Graph Theory

UNIT – II

Paths, Circuits, Cycles and Connectivity:

(1) Walk:

A walk in a graph G is a finite alternating sequence

$$v_0 - e_1 - v_1 - e_2 - v_2 - e_3 \ldots - e_n - v_n.$$

of vertices and edges of the graph. such that each edge e_i in the sequence joins vertices v_{i-1} and v_i. $(1 \leq i \leq n)$.

(i) The end vertices v_0 and v_n are the end or terminal vertices of the walk.

(ii) The vertices $v_1, v_2, \ldots, v_{n-1}$ are called its internal vertices.

(iii) The integer n, the no. of edges in the walk is called the length of the walk.

Note:– A walk may repeat both vertices and edges.

(2) Open Walk:

A walk is called open when the terminal vertices are distinct.

(3) Closed Walk:

For the same end terminal vertices, a walk is termed as closed.

(4) Trail:

A walk is called a trail if all its edges are distinct.

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Note: A trail is open or closed depends on whether its end vertices are distinct or not.

5. Circuit :
A closed trail is called a circuit.

6. Path :
A walk is called a path if all its vertices and edges are distinct.

Note: Every path is a trail.

7. Cycle :
A path in which only repeated vertex is the first vertex is called a cycle to describe such a closed path.

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	Repeated Edge	*Repeated Vertex*	*Starts and Ends at same points ?*
Walk (open)	allowed	allowed	no
Walk (closed)	allowed	allowed	yes
Trail	no	allowed	no
Circuit	no	allowed	yes
Path	no	no	no
Cycle	no	first and last only	yes

For example, in the graph given in Fig. 13.29.

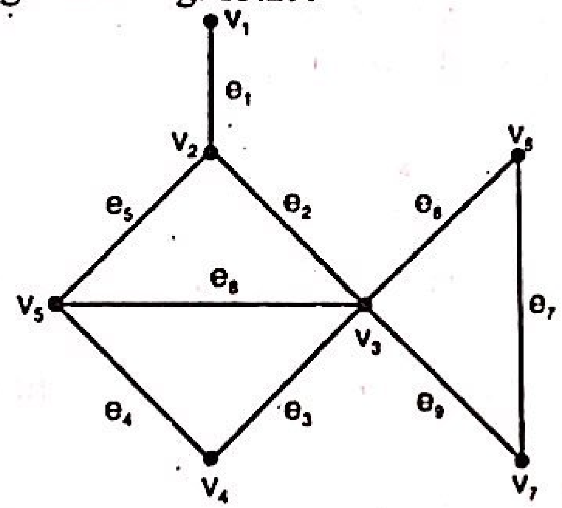

Fig. 13.29

(*i*) The sequence $v_1 - e_1 - v_2 - e_5 - v_5 - e_8 - v_3 - e_3 - v_4 - e_4 - v_5 - e_5 - v_2 - e_2 - v_3 - e_6 - v_6$, walk of length 8. It contains repeated vertices v_2, v_3 and v_5 and repeated edge e_5.

(*ii*) The sequence $v_1 - e_1 - v_2 - e_5 - v_5 - e_3 - v_3 - e_3 - v_4 - e_4 - v_5$ is a trail. It contains repeated vertex v_5 but does not contain repeated edge.

(*iii*) The sequence $v_1 - e_1 - v_2 - e_5 - v_5 - e_8 - v_3 - e_3 - v_4$ is a path. It does not contain repeated vertex and repeated edge.

(*iv*) The sequence $v_2 - e_2 - v_3 - e_3 - v_4 - e_4 - v_5 - e_5 - v_2$ is a cycle. It does not conatain repeated vertex and repeated edge except the first and last vertex.

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Reachability

A vertex v in a simple graph G is said to be reachable from the vertex u of G if there exists a path from u to v. The set of vertices which are reachable from a given vertex v is called the reachable set of v and is denoted by R(v).

