Graph Theory

13.1. Introduction

Many situations that occur in computer Science, Physical Science, Communication Science, Economics and many other areas can be analysed by using techniques found in a relatively new area of mathematics called graph theory. The graphs can be used to represent almost any problem involving discrete arrangements of objects, where concern is not with the internal properties of these objects but with relationship among them. In this chapter, we begin with some basic graph terminology and then discuss some important concepts in graph theory with many applications of graphs.

13.2. Basic Terminology

A graph G consists of two sets :

(*i*) a non-empty set V whose elements are called **vertices, nodes or points** of G. The set $V(G)$ is called the **vertex set** of G.

(*ii*) a set E of edges such that each edge $e \in$ E associated with ordered or unordered pairs of elements of V. The set E(G) is called the **edge set** of G.

The graph G with vertices V and edges E is written as G = (V, E) or G(V, E).

If an edge $e \in$ E is associated with an ordered pair (u, v) or an unordered pair (u, v), where u, $v \in V$, then e is said to **connect** u and v and u and v are called end points of e. An edge is said to be incident with the vertices it joins. Thus, the edge e that joins the nodes u and v is said to be **incident** on each of its end points u and v. Any pair of nodes that is connected by an edge in a graph is called **adjacent nodes.**

In a graph a node that is not adjacent to another node is called an **isolated node.**

A graph G (V, E) is said to be **finite** if it has a finite number of vertices and finite number of edges. (A graph with a finite number of vertices must also have finite number of edges): otherwise. it is a **infinite** graph, if G is a finite, | V (G) | denotes the number of vertices in G and is called the order of G. Similarly if G is finite, | E (G) | denotes the number of edges in G and is called the size of G. We shall often refer to a graph of order n and size m an (n, m) graph. If G be a (p, q) graph then G has p vertices and q edges.

Although graphs are frequently stored in a computer as list of vertices and edges, they are pictured as diagrams in the plane in a natural way. Vertex set of graph is represented as a set of points in a plane and edge is represented by a line segment or an arc (not necessarily straight). The objects shown in Fig. 13.1 are graphs.

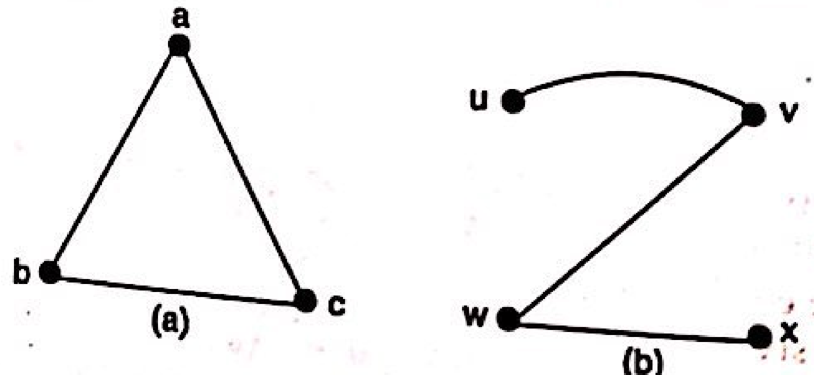


Fig. 13.1

It helps when discussing a graph to label each vertex, often with lower case letters as shown above. In Fig. 13.1 (a), V = {a, b, c,} and E = {(a, b), (a, c), (b, c)} the member of vertices and edges are |v(G)| = 3 and |E(G)| = 3 in this graph, the vertices a and b, a and c and b and c are adjacent vertices.

In Fig. 13.1 (b), V = {u, v, w, x} and E = {(u, v), (v, w), (w, x)}

Here vertices u and v, v and w, w and x are adjacent, whereas u and w, u and x and v and x are non adjacent. The number of vertices and edges are |v(G)| = 4 and |E(G)| = 3.

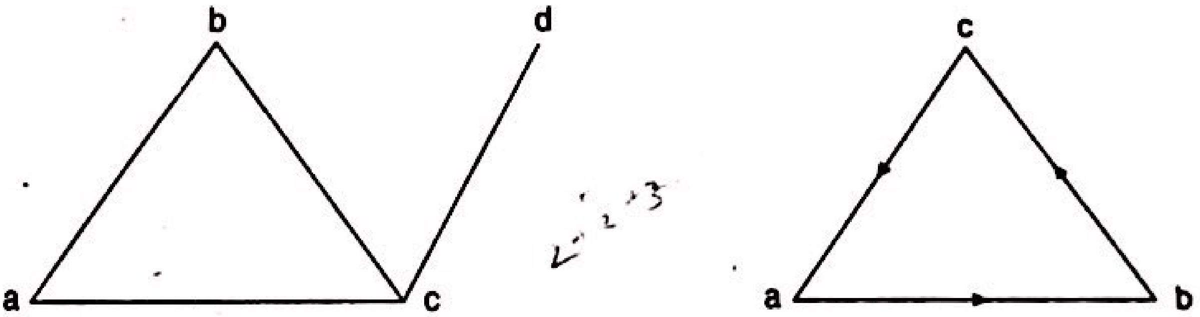
The definition of a graph contains no reference to the length or the shape and the positioning of the edge or arc joining any pair of nodes, nor does it prescribe any ordering of positions of the nodes. Therefore, for a given path, there is no unique diagram that represents the graph, and it can happen that two diagrams that look entirely different from one another may represent the same graph. It is to be noted that, in drawing a graph, it is immaterial whether the lines are drawn straight or curved, long or short, what is important is the incidence between edges and vertices are the same in both cases.

Undirected and Directed Graph

An **undirected** graph G consists of set V of vertices and a set E of edges such that each edge $e \in E$ is associated with an unordered pair of vertices.

Fig. 13.2 (a) is an example of an undirected graph we can refer to an edge joining the vertex pair i and j as either (i, j) or (j, i).

A **directed graph** (or **digraph**) G consists of a set V of vertices and a set E of edges such that $e \in E$ is associated with an ordered pair of vertices. In other words, if each edge of the graph G has a direction then the graph is called directed graph. In the diagram of directed graph, each edge $e = (u, v)$ is represented by an arrow or directed curve from initial point u of e to the terminal point v. Fig. 13.2 (b) is an example of a directed graph.



(a) Undirected graph (b) Directed graph

Fig. 13.2

Suppose $e = (u,v)$ is a directed edge in a digraph, then

(i) u is called the **initial vertex** of e and v is the **terminal vertex** of e

(ii) e is said to be **incident from** u and to be **incident** to v.

(iii) u is adjacent to v, and v is **adjacent from** u

In specifying any edge of a digraph by its end-points, the edge is understood to be directed from the first vertex towards the second.

Graphs, both directed and undirected, occur widely in all sorts of problems and before introducing more terminology we give examples of how graph arise in some familiar contexts.

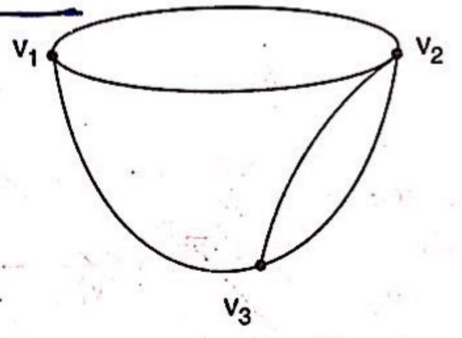
13.3. Simple Graph, Multigraph and Psuedograph

An edge of a graph that joins a node to itself is called a **loop** or **self loop** *i.e.*, a loop is an edge (v_i, v_j) where $v_i = v_j$.

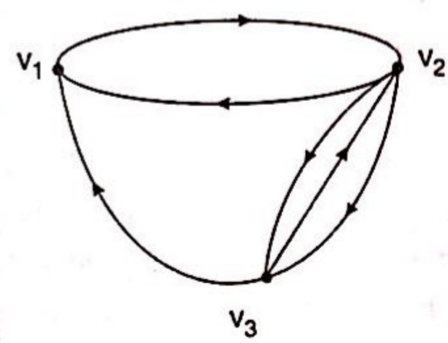
In some directed as well as undirected graphs, we may have contain pair of nodes joined by more than one edges, such edges are called **multiple** or **parallel** edges. Two edges (v_i, v_j) and (v_f, v_r) are parallel edges if $v_i = v_f$ and $v_j = v_r$. Note that in case of directed edges, the two possible edges between a pair of nodes which are opposite in direction are considered distinct. So more than one directed edge in a particular direction in the case of a directed graph is considered parallel.

