{13(-1+2i)}{13(5)} = \frac{-1+2i}{5}$

2(c) $z = u^3 e^{-4v}$ where $u = \sqrt{t}$ and $v = \frac{1}{t}$.

Now $\frac{dz}{dt} = \frac{dz}{du} \cdot \frac{du}{dt}$; $\frac{du}{dt} = \frac{1}{2\sqrt{t}}$, $\frac{dv}{dt} = \frac{-1}{t^2}$

~~$\frac{dz}{du} = u^3 \frac{d}{du}(e$~~ since $v = \frac{1}{t}$ and $u = \sqrt{t} \Rightarrow u^2 = t$

then $v = \frac{1}{u^2}$. then $z = u^3 e^{-4(1/u^2)}$

$\Rightarrow z = u^3 e^{-1/u}$

$\frac{dz}{du} = u^3 \frac{d}{du}(e^{-1/u}) + e^{-1/u}\left(\frac{d}{du}(u^3)\right)$

$= u^3\left(\frac{-1}{u^2}\right)e^{-1/u} + e^{-1/u}(3u^2)$

$= -ue^{-1/u} + 3u^2 e^{-1/u}$

Hence $\frac{dz}{dt} = \left[-(\sqrt{t})e^{-1/\sqrt{t}} + 3(\sqrt{t})^2 e^{-1/\sqrt{t}}\right] \cdot \left[\frac{1}{2\sqrt{t}}\right]$

$= \frac{-1}{2}e^{-1/\sqrt{t}} + \frac{3\sqrt{t}}{2} e^{-1/\sqrt{t}}$

$= \frac{1}{2}e^{-1/\sqrt{t}}\left[1 + 3\sqrt{t}\right]$

2. (d) (i) $\int \frac{3+x^2}{\sqrt{x}}\,dx = \int \frac{3}{\sqrt{x}}\,dx + \int \frac{x^2}{\sqrt{x}}\,dx$

$= \int 3(x^{-1/2})\,dx + \int x^{3/2}\,dx = 3\left[2x^{1/2}\right] + \frac{2}{5}x^{5/2} + c$

$= 6x^{1/2} + \frac{2}{5}x^{5/2} + C = 2\sqrt{x}\left[3 + \frac{1}{5}x^2\right] + c$

(ii) $\int \frac{\cos 3x}{(2+\sin 3x)^2}\,dx$; let $u = 2+\sin 3x$

$\Rightarrow \frac{du}{dx} = 3\cos 3x$

$\Rightarrow dx = \frac{du}{3\cos 3x}$

$\Rightarrow \int \frac{\cos 3x}{(2+\sin 3x)^2}\,dx$

$= \int \frac{\cancel{\cos 3x}}{u^2}\,\frac{du}{3\cancel{\cos 3x}} = \frac{1}{3}\int \frac{du}{u^2}$

$= \frac{1}{3}\left(\frac{-1}{u}\right) + c$; Substituting $u = 2+\sin 3x$.

$= \frac{1}{3}\left[\frac{-1}{2+\sin 3x}\right] + c$

$= \frac{-1}{3(2+\sin 3x)} + C$

2 (d) (iii) $\int x\sqrt{2+x}\,dx$

let $u = \sqrt{2+x}$ $\Rightarrow$ $u^2 = 2+x$ and $x = u^2-2$

$\frac{du}{dx} = \frac{1}{2\sqrt{2+x}} \Rightarrow dx = 2\sqrt{2+x}\,du$

$\Rightarrow dx = 2u\,du.$

$\therefore \int x\sqrt{2+x}\,dx = \int (u^2-2)u(2u)\,du$

$\Rightarrow \int (u^2-2)\,2u^2\,du = \int 2u^4 - 4u^2\,du.$

$\Rightarrow \frac{2}{5}u^5 - \frac{4}{3}u^3 + c$ at $u = \sqrt{2+x}$

$\Rightarrow \frac{2}{5}(\sqrt{2+x})^5 - \frac{4}{3}(\sqrt{2+x})^3 + c$

(e) $W = (x^2+2xy+2yz+z^2)^k$

for $k \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{R}^+$

$\frac{\partial W}{\partial x} = k(2x+2y)W^{k-1}$, $\frac{\partial W}{\partial y} = k(2x+2z)W^{k-1}$