For any subset U of the vertex set V, the reachable set of U is the set of all vertices which are reachable from any vertex set of S and this set is denoted by R(S). For example in the graph given on the next page.

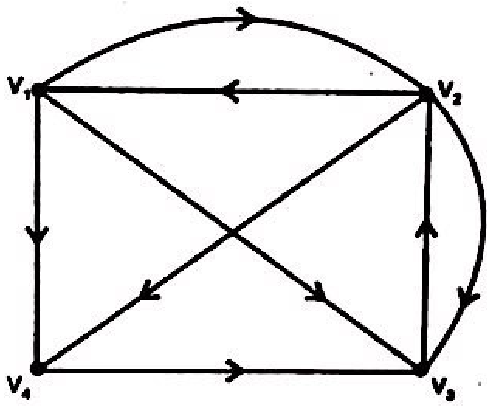

Fig. 13.30

$R(v_1) = \{v_2, v_3, v_4\}$, $R(v_2) = \{v_1, v_3, v_4\}$,
and $R(\{v_1, v_2\}) = \{v_3, v_4\}$.

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Connectedness in Undirected Graphs

A graph is connected if it is possible to travel from any vertex to any other vertex along a sequence of adjacent edges of the graph. Thus a graph G is **connected** if given any vertices v and w in G, there is a path from v to w. *i.e.*, if any pair of vertices are reachable from one another. A graph that is not connected is called **disconnected.** A disconnected graph is the union of two or more connected subgraphs each pair of which has no vertex in common. These disjoint connected subgraphs are called the **connected components** of the graph. A connected component of a graph is a connected subgraph of largest possible size. Thus a graph H is a connected component or simply component of a graph G if and only if

1. H is a subgraph of G;
2. H is connected;
3. No connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H.

The graph G and a connected component of the graph is shown in Fig. 13.31 and Fig. 13.32.

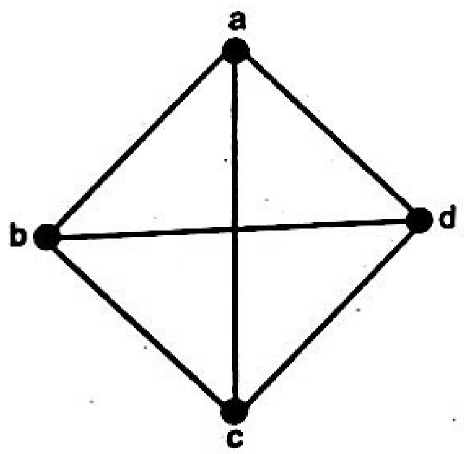


Fig. 13.31

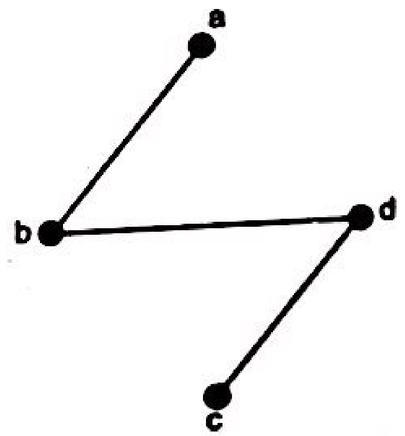
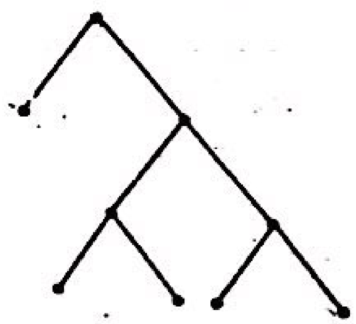
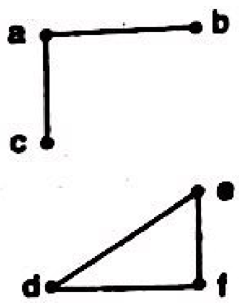


Fig. 13.32. Connected component of G

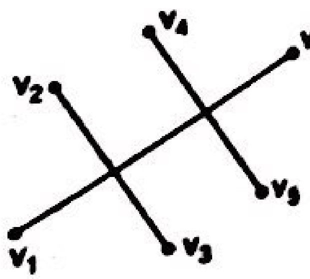
Example 24. Which of the graphs below are connected ?