A graph which has neither loops nor multiple edges *i.e.*, where each edge connects two distinct vertices and no two edges connect the same pair of vertices is called a **simple graph**. Fig. 13.2 (*a*) and (*b*) represents simple undirected and directed graph because the graphs do not contain loops and the edges are all distinct.

Any graph which contains some multiple edges is called a **multigraph.** In a multigraph, no loops are allowed.



(a) Undirected multigraph

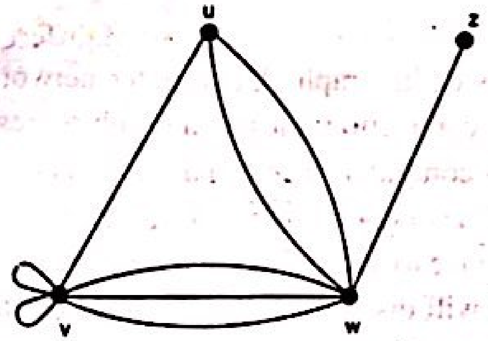


(b) Directed multigraph

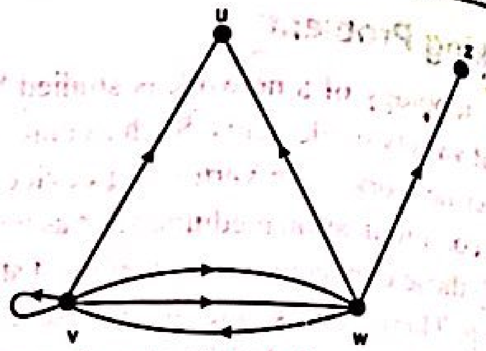
Fig. 13.7

In Fig. 13.7 (*a*) there are two parallel edges joining nodes v_1 and v_2 and v_2 and v_3. In Fig. 13.7. (*b*), there are two parallel edges associated with v_2 and v_3.

A graph in which loops and multiple edges are allowed, is called a **pseudograph.**



(a) Undirected pseudograph



(b) Directed pseudograph

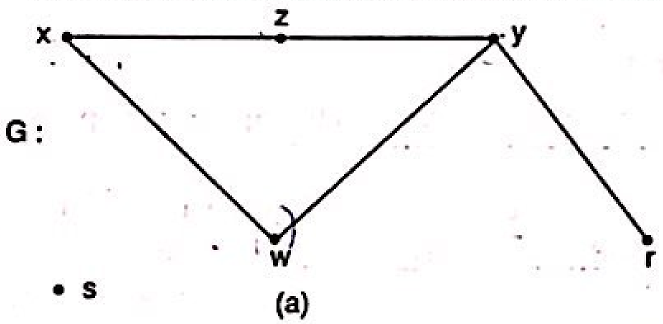
Fig. 13.8

It may be noted that there is some lack of standardisation of terminology in graph theory. Many words have almost obvious meaning, which are the same from book to book, but other terms are used differently by different authors.

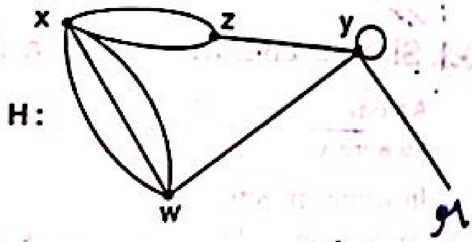
13.4. Degree of a Vertex

The degree of a vertex of an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v in a graph G may be denoted by $\deg_G(v)$.

The degrees of vertices in the graph G and H in Fig. 13.9 are given below.



(a)



(b)

Fig. 13.9

In G as shown in Fig. 13.9 (*a*)

$\deg_G(x) = 2 = \deg_G(z) = \deg_G(w), \deg_G(y) = 3$ and $\deg_G(r) = 1$ and in H as shown in Fig. 13.9 (*b*),
$\deg_H(x) = 5$, $\deg_H(z) = 3$, $\deg_H(y) = 5$, $\deg_H(w) = 4$ and $\deg_H(r) = 1$.

A vertex of degree 0 is called **isolated vertex**. A vertex is **pendant** if and only if it has a degree 1. Vertex s in the graph G is isolated and vertex r is pendant. A vertex of a graph is called odd vertex or even vertex depending on whether its degree is odd or even.

In any graph G, we define

$$\delta(G) = \min\{\deg v : \in V(G)\} \text{ and}$$
$$\Delta(G) = \max\{\deg v : \in V(G)\}$$

If $v_1, v_2, \ldots\ldots v_n$ are the n vertices of G, then the sequence $(d_1, d_2, \ldots\ldots d_n)$ where $d_i = \deg(v_i)$ is the degree sequence of G. In general, we order the vertices so that the degree sequence is monotonically increasing *i.e.*

$$\delta(G) = d_1 \le d_2 \le \ldots\ldots \le d_n = \Delta(G).$$

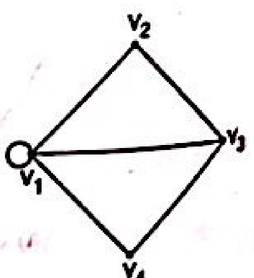


Fig. 13.10

For example, the degree sequence of the graph shown in Fig. 13.10 is (2,2,3,5) as $\deg(v_2) = \deg(v_4) = 2$, $\deg(v_3) = 3$ and $\deg(v_1) = 5$.

Theorem 13.1 A simple graph with at least two vertices has at least two vertices of same degree.

Proof. Let G be a simple graph with $n \ge 2$ vertices. The graph G has no loop and parallel edges.

Hence the degree of each vertices is $\leq n-1$. Suppose all the vertices of G are of different degrees. Hence the following degrees

$$0, 1, 2, 3, \ldots\ldots, n-1$$

are possible for n vertices of G. Let u be the vertex with degree 0. Then u is an isolated vertex. Let v be the vertex with degree $n-1$ then v has $n-1$ adjacent vertices. Since v is not an adjacent vertex of itself, therefore every vertex of G other than u is an adjacent vertex of G. Hence u cannot be an isolated vertex, this contradiction proves that a simple graph contains two vertices of same degree.

The converse of the above theorem is not true.

Theorem (the Handshaking theorem) 13.2. If G = (V, E) be an undirected graph with e edges. Then

$$\sum_{v \in V} \deg_G(v) = 2e$$

i.e., the sum of degrees of the vertices in an undirected graph is even.

Proof: Since the degree of a vertex is the number of edges incident with that vertex, the sum of the degree counts the total number of times an edge is incident with a vertex. Since every edge is incident with exactly two vertices, each edge gets counted twice, once at each end. Thus the sum of the degrees is equal twice the number of edges.

Note: This theorem applies even if multiple edges and loops are present. The above theorem holds this rule that if several people shake hands, the total number of hands shake must be even that is why the theorem is called handshaking theorem.

Corollary: In a non directed graph, the total number of odd degree vertices is even.

Proof: Let G = (V, E) a non directed graph. Let U denote the set of even degree vertices in G and W denote the set of odd degree vertices.

Then $\sum_{v_i \in V} \deg_G(v_i) = \sum_{v_i \in U} \deg_G(v_i) + \sum_{v_i \in W} \deg_G(v_i)$

$\Rightarrow$ $2e - \sum_{v_i \in U} \deg_G(v_i) = \sum_{v_i \in W} \deg_G(v_i)$ ...(1)

Now $\sum_{v_i \in U} \deg_G(v_i)$ is even as the sum of degrees of even degree vertices is always even.

Therefore, from (1)

$\sum_{v_i \in W} \deg_G(v_i)$ is even

$\therefore$ Since for each $v_i \in W$, $\deg_G(v_i)$ is odd, the number of odd vertices in G must be even.