$\frac{\partial W}{\partial z} = k(2y+2z)W^{k-1}$

$\Rightarrow \frac{\partial W}{\partial x} + \frac{\partial W}{\partial y} + \frac{\partial W}{\partial z} = k(2x+2y)W^{k-1} + k(2x+2z)W^{k-1} + k(2y+2z)W^{k-1}$

$= kW$

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(2)(c) $W = (x^2 + 2xy + 2yz + z^2)^k$

$$\frac{\partial W}{\partial x} = k(2x+2y)(x^2+2xy+2yz+z^2)^{k-1}$$

$$\frac{\partial W}{\partial y} = k(2x+2z)(x^2+2xy+2yz+z^2)^{k-1}$$

$$\frac{\partial W}{\partial z} = k(2y+2z)(x^2+2xy+2yz+z^2)^{k-1}$$

$$\Rightarrow \frac{\partial W}{\partial x} + \frac{\partial W}{\partial y} + \frac{\partial W}{\partial z} = k(x^2+2xy+2yz+z^2)^{k-1}\left[2x+2y+2x+2z+2y+2z\right]$$

$$= \frac{k(x^2+2xy+2yz+z^2)^k\left[4x+4y+4z\right]}{x^2+2xy+2yz+z^2}$$

$$= \frac{4k(x+y+z)W}{x^2+2xy+2yz+z^2}$$

$$\frac{4kW(x+y+z}{x^2+2xy+2yz+z^2}$$

③ ⓐ $X = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{pmatrix}$.

$$|X| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{vmatrix} = |(1)(6-16) - (1)(4-6) + (1)(16-9)|$$

$$= |-10+2+7| = \underline{\underline{1}}.$$

$$\text{Cofactor} = \begin{bmatrix} 6-16 & 4-6 & 16-9 \\ 2-8 & 2-3 & 8-3 \\ 2-3 & 2-2 & 3-2 \end{bmatrix} = \begin{bmatrix} -10 & -2 & 7 \\ -6 & -1 & -5 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\text{Adj}(X) = \begin{bmatrix} -10 & 6 & 1 \\ 2 & -1 & 0 \\ 7 & 5 & 1 \end{bmatrix}$$

$$X^{-1} = \begin{bmatrix} -10 & 6 & 1 \\ 2 & -1 & 0 \\ 7 & 5 & 1 \end{bmatrix}$$

③ ⓑ $u = (2, 3, 4)$ and $v = (1, k, 1)$.

$\Rightarrow u \cdot v = 0$

$\Rightarrow 2(1) + 3(k) + 4(1) = 0$

$\Rightarrow 2 + 3k + 4 = 0$

$\Rightarrow 3k = -6$

$\Rightarrow \underline{\underline{k = -2}}$

Hence $\underline{\underline{v = (1, -2, 1)}}$.

3 (b) (ii) $u = (2,3,4)$ $v = (1,-2,1)$; let the nonzero vector be (i, j, k).

ie $\begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ 1 & -2 & 1 \end{vmatrix} = i(3+8) - j(2-4) + k(-4-3).$

$= 11i + 2j - 7k$

Now we find mod of coefficients.

$\Rightarrow \sqrt{11^2 + 2^2 + (-7^2)} = \sqrt{121+4+49} = 13$

Hence the vector is $\left(\frac{11}{13}, \frac{2}{13}, \frac{-7}{13}\right)$

3 (c). Eigenvalues for A

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}.$

$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)(2-\lambda)$

$\Rightarrow \lambda = 1, \lambda = 1,$ or $\lambda = 2.$

$\therefore$ Eigenvalues of A are 1, 1, and 2.

Eigenvalue for B

$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}.$

$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)(1-\lambda)$

$\Rightarrow \lambda = 1, \lambda = 1, \lambda = 2$

Eigenvalues of B are 1, 1, 2.

④ ⓐ (i) $\sum_{r=1}^{n} r(r+2) = \sum_{r=1}^{n} (r^2 + 2r)$

$= \sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r = \frac{n(n+1)(2n+1)}{6} + \cancel{2}\left[\frac{n(n+1)}{\cancel{2}}\right]$

$= n(n+1)\left[\frac{2n+1}{6} + 1\right] = n(n+1)\left[\frac{2n+7}{6}\right]$.