(a)



(b)



(c)

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Solution. The graph shown in 13.33 (*a*) is a connected graph since for every pair of distinct vertices there is a path between them.

The graph shown in Fig. 13.33 (*b*) is not connected since there is no path in the graph between vertices *b* and *d*.

The graph shown in Fig. 13.33 (*c*) is not connected. In drawing a graph two edges may cross at a point which is not a vertex. The graph can be redrawn as

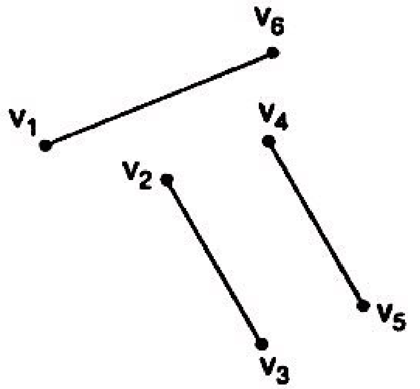

Fig. 13.34

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Connectedness in Directed Graphs

The definition of connectedness in undirected graphs can not be applied to directed graphs without some further modification because in a directed graph if a vertex u is reachable from another vertex v, the vertex v may not be reachable from u. To overcome this difficulty we define the connectedness in a diagraph as follows.

Connected or weakly connected: A directed graph is called connected or weakly connected if it is connected as an undirected graph in which each directed edge is converted to an undirected graph.

Unilaterally connected: A simple directed graph is said to be unilaterally connected if for any pair of vertices of the graph at least one of the vertices of the pair is reachable from other vertex.

Strongly connected: A directed graph is called strongly connected if for any pair of vertices of the graph both the vertices of the pair are reachable from one another.

For the digraphs in Fig. 13.37 the digraph in (*a*) is strongly connected, in (*b*) it is weakly connected, while in (*c*) it is unilateraly connected but not strongly connected.

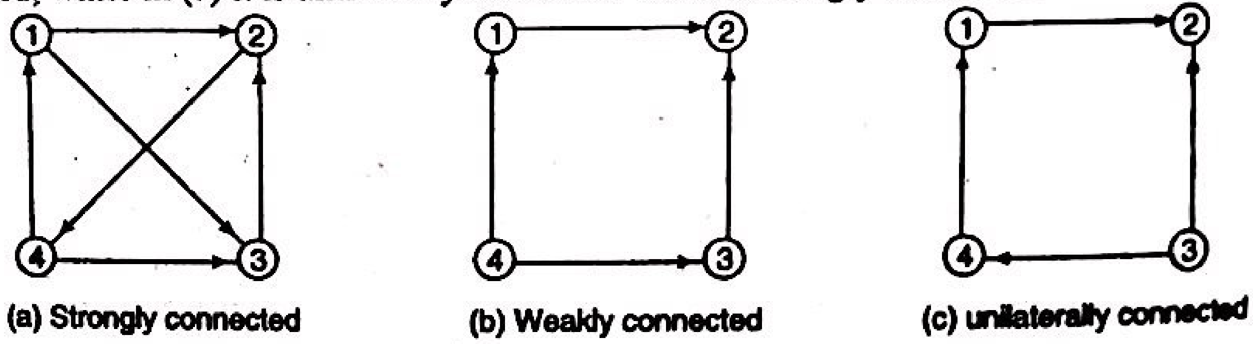

Fig. 13.37. Connectivity in digraphs

Note that a unilaterally connected digraph is weakly connected but a weakly connected digraph is not necessarily unilaterally connected. A strongly connected digraph is both unilaterally and weakly connected.