In degree and out degree

In a directed graph G, the out degree of a vertex v of G, denoted by $\text{outdeg}_G(v)$ or $\deg_G^+(v)$, is the number of edges beginning at v and the indegree of v, denoted by $\text{indeg}_G(v)$ or $\deg_G^{-1}(v)$, is the number of edges ending at v. The sum of the in degree and out degree of a vertex is called the **total degree** of the vertex. A vertex with zero in degree is called a **source** and a vertex with zero out degree is called a **sink.** Since each edge has an initial vertex and terminal vertex, the immediate theorem follows:

Theorem 13.3. If G = (V,E) be a directed graph with e edges, then

$$\sum_{v \in V} \deg_G^+(v) = \sum_{v \in V} \deg_G^-(v) = e$$

i.e., the sum of the outdegrees of the vertices of a diagraph G equals the sum of in degrees of the vertices which equals the number of edges in G.

Proof : Any directed edge (u,v) contributes 1 to the in degree of u and 1 to the out degree of v. Further, a loop at v contributes 1 to the in degree and 1 to the out degree of v. Hence the proof.

In the directed graph G in Fig. 13.11.

$\text{Indeg}_G(a) = 2$, $\text{Indeg}_G(b) = 1$, $\text{Indeg}_G(c) = 2$, $\text{Indeg}_G(d) = 3$.

$\text{Outdeg}_G(a) = 1$, $\text{Outdeg}_G(b) = 5$, $\text{Outdeg}_G(c) = 1$, $\text{Outdeg}_G(d) = 1$.

Note that, the sum of the in degrees and the sum of the out degrees each equal to 8, the number of edges.

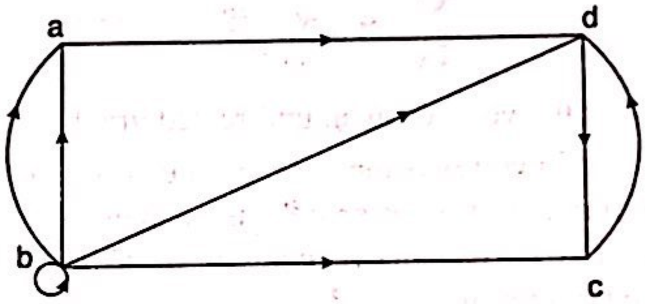
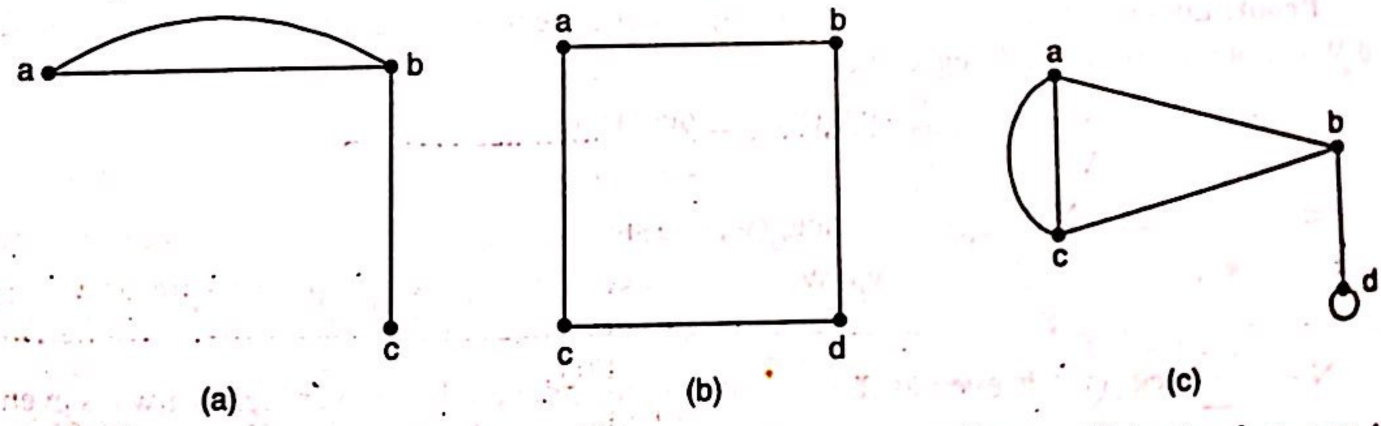


Fig. 13.11

SOLVED EXAMPLES

Example 1. State which of the following graphs are simple ?

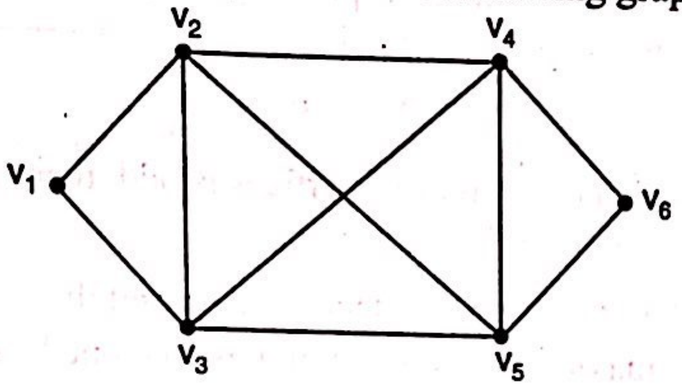


Solution. (*a*) The graph is not a simple graph, since it contains parallel edge between tv vertices *a* and *b*.

(*b*) The graph is a simple graph, it does not contain loop and parallel edge.

(*c*) The graph is not a simple graph since it contains parallel edge and a loop.

Example 3. Find the degree of each vertex of the following graph.

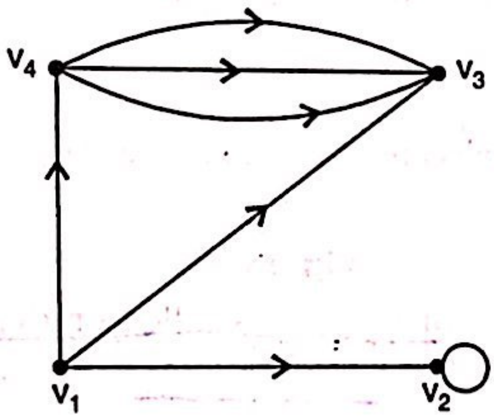


Solution. It is an undirected graph. Then

$\deg(v_1) = 2,\quad \deg(v_2) = 4,\quad \deg(v_3) = 4$

$\deg(v_4) = 4\quad \deg(v_5) = 4,\quad \deg(v_6) = 2$

Example 4. Find the in degree out degree and of total degree of each vertex of the following graph.



Solution. It is a directed graph

in deg $(v_1) = 0$	out degree $(v_1) = 3$	total deg $(v_1) = 3$
in deg $(v_2) = 2$	out degree $(v_2) = 1$	total deg $(v_2) = 3$
in deg $(v_3) = 4$	out degree $(v_3) = 0$	total deg $(v_3) = 4$
in deg $(v_4) = 1$	out degree $(v_4) = 3$	total deg $(v_4) = 4$

13.5. Types of Graphs

Some important types of graphs are introduced here. These graphs are often used as examples and arise in many applications.

Null Graph

A graph which contains only isolated node is called a null graph *i.e.*, the set of edges in a null graph is empty. Null graph is denoted on *n* vertices by N_n : N_4 is shown in Fig. 13.12. Note that each vertex of a null graph is isolated.

K_n = max. edge $n\frac{(n-1)}{2}$

degree = $(n-1)$

Fig. 13.12

Complete Graph

A simple graph G is said to be complete if every vertex in G is connected with every other vertex *i.e.*, if G contains exactly one edge between each pair of distinct vertices. A complete graph is usually denoted by K_n. It should be noted that K_n has exactly $\frac{n(n-1)}{2}$ edges. The graphs K_n for $n = 1,2,3,4,5,6$ are shown in Fig. 13.13.