(ii) $\cancel{\sum_{r=1}^{n}}$ $\sum_{r=1}^{n} \frac{1}{r(r+2)}$. Since $\sum_{r=1}^{n} r(r+2) = \frac{n(n+1)(2n+1)}{6} + n(n+1)$

$= \frac{\sum_{r=1}^{n} 1}{\sum_{r=1}^{n} r(r+2)} = \frac{n}{n(n+1)\left[\frac{2n+7}{6}\right]} = n \div n(n+1)\left[\frac{2n+7}{6}\right]$

$= \frac{6n}{n(n+1)(2n+7)} = \frac{6}{n(n+1)(2n+7)}$

④ ⓑ (i) ~~For~~ If repetitions are allowed, the first digit (digit in hundredt thousandth position) has 4 possibilities (excluding 0 and 5) and the last digit has only 3 possibilities because it has to be an odd number (5, 7, 9). The rest of the positions have 6 possibilities each.

Hence: $4 \times 6 \times 6 \times 6 \times 6 \times 3$

$= 15,552$

④ b(ii) If repetitions are not allowed, with the preamble from (i), a few changes need to happen, the various combinations of the first and last digit to make the figure greater than 600,000 and odd, will mean there are 10 ways to combine the first and last digits. With that said, any time they are filled, the ~~second pos~~ rest of the poisitions have lesser possibilities.

ie: $10 \times 4 \times 3 \times 2 \times 1 = 240$

4 (c) $11^3 \bmod 37 = (11^2 \bmod 37 \times 11 \bmod 37) \bmod 37$

$= (10 \times 11) \bmod 37 = 110 \bmod 37 = 36.$

$31^4 \bmod 37 = (31^2 \bmod 37 \times 31^2 \bmod 37) \bmod 37$

$= (961 \bmod 37 \times 961 \bmod 37) \bmod 37 = (36 \times 36) \bmod 37$

$= 1$

Now $31^{29} + 48^{29}$; $29 = 1 + 4 + 8 + 16$.

Hence

$31^{29} \bmod 37 = (31 \bmod 37 \times 31^4 \bmod 37 \times 31^8 \bmod 37 \times 31^{16} \bmod 37) \bmod 37$

$= (31 \times 1 \times (31^4 \bmod 37)^2 \bmod 37 \times (31^4 \bmod 37)^4 \bmod 37) \bmod 37$

$= (31 \times 1 \times 1 \bmod 37 \times 1 \bmod 37) \bmod 37.$

$= 31$

$48^{29} \bmod 37 = (48 \bmod 37)^{29} \bmod 37 = 11^{29} \bmod 37.$

$\Rightarrow 11^{29} \bmod 37 = (11 \bmod 37 \times 11^2 \bmod 37 \times 11^3 \bmod 37 \times 11^6 \bmod 37 \times 11^8 \bmod 37 \times 11^9 \bmod 37) \bmod 37$

$= (11 \times 10 \times 36 \times 36^2 \times 10^4 \times 36^3) \bmod 37.$

$= (110 \bmod 37 \times 10^4 \bmod 37 \times 36^6) \bmod 37$

$= (36^7 \times 10^4 \bmod 37) \bmod 37.$

$= (11 \times 26) \bmod 37 = 27$

Hence $(31^{29} + 48^{29}) \bmod 37 = (31 + 27) \bmod 37 = 58 \bmod 37 = 21.$

(d)

5. (a)

(i) By Taylor Series

$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \ldots$

where $f', f'', f''', \ldots$ are the derivatives of f.

(ii) $f(x) = \sin x$ and $x_0 = 0$.

$f(x_0) = \sin(x_0)$; for $f'(x_0) = \cos(x_0)$.

$f''(x_0) = -\sin(x_0)$; $f'''(x_0) = -\cos(x_0)$.

$f^{(4)}(x_0) = \sin(x_0)$; $f^{(5)}(x_0) = \cos(x_0)$.

$f^{(6)}(x_0) = -\sin(x_0)$; $f^{(7)}(x_0) = -\cos(x_0)$

$f^{(8)}(x_0) = \sin(x_0)$.

The pattern repeats after 3 consecutive deritives.

Now at $x_0 = 0$

$f(0) = \sin(0) = 0$; $f'(0) = \cos(0) = 1$; $f''(0) = -\sin(0) = 0$

$f^{(3)}(0) = -\cos(0) = -1$; $f^{(4)}(0) = \sin(0) = 0$

Hence the coefficient is $(-1)^n$, where $n = 2k$ for even, and $2k+1$ for odd.