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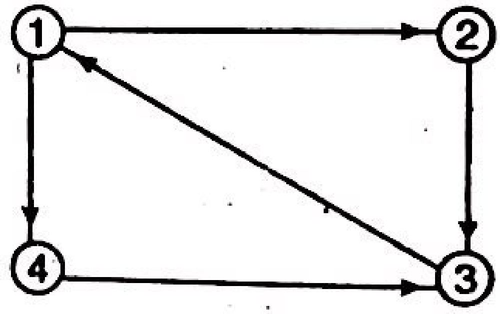
Example 26. Is the directed graph given below strongly connected ?



Solution. The possible pairs of vertices and the forward and the backward paths between then are shown below for the given graph

Pairs of vertices	*Forward path*	*Backward path*
(1,2)	1–2	2–3–1
(1,3)	1–2–3	3–1
(1,4)	1–4	4–3–1
(2,3)	2–3	3–1–2
(2,4)	2–3–1–4	4–3–1–2
(3,4)	3–1–4	4–3

Therefore, we see that between every pair of distinct vertices of the given graph there exists a forward as well as backward path, and hence it is strongly connected.

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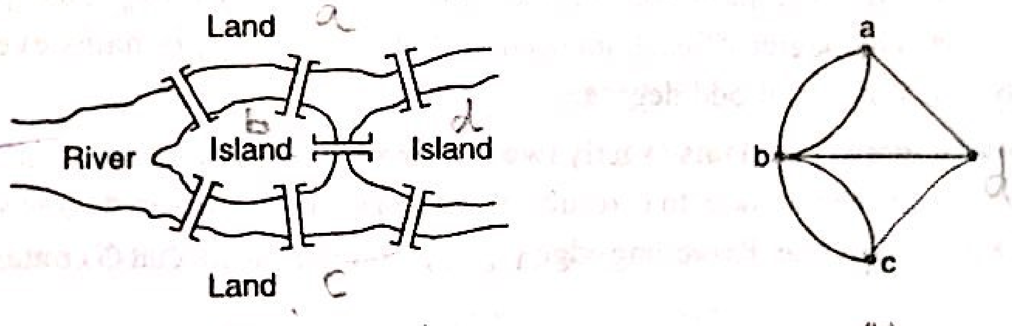
Bridges of konigsberg:

One of the oldest problem involving graphs is the Konigsberg bridge probelm. There were two islands linked to each other and to the banks of the Pregel River (earlier known as Konigsberg) by seven bridges shown in Fig. 13.38 (*a*). The problem was to begin at any of the four land areas to walk across each bridge once and to return the starting point. Euler drew a graph like Fig. 13.38 (*b*) for the problem in which *a* and *c* represent the two river banks; *b* and *d* the two islands. The arcs joining them represent the seven bridges.



(a) (b)

Fig. 13.38. Konigsberg graph

It is clear that the problem of walking each of the seven bridges exactly once and returning to the starting point is equivalent to finding a circuit in graph (*b*) that traverses each of the edge exactly once. Euler discovered a very simple criterion for determining whether such a circuit exists in a graph.

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Eulerian Graph

A circuit in a connected graph is an Euler circuit if it contains every edge of the graph exactly once. A connected graph with an Euler circuit is called an Euler graph or Eulerian graph.

If there is an open trail from a to b in G and this trail traverses each edge in G exactly once, then the trail is called an Euler trail.

The existence of Euler circuit and trail depends on the degree of vertices.

The next theorem provides necessary and sufficient condition for characterising Euler graph.

Theorem 13.13. A nonempty connected graph G is Eulerian if and only if its vertices are all of even degree.

Theorem 13.14. A connected graph contains an Eulerian trail, but *not* an Eulerian circuit, if and only if it has exactly two vertices of odd degree.