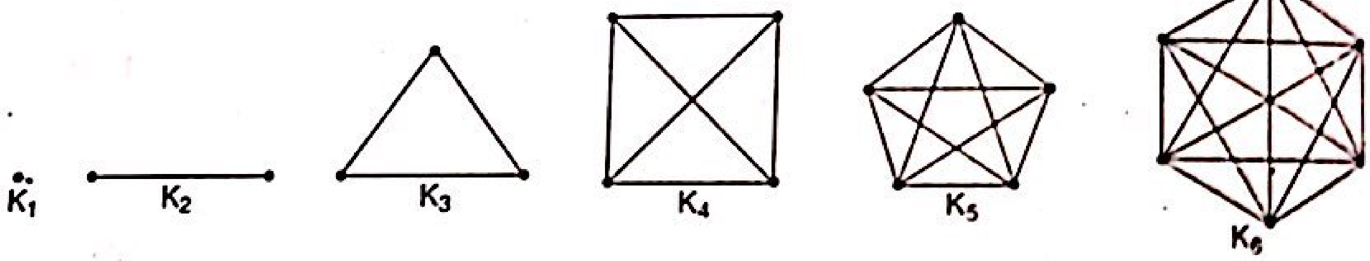
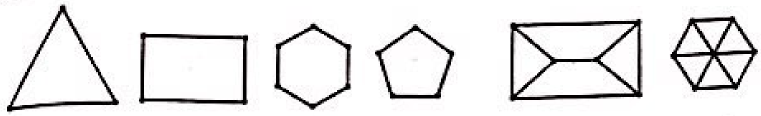


Fig. 13.13

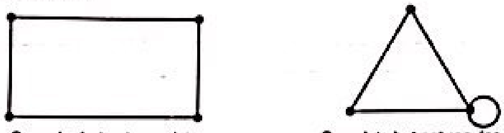
Regular Graph (equal vertice degree)

A graph in which all vertices are of equal degree is called a regular graph. If the degree of each vertex is *r*, then the graph is called a regular graph of degree *r*. Note that every null graph is regular of degree zero, and that the complete graph K_n is a regular of degree $n-1$. Also, note that if G has *n* vertices and is regular of degree *r*, then G has $(1/2)\, r\, n$ edges.



Regular graph of degree 2 Regular graph of degree 3

Note that complete graph need not be regular and a regular graph need not be complete as the following examples illustrate.



2 regular but not complete Complete but not regular

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Cycles $n \geq 3$

The cycle graph C_n $(n \geq 3)$, of length n is a connected graph which consists of n vertices $v_1, v_2 \ldots . v_n$ and n edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, ... $\{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$. The cycles C_3, C_4, C_5, and C_6 are shown in Fig. 13.15.

C_n

n vertices
n edges

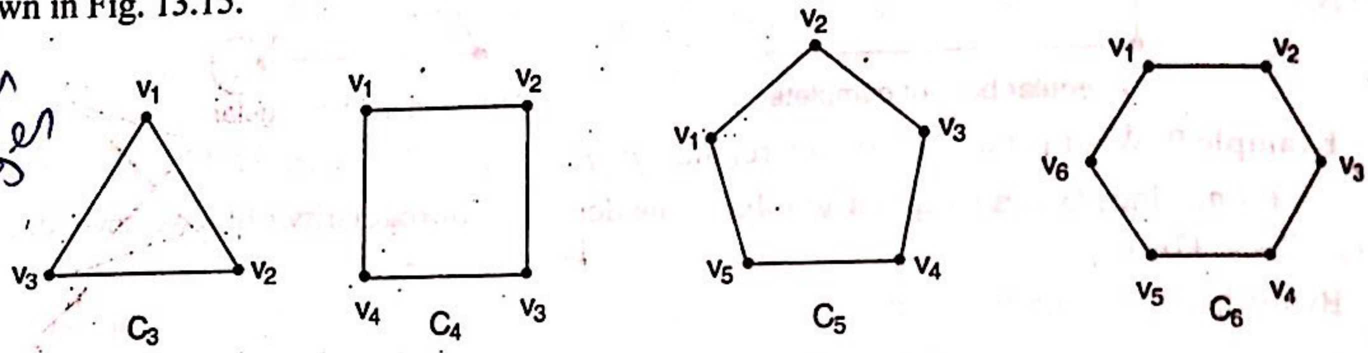


Fig. 13.15. Cycles C_3, C_4, C_5 and C_6

C_n is a regular graph of degree 2.

n+1 vertices
2n edges

Wheels $n > 3$ W_n

The wheel graph W_n $(n > 3)$ is obtained from C_n by adding a vertex v inside C_n and connectin it to every vertex in C_n. The wheels W_3, W_4, W_5, and W_6 are displayed in Fig. 13.16.

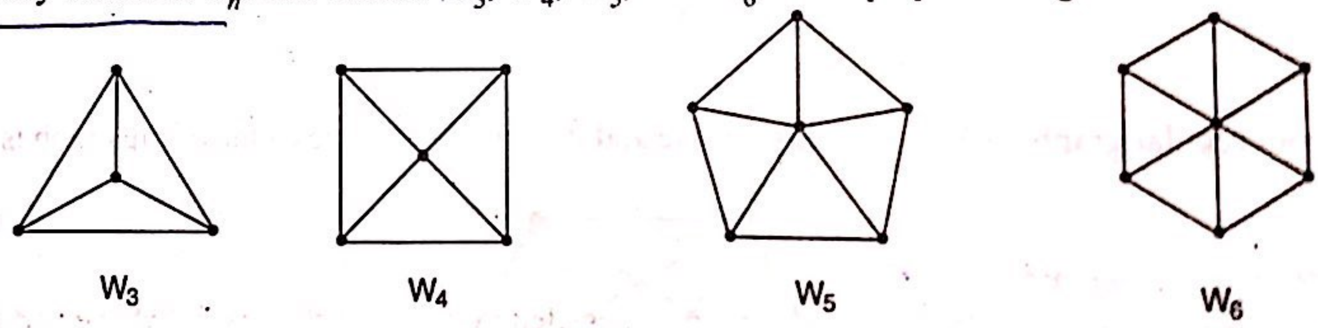
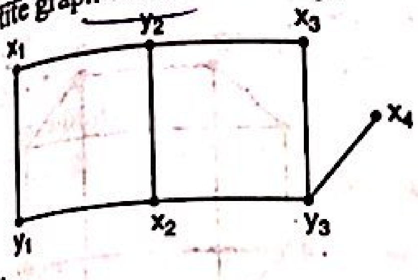


Fig. 13.16. The Wheels W_3, W_4, W_5 and W_6

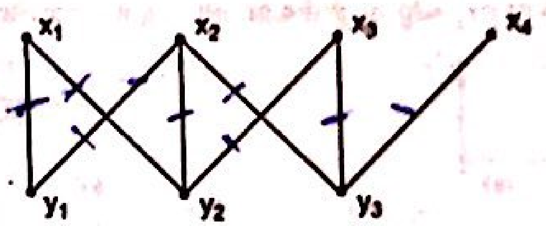
W_n is a regular graph for $n = 3$. It has $n + 1$ vertices and $2n$ edges.

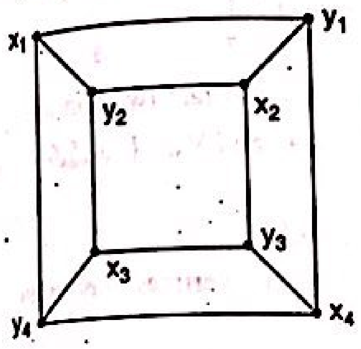
Bipartite Graph

A graph $G = (V, E)$ is bipartite if the vertex set V can be partitioned ino two subsets (disjoint) V_1 and V_2 such that every edge in E connects a vertex in V_1 and a vertex V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). (V_1, V_2) is called a bipartition of G: Obviously, a bipartite graph can have no loop.



which redrawn as:





which redrawn as:

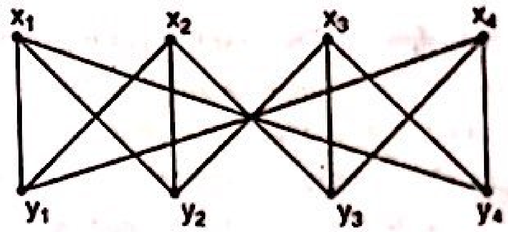
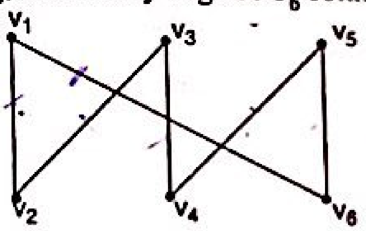


Fig. 13.18. Some bipartite graphs

Example 11. Show that C_6 is a bipartite graph.