Substituting this into the general formula from (i); we have

$f(x) = \sin x = 0 + (1)(x-0) + \frac{0}{2!}(x-0)^2 + \frac{(-1)}{3!}(x-0)^3 + \frac{0}{4!}(x-0)^4 + \ldots$

$= 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \ldots$

In that pattern; we have

$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots$

~~ie~~ $\cancel{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n x^{2n+1}}$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

From the last term $\frac{x^{2n+1}}{(2n+1)!}$; we have $\frac{x^{2n} \cdot x}{(2n+1)!}$

⑤ a (ii) Cont'd

we have $\dfrac{x^{2n} \cdot x}{(2n+1)!}$

if $n=0$, $\Rightarrow \dfrac{x}{1!} = x$.

If $n=1$, $\Rightarrow \dfrac{x^3}{3!} = \dfrac{x^3}{6} = \dfrac{x \cdot x \cdot x}{1 \times 2 \times 3}$

if $n=2$ $\Rightarrow \dfrac{x^5}{5!} = \dfrac{x^5}{5!} = \dfrac{x \cdot x \cdot x \cdot x \cdot x}{1 \times 2 \times 3 \times 4 \times 5}$;

Hence

$$\frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1} \cdot \frac{x}{2} \cdot \frac{x}{3} \cdots \frac{x}{2n+1}$$

So $\lim\limits_{n \to \infty} \left(\dfrac{x}{2n+1}\right) = \lim\limits_{n \to \infty} \left(\dfrac{x/n}{2 + 1/n}\right)$; where $\{ \frac{1}{n} \}^{\infty} \to 0$

$$= \frac{0}{2+0} = \underline{\underline{0}}$$

Hence we can conclude that the Taylor Series converges to

0 as $n \to \infty$.

⑤ ⑥ $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 6e^{-3x}$

Kindly note that the second derivative $\frac{d^2y}{dx^2}$ was not typed completely. it omits the dx^2

Now let $r = \frac{dy}{dx}$.

$\Rightarrow \quad r^2 + 2r + 5 = 0$ for the Characteristic equation.

ie $\quad r = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2}$

$r = \frac{-2 \pm 4i}{2} = -1 + 2i$.

Since it is a complex root, the homogenous solution will be

$y = e^{-x}\left[c_1 \cos 2x + ic_2 \sin 2x\right]$ for some constants $c_1, c_2 \in \mathbb{R}$.

~~$\frac{dy}{dx} = (-1)e^{-x}\left[c_1 \cos 2x + ic_2 \sin 2x\right] + e^{-x}\left[-c_1 \sin 2x(2) + ic_2 \cos 2x(2)\right]$~~

~~$= e^{-x}\left[-2c_1 \sin 2x + 2ic_2 \cos 2x\right] - e^{-x}\left[c_1 \cos 2x + ic_2 \sin 2x\right]$~~

~~$= e^{-x}[$~~

For particular solution; we assume $y = axe^{-3x}$; $a \in \mathbb{R}$.

$\frac{dy}{dx} = -3axe^{-3x} + ae^{-3x}$ ~~$-3a$~~

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$\frac{d^2y}{dx^2} = 9axe^{-3x} - 3ae^{-3x} + a(-3)e^{-3x}$

$= 9axe^{-3x} - 6ae^{-3x} = 3ae^{-3x}[3x-2]$.

For particular solution, we assume $y = ae^{-3x}$; $a \in \mathbb{R}$

$\frac{dy}{dx} = -3ae^{-3x}$ $\qquad \frac{d^2y}{dx^2} = 9ae^{-3x}$

we have; $\quad 9ae^{-3x} + 2(-3)ae^{-3x} + 5ae^{-3x} = 6e^{-3x}$

$8ae^{-3x} = 6e^{-3}$

$a = \frac{6}{8} = \frac{3}{4}$

Hence, the general solution: $y = e^{-x}\left[c_1 \cos 2x + ic_2 \sin 2x\right] + \frac{3}{4}e^{-3x}$

5 (c) $V_n = \{a_0 + a_1 x^1 + \cdots + a_n x^n : a_0, a_1, \ldots a_n \in \mathbb{R}\}$

(i)

Let $f, g \in V_n$ and $a_i, b_i \in \mathbb{R}$ for $i = 0, 1, \ldots, n$.

$f = a_0 + a_1 x^1 + \cdots + a_n x^n$

$g = b_0 + b_1 x^1 + \cdots + b_n x^n$

If V_n is a subspace of $\mathbb{R}[x]$, then

~~1. $f+g \in \mathbb{R}$~~

~~2. $cf \in \mathbb{R}$~~

1. $f+g \in V_n$
2. $cf \in V_n$ for some $c \in \mathbb{R}$.