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Example 28(a). Let G be a graph of Fig. 13.39. Verify that G has an Eulerian circuit,

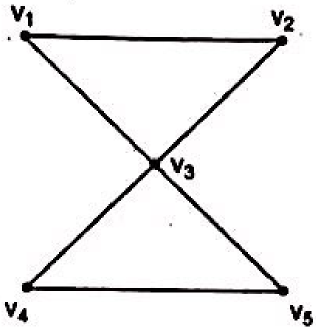

Fig. 13.39

Solution. We observe that G is connected and all the vertices are having even degree deg $(v_1) = \deg(v_2) = \deg(v_4) = \deg(v_5) = 2$.

Thus G has a Eulerian circuit. By inspection, we find the Eulerian circuit.

$$v_1 - v_3 - v_5 - v_4 - v_3 - v_2 - v_1$$

Example 28 (b). Show that the graphs in Fig. 13.40 contain no Eulerian circuit.

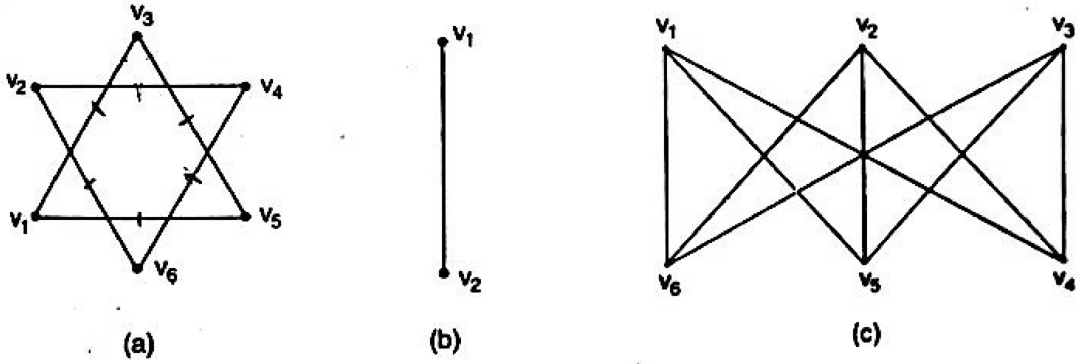

Fig. 13.40

Solution. The graph shown in Fig. 13.40 (*a*) does not contain Eulerian circuit since it is not connected.

The graph shown in Fig. 13.40 (*b*) is connected but vertices v_1 and v_2 are of degree 1. Hence it does not contain Eulerian circuit.

All the vertices of the graph shown in Fig. 13.40 (*c*) are of degree 3, hence it does not contai Eulerian circuit.

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Example 29. Show that the graph shown in Fig. 13.41 has no Eulerian circuit but has a Eulerian trail.

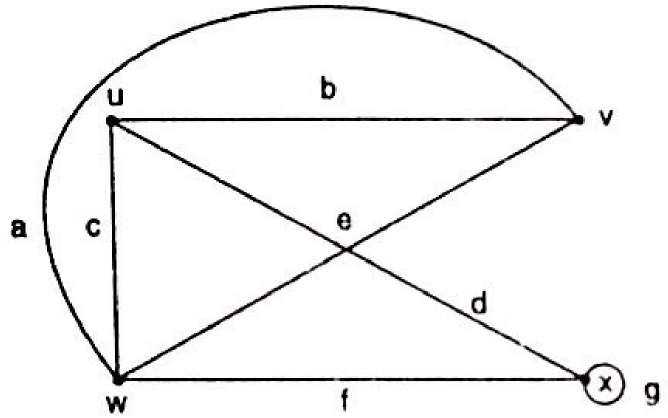

Fig. 13.41

Solution. Here deg (u) = deg (v) = 3 and deg (w) = 4, deg (x) = 4. Since u and v have only two vertices of odd degree, the graph shown in Fig. 13.41 does not contain Eulerian circuit, but the path $b-a-c-d-g-f-e$ is an Eulerian trail.