Solution. C_6 in Fig. 13.15 is a bipartite graph since its vertex set can be partitioned into the two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$, and every edge of C_6 connects a vertex in V_1 and a vertex in V_2.



Note : Qn is a bipartite graph

Complete bipartite graph $K_{m,n}$

r + s vertices
rs edges.

The complete bipartite graph on m and n vertices, denoted $K_{m,n}$ is the graph, whose vertex set is partitioned into sets V_1 with m vertices and V_2 with n vertices in which there is an edge between each pair of vertices v_1 and v_2 where v_1 is in V_1 and v_2 is in V_2. The complete bipartite graphs $K_{2,3}$, $K_{2,4}$, $K_{3,3}$, $K_{3,5}$ and $K_{2,6}$ are shown in Fig. 13.19. Note that $K_{r,s}$ has $r + s$ vertices and rs edges.

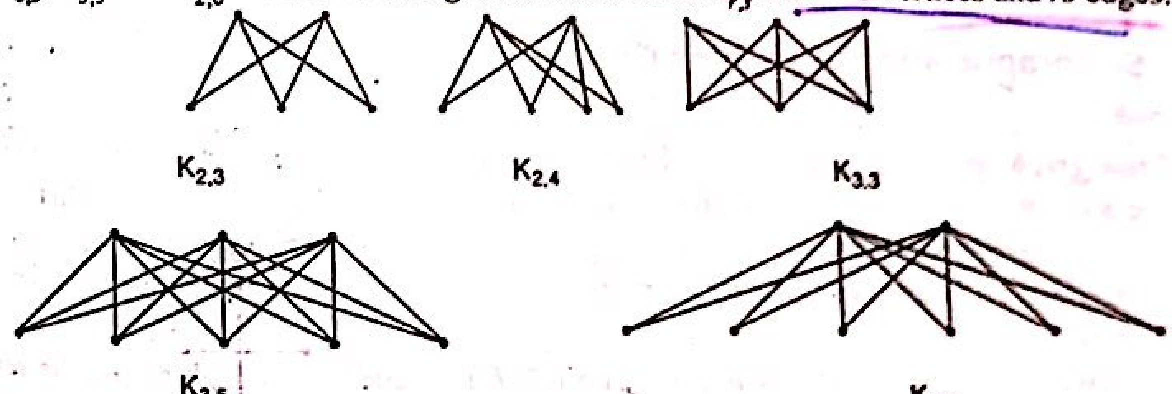


Fig. 13.19. Some complete bipartite graphs

Note :

1. Any graph $K_{1,n}$ is called a star graph.
2. A complete bipartite graph $K_{m,n}$ is not a regular if $m \neq n$.
3. K_5 and $K_{3,3}$ are called kuaratowski graphs.

13.6. Subgraphs and Isomorphic Graphs

Subgraph

Some graph applications are concerned with only parts of a graph. For instance, we may be interested about the part of a large computer network that involves the computer centres. In the graph model, we can remove the vertices corresponding to the computer centre other than of our interest and we can remove all edges incident with a vertex that was removed. A smaller graph is obtained. Such graph is called a subgraph of the original graph.

Consider a graph $G = (V, E)$. A graph $H = (V', E')$ is called a subgraph of G if the vertices and edges of H are contained in the vertices and edges of G, that is, if $V' \subseteq V$ and $E' \subseteq E$. In particular.

(i) If $V' \subset V$ and $E' \subset E$, then H is called a **proper subgraph** of G.

(ii) A subgraph H of G, is called a **spanning subgraph** of G if and only if $V(H) = V(G)$.

(iii) A subgraph H (V', E') of $G(V, E)$ is called the **induced subgraph** of G if its edge set E' contain all edges in G whose end points belong to vertices in G.

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(iv) If a subset U of V and all the edges incident on the elements of U are deleted from a graph $G(V, E)$, then the resulting subgraph is called a **vertex deleted subgraph** of $G(V, E)$.

(v) If a subset S of E from a graph $G(V, E)$ is deleted, then the resulting subgraph is called an **edge deleted subgraph** *of* G.

For a given graph G, there can be many subgraphs. Let $|V| = m$ and $|E| = n$. The total non-empty subsets of V is $2^m - 1$ and total subsets of E is 2^n. Thus, number of subgraph is equal to $(2^m - 1) \times 2^n$. Then number of spanning subgraph is equal to 2^n because all vertices are to be included in a spanning subgraph. For example, in Q_3 we have $|V| = 8, |E| = 12$. Then total number of subgraphs is

$$(2^8 - 1) \times 2^{12} = 127 \times 4096 = 520192$$

and the total number of spanning subgraphs is $2^{12} = 4096$.

So, if H is a subgraph of G, then Note.

(i) All the vertices of H are in G.

(ii) All the edges of H are in G and

(iii) Each edge of H has the same end points in H as in G.

Any subgraph of a graph G can be obtained by removing certain vertices and edges from G. It is understood that the removal of an edge leaves its points in place, whereas the removal of a vertex necessitates the removal of any edges with that vertex as an end point.

...le 14. Consider the graph shown in Fig. 13.20. Find the different subgraphs of the graph

Example 14. Consider the graph shown in Fig. 13.20. Find the different subgraphs of this graph.

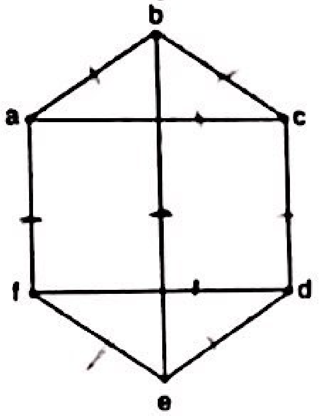
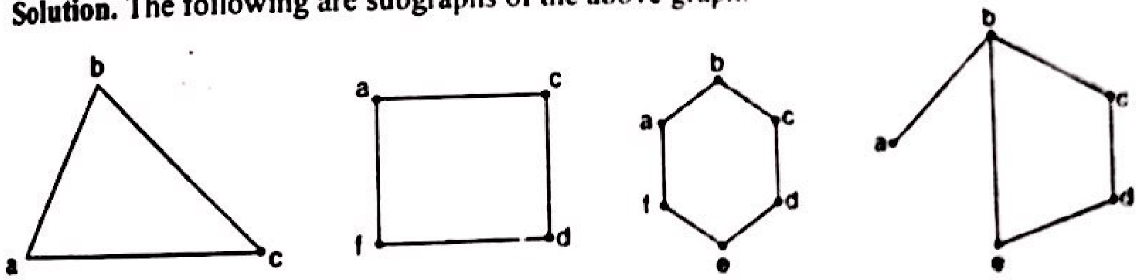
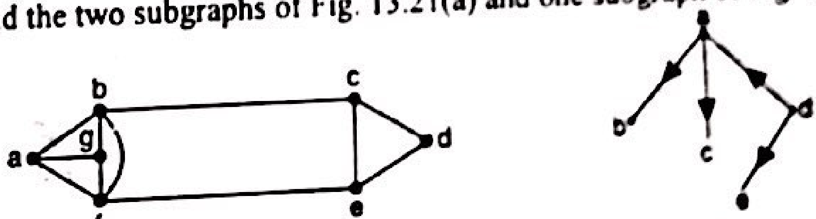


Fig. 13.20

Solution. The following are subgraphs of the above graph.



Example 15. Find the two subgraphs of Fig. 13.21(a) and one subgraph of Fig. 13.21(b).



(a) (b)

Fig. 13.21

Solution : Two subgraphs of the undirected graph and one subgraph of the directed graph are shown in Fig. 13.22.

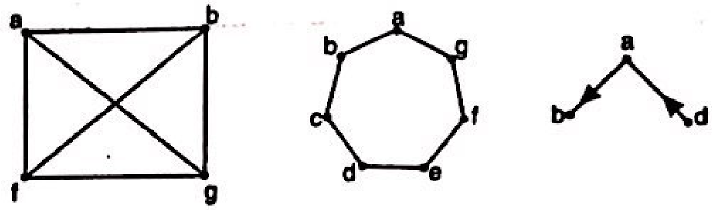


Fig. 13.22

Note. that the subgraphs do not have to be drawn the same way they appear in the presentation of *G*.