1. $f+g = a_0 + a_1 x^1 + \cdots + a_n x^n + b_0 + b_1 x^1 + \cdots + b_n x^n$

$= a_0 + b_0 + (a_1 + b_1) x^1 + (a_2 + b_2) x^2 + \cdots + (a_n + b_n) x^n$

Since $a_i + b_i \in \mathbb{R}$ for $i = 0, 1, 2, \ldots n$ and $x^i \in V_n$,

$\Rightarrow a_0 + b_0 + (a_1 + b_1) x^1 + \cdots + (a_n + b_n) x^n \in V_n$.

2. $cf = c(a_0 + a_1 x^1 + \cdots + a_n x^n)$.

$= ca_0 + ca_1 x^1 + \cdots + ca_n x^n$

Assume $ca_i = b_i$

$\Rightarrow cf = b_0 + b_1 x^1 + \cdots + b_n x^n$

Hence $V_n \subset \mathbb{R}[x]$

That is V_n is a subspace of $\mathbb{R}[x]$. ⊠

5 c (ii) For B_n to be a basis for V_n; we show that
1. B_n spans V_n
2. B_n is linearly independent

ie 1. Let $c_0, c_1, c_2, \ldots, c_n \in \mathbb{R}$.

such that $c_0 B_n$ is a vector.

$$\Rightarrow \quad c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$$

Since ~~the~~ above shows in the form of V_n,
B_n spans V_n.

2. ~~line~~ Let $c_0, c_1, c_2, \ldots, c_n \in \mathbb{R}$.

$$c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n = 0$$

$$\Rightarrow \quad c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n = 0 + 0x + 0x^2 + \ldots + 0x^n$$

$$\Rightarrow \quad c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$$

$$\Rightarrow \quad c_i = 0 \quad \text{for } i = 0, 1, 2, \ldots, n.$$

Hence it is sufficient to say B_n is a basis for V_n.

(5) c. (iii)

$$\delta : V_3 \longrightarrow V_2$$

let $f = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$g = b_0 + b_1 x + b_2 x^2 + b_3 x^3$

$$\frac{df}{dx} = a_1 + 2a_2 x + 3a_3 x^2$$

$$\frac{dg}{dx} = b_1 + 2b_2 x + 3b_3 x^2$$

If $\delta(f)$ is a linear transformation, then

1. $L(f+g) = L(f) + L(g)$
2. $L(cf) = c\,L(f)$.

(1) $L(f+g) = L(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + b_0 + b_1 x + b_2 x^2 + b_3 x^3)$

$= L(a_0 + b_0) + L[(a_1 + b_1)x] + L[(a_2 + b_2)x^2] + L[(a_3 + b_3)x^3]$

$= \frac{d}{dx}(a_0 + b_0) + \frac{d}{dx}[(a_1 + b_1)x] + \frac{d}{dx}[(a_2 + b_2)x^2] + \frac{d}{dx}[(a_3 + b_3)x^3]$

$= a_1 + b_1 + 2(a_2 + b_2)x + 3(a_3 + b_3)x^2$

$= a_1 + 2a_2 x + 3a_3 x^2 + b_1 + 2b_2 x + 3b_3 x^2$

$= L(f) + L(g)$.

(2) $L(cf)$, $c \in \mathbb{R}$.

$L(cf) = L[c(a_0 + a_1 x + a_2 x^2 + a_3 x^3)]$

$= L[ca_0 + ca_1 x + ca_2 x^2 + ca_3 x^3]$

$= L(ca_0) + L(ca_1 x) + L(ca_2 x^2) + L(ca_3 x^3)$

$= \frac{d}{dx}(ca_0) + \frac{d}{dx}[ca_1 x] + \frac{d}{dx}[ca_2 x^2] + \frac{d}{dx}[ca_3 x^3]$

$= ca_1 + 2ca_2 x + 3ca_3 x^2$

$= c(a_1 + 2a_2 x + 3a_3 x^2)$

$= c\,L(f)$ Hence δ is a linear transformation.

5 (c) (iv) $\ker f = \{L(f) = 0,\ f \in V_n\}$.