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Hamiltonian Graphs

Hamiltonian graphs are named after Sir William Hamilton, an Irish mathematician who introduced the problems of finding a circuit in which all vetices of a graph appear exactly once.

A circuit in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice is known as **Hamiltonian cycle.**

A graph G is called a **Hamiltonian cycle** if it contains a Hamiltonian cycle.

A **Hamiltonian path** is a simple path that contains all vertices of G where the end points may be distinct.

Example 33. Which of the graphs given in Fig. 13.43 is Hamiltonian cycle. Give the cycles on the graphs that contain them.

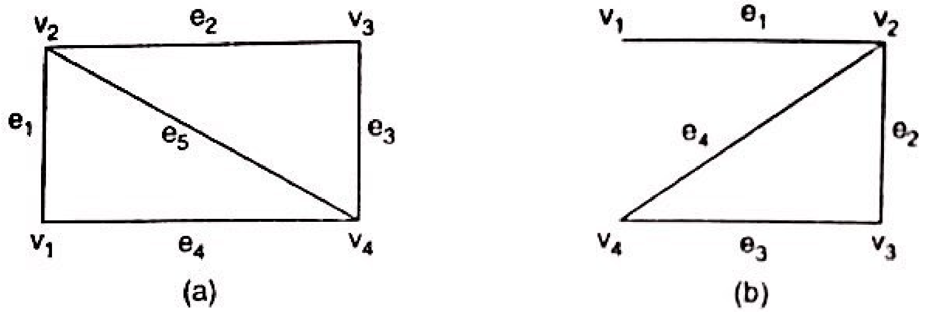

Fig. 13.43

Solution. The graph shown in Fig. 13.43 (a) has Hamilitonian cycle given by $v_1 e_1 v_2 e_2 v_3 e_3 v_4$ $e_4 v_1$. Note that all vertices appear in this a cycle but not all edges. The edge e_5 is not used in the cycle.

The graph shown in Fig.13.43 (b) does not contain Hamiltonian cycle since every cycle containing every vertex must contain the e_1 twice. But the graph does have a Hamiltonian path $v_1 - v_2 - v_3 - v_4$.

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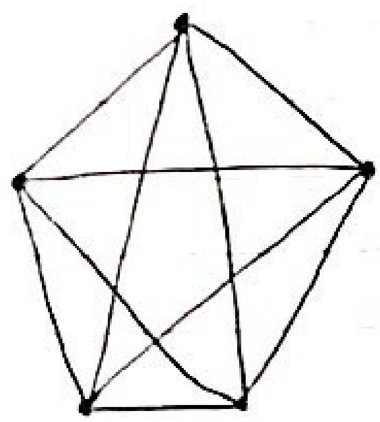
Hamiltonian Graphs

Theorems:

(1) A simple connected graph G with $n \geq 3$ vertices is Hamiltonian if $\deg(v) \geq \frac{n}{2}$ for every vertex v in G.

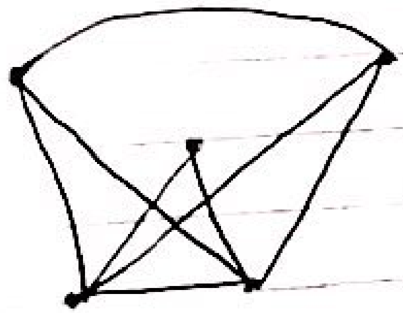
(2) A simple connected graph with n vertices and m edges is Hamiltonian if $m \geq [(n-1)(n-2)]/2 + 2$.

(a)



satisfies Theorem 1.

(b)



satisfies Theorem 2.

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Seating Problem: Nine members of a new club meet each day for lunch at a round table. They decide to sit such that every member has different neighbors at each lunch. How many days can this arrangement last?