Isomorphic Graph

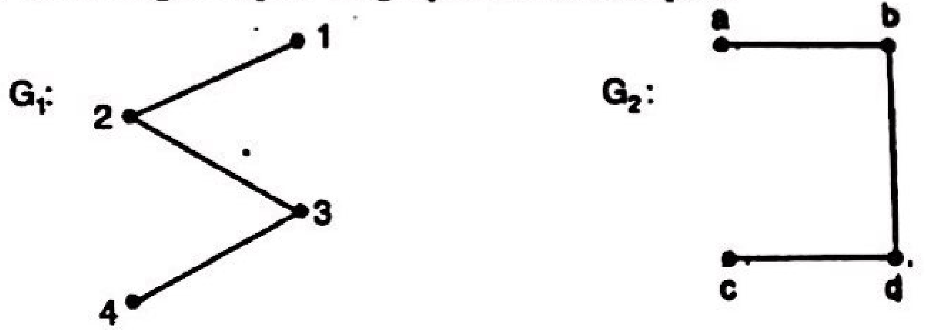
Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists a functions $f: V_1 \to V_2$ such that

(*i*) f is one-to-one onto *i.e.*, f is bijective.

(*ii*) $\{a, b\}$ is an edge in E_1, if and only if $\{f(a), f(b)\}$ is an edge in E_2 for any two elements a, $b \in V_1$.

The condition (*ii*) says that if vertices a and b are adjacent in G_1 then $f(a)$ and $f(b)$ are adjacent in G_2. In other words the function f preserves adjacency relationship and consequently the corresponding vertices in G_1 and G_2 will have the same degree. Any function f with the above properties is called an **isomorphism** between G_1 and G_2.

Example 19. Show that the given pair of graphs are isomorphic



Solution. Here, $V(G_1) = \{1,2,3,4\}$, $V(G_2) = \{a,b,c,d\}$, $E(G_1) = \{\{1,2\}, \{2,3\}, \{3,4\}\}$ and $E(G_2) = \{\{a,b\}, \{b,d\}, \{d,c\}\}$. Hence $|V(G_1)| = |V(G_2)|$ and $|E(G_1)| = |E(G_2)|$

The vertices of degree 1 in G_1 are $\{1, 4\}$ and

The vertices of degree 1 in G_2 are $\{a, c\}$

The vertices of degree 2 in G_1 are $\{2, 3\}$ and

The vertices of degree 2 in G_2 are $\{b, d\}$

Define a function $f: V(G_1) \to V(G_2)$ as

$f(1) = a, f(2) = b, f(3) = d$ and $f(4) = c$.

f is clearly one-one and onto.

Further,

$\{1,2\} \in E(G_1)$ and $\{f(1), f(2)\} = \{a, b\} \in E(G_2)$

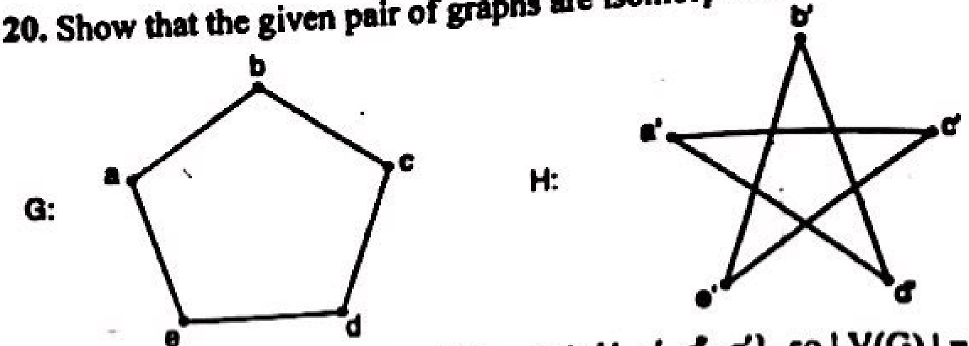
$\{2,3\} \in E(G_1)$ and $\{f(2), f(3)\} = \{b, d\} \in E(G_2)$

$\{3,4\} \in E(G_1)$ and $\{f(3), f(4)\} = \{d, c\} \in E(G_2)$

Hence f preserves adjacency of the vertices.

Therefore, G_1 and G_2 are isomorphic.

Example 20. Show that the given pair of graphs are isomorphic.



Solution: Here $V(G) = \{a, b, c, d, e\}$, $V(H) = \{a', b', c', d', e'\}$, so $|V(G)| = |V(H)|$

Also, $|E(G)| = |E(H)|$

The vertices of degree 2 in G are $\{a, b, c, d, e\}$ and

The vertices of degree 2 in H are $\{a', b', c', d', e'\}$

Define a function $f: V(G) \rightarrow V(H)$ as

$f(a) = a', f(b) = c', f(c) = e' f(d) = b'$ and $f(e) = d'$

f is clearly one-one and onto.

Further,

$\{a, b\} \in E(G)$ and $\{f(a), f(b)\} = \{a', c'\} \in E(H)$

$\{a, e\} \in E(G)$ and $\{f(a), f(e)\} = \{a', d'\} \in E(H)$

$\{b, c\} \in E(G)$ and $\{f(b), f(c)\} = \{c', e'\} \in E(H)$

$\{c, d\} \in E(G)$ and $\{f(c), f(d)\} = \{e', b'\} \in E(H)$

$\{d, e\} \in E(G)$ and $\{f(d), f(e)\} = \{b', d'\} \in E(H)$

Hence f preserves adjacency of the vertices. Therefore, G and H are isomorphic.

We hereby give some examples of isomorphic graphs.

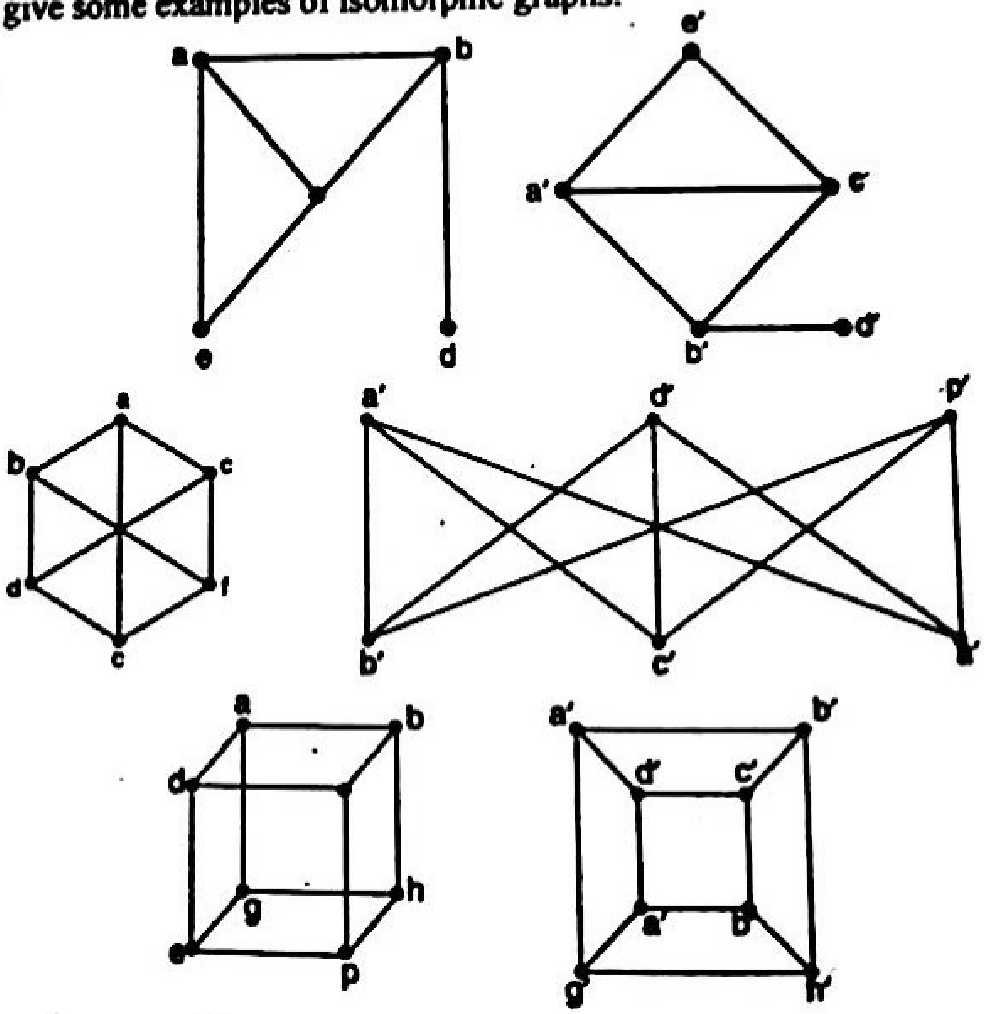


Fig. 13.25. Isomorphic pairs of graphs

A property shared by isomorphic graphs is called an **isomorphism invariant.** We may list some of the following invariants.