$\Rightarrow L(f) = a_1 + 2a_2 x + 3a_3 x^2 = 0$

$\Rightarrow a_1 = 0,\ 2a_2 = 0 \Rightarrow a_2\ 0$

$3a_3 = 0 \Rightarrow a_3 = 0$

hence for $L(f)$; The $\ker f = \{a_0\}$.

The nullity is $\underline{1}$.

$a_0 = b_1 \quad a_1 = 2b_2 \quad a_2 = 3b_3 \quad a_3 = 0$

$\Rightarrow \operatorname{Im} f = \{b_1 + 2b_2 + 3b_3 + 0\}$.

$= \{b_1 + 2b_2 x + 3b_3 x^2;\ b_1, b_2, b_3 \in R\}$.

Rank $= \dim(\operatorname{Im} f) = 3$

5 (c) (v) - For $B_2 = \{1, x, x^2, x^3\}$.

$a_0(1,0,0,0) + a_1(0,1,0,0) + a_2(0,0,1,0) + a_3(0,0,0,1)$.

Matrix $= a_0(1,0,0,0) + a_1(0,1,0,0)$

$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$.

For $B_2 = \{1, x, x^2\}$ for $V_2 = a_1 + 2a_2 x + 3a_3 x^2$

Matrix $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

6 a

$m \sim n : m^q = n$ for $q \in \mathbb{Q}^+$

(i) To show that $\sim$ is an equivalence relation on $\mathbb{Z}^+$, we show

① Reflexive : $a \sim b \in \mathbb{Z}^+$ for $a, b \in \mathbb{Z}^+$

② Symmetry if $a \sim b$ then $b \sim a$

③ Transitive : if $a \sim b$ and $b \sim c$ then $a \sim c$

① Reflexive

$a^q = b$ for $a, b \in \mathbb{Z}^+$.

if $q = 1$, $a^1 = b = a$, Hence true.

② Symmetry

$a \sim b \Rightarrow a^q = b$

$\Rightarrow a^{q \times 1/q} = b^{1/q}$

$\Rightarrow a = b^{1/q}$; let $1/q = r$; $1/q \in \mathbb{Q}^+$ so $r \in \mathbb{Q}^+$

$\Rightarrow b^r = a$

$\Rightarrow b \sim a$

(iii) Transitive : $a \sim b$ and $b \sim c$

$\Rightarrow a^q = b$ and $b^q = c$

$\Rightarrow (a^q = (b))^q \Rightarrow a^{q^2} = b^q$ since $b^q = c$.

$\Rightarrow a^{q^2} = c$ let $q^2 = p$; $p \in \mathbb{Q}^+$

$\Rightarrow a^p = c \Rightarrow a \sim c$.

Hence $\sim$ is an equivalence relation on $\mathbb{Z}^+$.

6. (b)

$\alpha = (125)(476)$

$B = (142\ (3567)$

$\alpha\beta = (26)(354)(7)$

$\alpha^{-1} = (152)(467)$

$\alpha^{2} = (152)(467)$

$(\beta\alpha) = (72154)(3)$

6.ⓒ To show that $\circ$ is a group under Q^* when $r \circ s$.
for $r, s \in Q^*$; we show that

① $\circ$ is associative
② $\circ$ has an identity element in Q^*
③ $\circ$ has an inverse element in Q^*

①

ie: $r \circ (s \circ t) = (r \circ s) \circ t$

$$\Rightarrow r \circ (s \circ t) = r \circ (3st) = 3r(3st).$$
$$= 9rst$$
$$= 3rs[3t].$$
$$= r \circ s\,[3t] = 3[r \circ s]t$$
$$= (r \circ s) \circ t$$

Hence, $\circ$ is associative.

②

ie $\exists$ an element $e \in Q^*$ such that $r \circ e = r$

$$\Rightarrow \quad r \circ e = 3re = r$$
$$\Rightarrow \quad e = r/3r = 1/3 \in Q^*$$

Hence, $\circ$ has an identity element.

③

ie For any $r \in Q^*$; $\exists$ an r^{-1} (called the inverse of r) such that
$r \circ r^{-1} = e$ where $r^{-1} \in Q^*$

$$\Rightarrow r \circ r^{-1} = 3rr^{-1} = 1/3$$
$$= r^{-1} = 1/3 \times \frac{1}{3r} = 1/9r \quad ; \text{ where } 1/9r \in Q^*.$$

Hence, for every $r \in Q^*$, there is $1/9r \in Q^*$; where $r^{-1} = 1/9r$.

$\therefore$ $\circ$ is a group under Q^*