This situation can be represented by a graph with nine vertices such that each vertex represents a member, and an edge joining two vertices represents the relationship of sitting next to each other. Figure 1-9 shows two possible

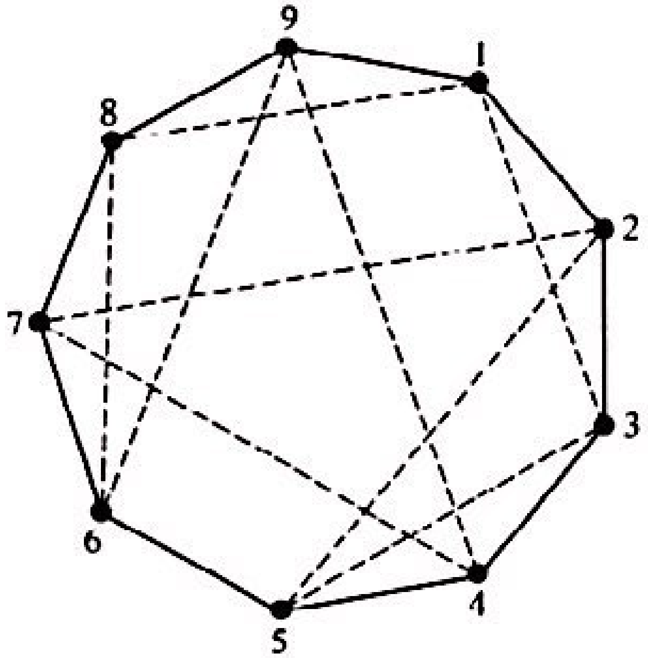

Fig. 1-9 Arrangements at a dinner table.

seating arrangements—these are 1 2 3 4 5 6 7 8 9 1 (solid lines), and 1 3 5 2 7 4 9 6 8 1 (dashed lines). It can be shown by graph-theoretic considerations that there are only two more arrangements possible. They are 1 5 7 3 9 2 8 4 6 1 and 1 7 9 5 8 3 6 2 4 1. In general it can be shown that for n people the number of such possible arrangements is

$$\frac{n-1}{2}, \quad \text{if } n \text{ is odd,}$$

and

$$\frac{n-2}{2}, \quad \text{if } n \text{ is even.}$$

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Travelling Salesman Problem

Suppose a salesman territory includes several cities with highways connecting certain pair of these cities. He is required to visit each city personally exactly once. Graph theory can be used to solve this transportation system. The system can be represented by a graph G whose vertices correspond to the cities and such that two vertices are joined by an edge if and only if a highway connects the corresponding cities.

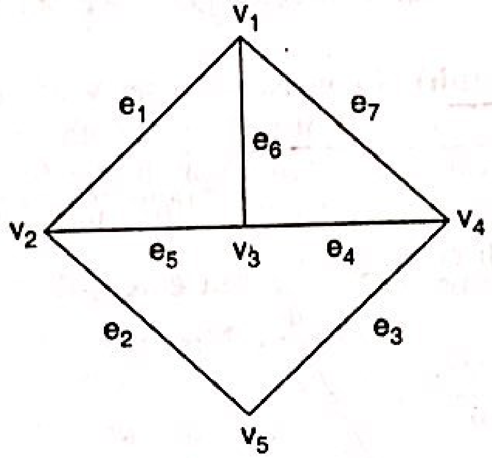

Fig. 13.4 Travelling salesman's territory graph

Starting at vertex v_1, salesman can visit each vertex by taking the edges e_1, e_2 e_3, e_4 and e_6 and back to v_1.

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Utilities Problem

An old problem concerns with three houses H_1, H_2, H_3, each to be connected to each of the three utilities – Water (W), Gas (G) and Electricity (E). Is it possible to connect each utility with each of the three houses without any two connections crossing each other ?