If $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are isomorphic graphs, then G_1 and G_2 have the

(*i*) same number of vertices, *i. e.*, $|V_1| = |V_2|$

(*ii*) same number of edges, *i.e.*, $|E_1| = |E_2|$

(*iii*) same degree sequences *i.e.*, if the degree of a vertex v_i in G_1 is m, then the degree of the vertex $f(v_i)$ in G_2 must also be m.

(*iv*) If $\{v, v\}$ is a loop in G_1 then $\{f(v), f(v)\}$ is also a loop in G_2.

If any of these quantities differ in two graphs, they can not be isomorphic. However, these conditions are by no means sufficient. For instance, the two graphs shown in Fig. 13.26 satisfy all three conditions, yet they are not isomorphic. That the graphs in Fig. 13.26 (*a*) and (*b*) are not isomorphic can be shown as follows : if the graph in Fig. 13.26 (*a*) are to be isomorphic to the one in (*b*), vertex x must correspond to y, because there are no other vertices of degree three. Now in (*b*) there is only one pendant vertex w, adjacent to y, while in (*a*) there are two pendant vertices u and v, adjacent to x.

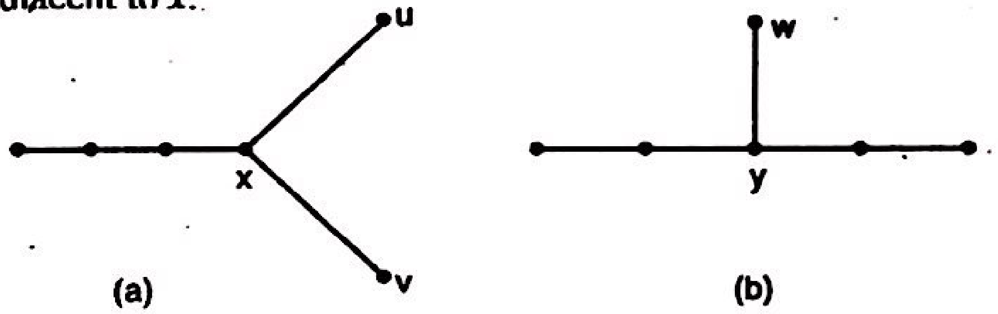
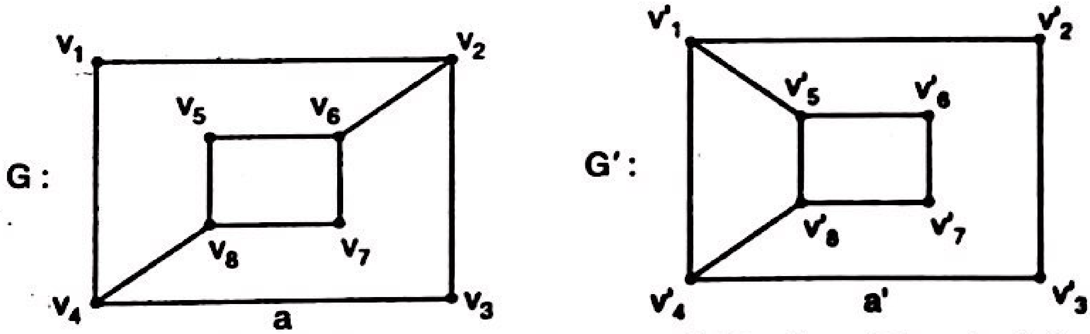


Fig. 13.26. Two graphs that are not isomorphic.

Example 21. Determine whether the following graphs are isomorphic.



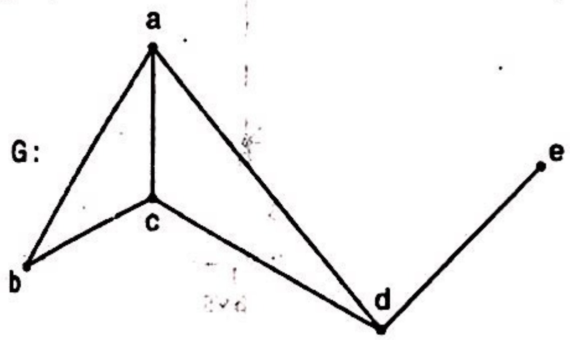
Solution. The graphs G and G' both have 8 vertices and 10 edges. They both have 4 vertices each of degree 3 and 4 vertices each of degree 2.

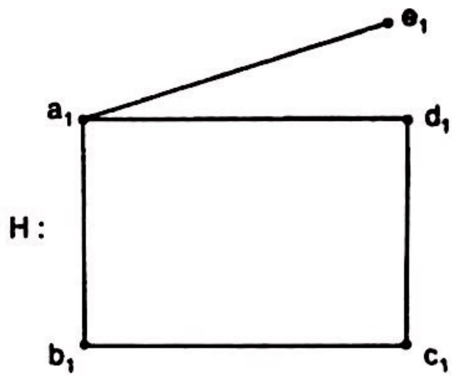
Now consider deg $(v_1) = 2$ in G. Then v_1 must correspond to either v_2', v_3', v_6', v_7', since these are vertices of degree 2 in G.

However, each of these vertices in G' is adjacent to another vertex of degree 2 in G' viz. v_2' is adjacent to v'_3, v'_6 is adjacent to v'_7 but v_1 is adjoint to v_2 and v_4 in G which are of degree 3. Thus the preservation of adjacency of the vertices is not maintained.

$\therefore$ G and G' are not isomorphic graphs.

Example 22. Examine G and H for isomorphism.





Solution. The graph G has 5 vertices and 6 edges. The graph H has 5 vertices and 5 edges. So $|E(G)| \neq |E(H)|$ *i.e.* the number of edges in G is not equal the number of edges in H.
$\therefore$ G and H are not isomorphic.

13.11. Representation of Graphs

Although a diagrammatic representation of a graph is very convenient for a visual study but this is only possible when the number of nodes and edges is reasonably small. Two types of representation are given below.

Matrix Representation. The matrix are commonly used to represent graphs for computer processing. The advantages of representing the graph in matrix form lies on the fact that many results of matrix algebra can be readily applied to study the structural properties of graphs from an algebraic point of view. There are number of matrices which one can associate with any graph. We shall discuss adjacency matrix and the incidence matrix.

Adjacency Matrix

(*a*) Representation of Undirected Graph

The adjacency matrix of a graph G with n vertices and no parallel edges is an n by n matrix $A = [a_{ij}]$ whose elements are given by

$a_{ij} = 1$, if there is an edge between ith and jth vertices, and

$= 0$, if there is no edge between them.

Note that for a given graph, the adjacency matrix is based on ordering chosen for the vertices. Hence, there are as many as $n\,!$ different adjacency matrices for a graph with n vertices, since there are $n\,!$ different ordering of n vertices. However, any two such adjacency matrices are closely related in that one can be obtained from the other by simply interchanging rows and colums.

There are a number of observations that one can make about the adjacency matrix A of a graph G.

(*i*) A is symmetric *i.e.* $a_{ij} = a_{ji}$ for all i and j.

(*ii*) The entries along the principal diagonal of A all 0's if and only if the graph has no self loops. A self loop at the vertex corresponds to $a_{ii} = 1$.

(*iii*) If the graph is simple (no self loop, no parallel edges), the degree of vertex equals the number of 1's in the corresponding row or column of A.

(*iv*) The (i, j) entry of A^m is the number of paths of length (no. of occurence of edges) m from vertex v_i to vertex v_j.