We can represent the connection of the three houses to the utilities by the graph of Fig. 13.3. Here we have diagram of six vertices, three of which represent the houses, denoted by H_1, H_2, H_3 the other three represent the utilities, denoted by G, W, E. An edge joins two vertices if and only if one vertex denotes a house and other vertex a utility. We shall see that it is not possible to draw this graph without edges crossing over. Thus the answer to this problem is no.

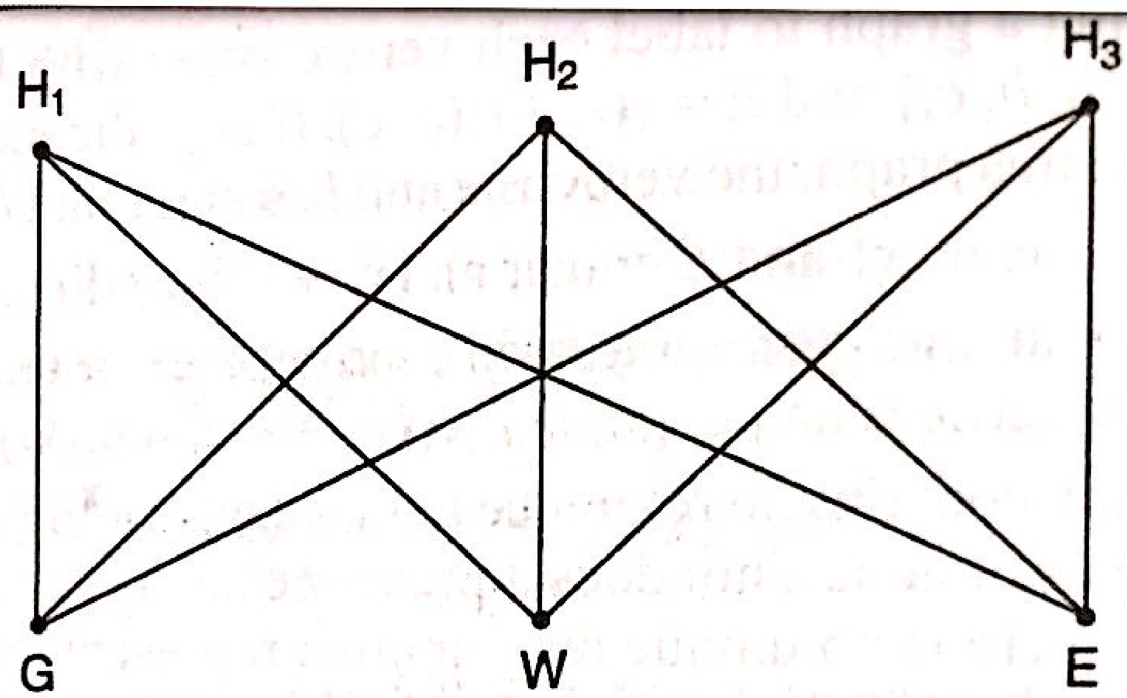

Fig. 13.3

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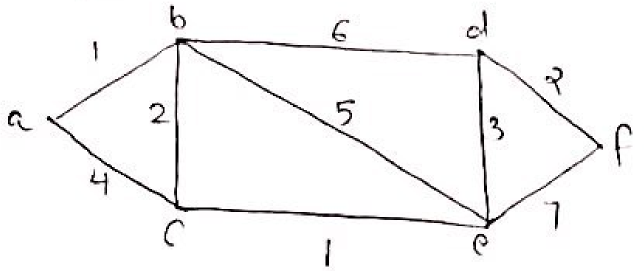
Weighted Graphs :
Graphs that have a number assigned to each edge are called weighted graphs.
(Here, the number may represent mileage cost, computer time or some other quantity).

The length of a path in a weighted graph is the sum of the weights of the edges of this path and the shortest path between the two vertices is the minimum length of the path.

example :



Shortest Path Problem :
The shortest path problem is the problem of finding a path between two vertices in a graph such that the sum of the weights of its constituent edges is minimized.

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