(*v*) If G be a graph with n vertices $v_1, v_2 \ldots\ldots\ldots\ldots v_n$ and let A denote the adjacency matrix of G with respect to this listing of the vertices. Let B be the matrix.

$$B = A + A^2 + A^3 + \ldots\ldots\ldots\ldots + A^n \; (n > 1)$$

Then G is a connected graph iff B has no zero entries.

This result can be used to check the connectedness of a graph by using its adjacency matrix.

Adjacency can also be used to represent undirected graphs with loops and multiple edges. A loop at the vertex v_i must have the element a_{ii} equal to 1 in the the adjacency matrix. When multiple edges are present, the adjacenmcy matrix is no longer a zero - one matrix, since the (i, j) th entry equals the number of edges that are associated between v_i and v_j. All undirected graphs, including multigraphs and pseudographs, have symmetric adjacency matrices.

(b) Representation of Directed Graph

The adjacency matrix of a diagraph D, with n vertices is the matrix $A = [a_{ij}]_{n \times n}$ in which

$$a_{ij} = 1 \text{ if arc } (v_i, v_j) \text{ is in D}$$
$$= 0 \text{ otherwise.}$$

One can make a number of observations about the adjacency matrix of a diagraph.

(i) A is not neccessary symmetric, since there may not be an edge from v_i to v_j when there is an edge from v_j to v_i

(ii) The sum of any column of j of A is equal to the number of arcs directed towards v_j.

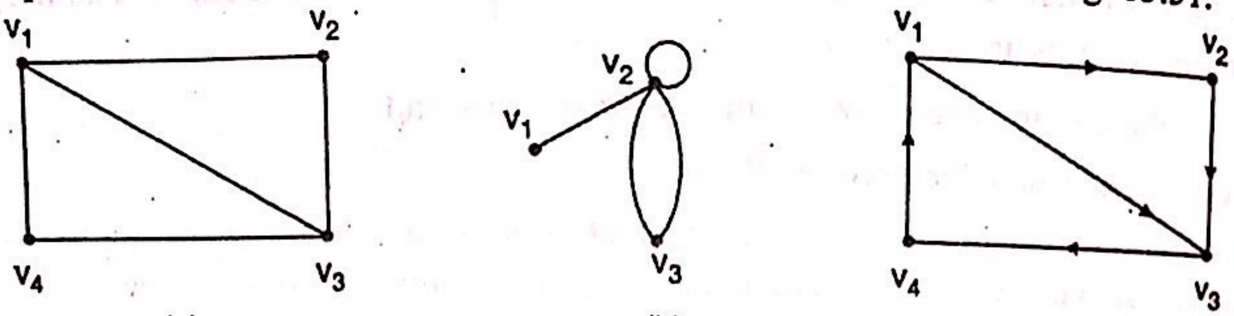
(iii) The sum of entries in row i is equal to the number of arcs directed away from vertex v_i (out degree of vertex v_i)

(iv) The (i, j) entry of A^m is equal to the number of path of length m from vertex v_i to vertex v_j

(v) The diagonal elements of A. A^T show that out degree of the vertices. The diagonal entries of A^T. A shows the in degree of the vertices.

The adjacency matrices can also be used to represent directed multigraphs. Again such matrices are not zero – one matries when there are multiple edges in the same direction connecting two vertices. In the adjacency matrix for a directed multigraph. a_{ij} equals the number of edges that are associated to (v_i, v_j).

Example 38 : Use adjacency matrix to represent the graphs shown in Fig. 13.51.



(a) (b) (c)

Fig. 13.51

Solution. We order the vertices in Fig 13.51 (a) as v_1, v_2, v_3 and v_4. Since there are four vertices, the adjacency matrix representing the graph will be a square matrix of order four. The required adjacency matrix A is

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

We order the vertices in Fig. 13.51 (b) as v_1, v_2 and v_3. The adjacency matrix representing the graph with loop and multiple edges is given by

$$\begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ A = \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix} \end{array}$$

Taking the order of the vertices in Fig.13.51 (c) as v_1, v_2, v_3 and v_4. The adjacency matrix epresenting the digraph is given by

$$\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ A = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example 39 : Draw the undirected graph represented by adjacency matrix A given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution. Since the given matrix is a square of order 5, the graph G has five vertices v_1, v_2, v_3, v_4, and v_5. Draw an edge from v_i to v_j where $a_{ij} = 1$. The required graph is drawn in Fig. 13.52.

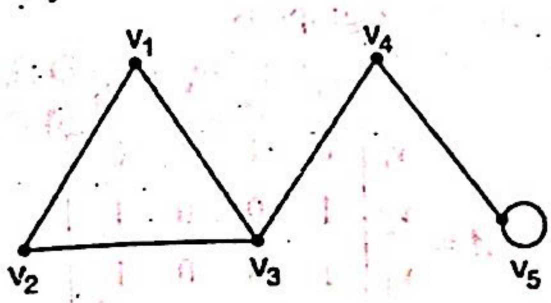


Fig. 13.52

Incidence Matrix

(a) Representation of Undirected Graph

Consider a undirected graph G = (V, E) which has n vertices and m edges all labelled. The incidence matrix $I(G) = [b_{ij}]$, is then $n \times m$ matrix, where

$$b_{ij} = 1 \quad \text{when edge } e_j \text{ is incident with } v_i$$
$$= 0 \quad \text{otherwise}$$

A graph and its incidence matrix are shown in examples. We can make a number of observations about the incidence matrix of G.

(i) Each column of B comprises exactly two unit entries.

(ii) A row with all 0 entries corresponds to an isolated vertex.

(iii) A row with a single unit entry corresponds to a pendant vertex.

(iv) The number of unit entries in row i of B is equal to the degree of the corresponding vertex v_i.

(v) The permutation of any two rows (any two columns) of I(G) corresponds to a relabelling of the vertices (edges) of G.

(vi) Two graphs are isomorphic if and only if their corresponding incidence matrices differ only by a permutation of rows and columns.

(vii) If G is connected with n vertices then the rank of I(G) is $n - 1$.

Incidence matrices can also be used to represent multiple edges and loops. Multiple edges are represented in the incidence matrix using columns with idential entries. Since these edges are incident with the same pair of vertices. Loops are represented using a column with exactly one entry equal to 1, corresponding to the vertex that is incident with this loop.

(b) Representation of Directed Graph

The incidence matrix $I(D) = [b_{ij}]$ of digraph D with n vertices and m edges is the $n \times m$ matrix in which.

$$b_{ij} = 1 \text{ if arc } j \text{ is directed away from } a \text{ vertex } v_i$$
$$= -1 \text{ if arc } j \text{ is directed towards vertex } v_i$$
$$= 0 \text{ otherwise.}$$

Example 44 : Find the incidence matrix to represent the graph shown in Fig.13.56.

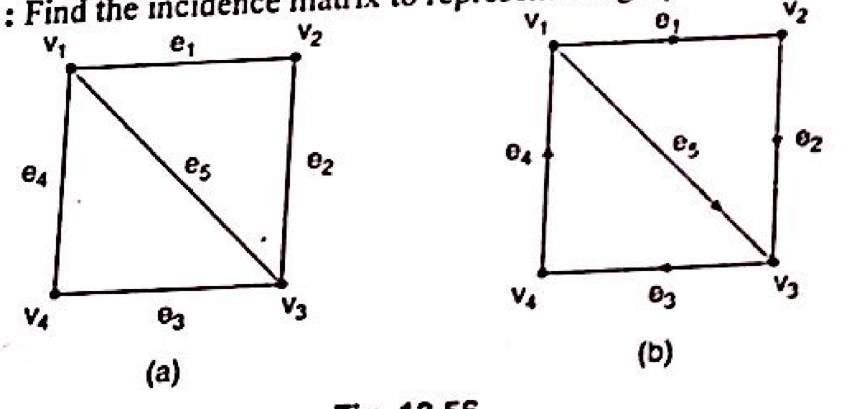


Fig. 13.56

Solution. The incidence matrix of Fig.(*a*) is obtained by entering for row v and column e is 1 if e is incident on v and 0 otherwise. The incidence matrix is.

$$I(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The incidence matrix of the digraph of Fig.(b) is

$$I(D) = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$