Set Theory

1.1 INTRODUCTION

The concept of a *set* appears in all mathematics. This chapter introduces the notation and terminology of set theory which is basic and used throughout the text.

Though logic is formally treated in Chapter 4, we introduce Venn diagram representation of sets here, and we show how it can be applied to logical arguments. The relation between set theory and logic will be further explored when we discuss Boolean algebra in Chapter 15.

This chapter closes with the formal definition of mathematical induction, with examples.

1.2 SETS AND ELEMENTS

A *set* may be viewed as an unordered collection of distinct objects, called as *elements* or *members* of the set. We ordinarily use capital letters, A, B, X, Y, ..., to denote sets, and lowercase letters, a, b, x, y, ..., to denote elements of sets. The statement "p is an element of A", or, equivalently, "p belongs to A" is written

$$p \in A$$

The statement that p is not an element of A, that is, the negation of $p \in A$, is written

$$p \notin A$$

The fact that a set is completely determined when its members are specified is formally stated as the principle of extension.

Principle of Extension

Two sets A and B are equal if and only if they have the same members.

As usual, we write $A = B$ if the sets A and B are equal, and we write $A \neq B$ if the sets are not equal.

Specifying Sets

There are essentially two ways to specify a particular set. One way, if possible, is to list its members. For example,

$$A = \{a, e, i, o, u\}.$$

denotes the set A whose elements are the letters a, e, i, o, u. Note that the elements are separated by commas and enclosed in braces { }. The second way is to state those properties which characterized the elements in the set. For example,

$$B = \{x: x \text{ is an even integer}, x > 0\}$$

which reads "B is the set of x such that x is an even integer and x is greater than 0", denotes the set B whose elements are the even positive integers. A letter, usually x, is used to denote a typical member of the set; the colon is read as "such that" and the comma as "and".

Example 1.1

(a) The set A above can also be written as

$$A = \{x: x \text{ is a letter in the English alphabet}, x \text{ is a vowel}\}.$$

Observe that $b \notin A$, $e \in A$, and $p \notin A$.

(b) We could not list all the elements of the above set B although frequently we specify the set by writing

$$B = \{2, 4, 6, \ldots\}$$

here we assume that everyone knows what we mean. Observe that $8 \in B$ but $-7 \notin B$.

(c) Let $E = \{x: x^2 - 3x + 2 = 0\}$. In other words, E consists of those numbers which are solutions of the equation $x^2 - 3x + 2 = 0$, sometimes called the *solution set* of the given equation. Since the solutions of the equation are 1 and 2, we could also write $E = \{1, 2\}$.

(d) Let $E = \{x: x^2 - 3x + 2 = 0\}$, $F = \{2, 1\}$ and $G = \{1, 2, 2, 1, \frac{6}{3}\}$. Then $E = F = G$. Observe that a set does not depend on the way in which its elements are displayed. A set remains the same if its elements are repeated or rearranged. Here G is multiset.

Some sets will occur very often in the text and so we use special symbols for them. Unless otherwise specified, we will let

$\mathbb{N}$ = the set of positive integers: 1, 2, 3, ...

$\mathbb{Z}$ = the set of integers: ..., −2, −1, 0, 1, 2, ...

$\mathbb{Q}$ = the set of rational numbers

$\mathbb{R}$ = the set of real numbers

$\mathbb{C}$ = the set of complex numbers

Even if we can list the elements of a set, it may not be practical to do so. For example, we would not list the members of the set of people born in the world during the year 1976 although theoretically it is possible to compile such a list. That is, we describe a set by listing its elements only if the set contains a few elements; otherwise we describe a set by the property which characterizes its elements.

The fact that we can describe a set in terms of a property is formally stated as the *principle of abstraction*.

Principle of Abstraction

Given any set U and any property P, there is a set A such that the elements of A are exactly those members of U which have the property P.

1.3 UNIVERSAL SET AND EMPTY SET

In any application of the theory of sets, the members of all sets under investigation usually belong to some fixed large set called the *universal set*. For example, in plane geometry, the universal set consists of all the points in the plane, and in human population studies the universal set consists of all the people in the world. We will let the symbol

$$U$$

denote the universal set unless otherwise stated or implied.

For a given set U and a property P, there may not be any elements of U which have property P. For example, the set

$$S = \{x : x \text{ is a positive integer}, x^2 = 3\}$$

has no elements since no positive integer has the required property.

The set with no elements is called the *empty set* or *null set* and is denoted by

$$\emptyset$$

There is only one empty set. That is, if S and T are both empty, then $S = T$ since they have exactly the same elements, namely, none.

1.4 SUBSETS

If every element in a set A is also an element of a set B, then A is called a *subset* of B. We also say that A is *contained* in B or that B *contains* A. This relationship is written

$$A \subseteq B \quad \text{or} \quad B \supseteq A$$

If A is not a subset of B, i.e. if at least one element of A does not belong to B, we write $A \not\subseteq B$ or $B \not\supseteq A$.

If $A \subseteq B$, then it is still possible that $A = B$. When $A \subseteq B$ but $A \neq B$, we say A is *proper subset of* B. We will write $A \subset B$ when A is a proper subset of B. For example suppose

$$A = \{1, 3\} \quad B = \{1, 2, 3\} \quad C = \{1, 3, 2\}$$

Then A and B are both subsets of C; but A is a proper subset of C, whereas B is not a proper subset of C since $B = C$.

Example 1.2

(a) Consider the sets

$$A = \{1, 3, 4, 5, 8, 9\} \quad B = \{1, 2, 3, 5, 7\} \quad C = \{1, 5\}$$

Then $C \subseteq A$ and $C \subseteq B$ since 1 and 5, the elements of C, are also members of A and B. But $B \subseteq A$ since some of its elements, e.g., 2 and 7, do not belong to A. Furthermore, since the elements of A, B, and C must also belong to the universal set U, we have that U must at least contain the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(b) Let N, Z, Q, and R be defined as in Section 1.2. Then

$$\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R}$$

(c) The set $E = \{2, 4, 6\}$ is a subset of the set $F = \{6, 2, 4\}$, since each number 2, 4, and 6 belonging to E also belong to F. In fact, $E = F$. In a similar manner it can be shown that every set is a subset of itself.

The following properties of sets should be noted:

(i) Every set A is a subset of the universal set U since, by definition, all the elements of A belong to U. Also the empty set $\varnothing$ is a subset of A.

(ii) Every set A is a subset of itself since, trivially, the elements of A belong to A.

(iii) If every element of A belongs to a set B, and every element of B belongs to a set C, then clearly every element of A belongs to C. In other words, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(iv) If $A \subseteq B$ and $B \subseteq A$, then A and B have the same elements i.e., $A = B$. Conversely, if $A = B$ then $A \subseteq B$ and $B \subseteq A$.

(d) Give an example one each for sets A, B and C such that $A \in B$, $B \in C$ and $A \notin C$.

Let $A = \{a, b\}$ and $B = \{\{a, b\}, a, e, d\}$

$C = \{\{\{a, b\}, a, e, d\}, g, h, \{i\}\}$

Here $A \in B$, $B \in C$ and $A \notin C$

(e) Give an example one each for sets A, B and C such that $A \in B$, $B \in C$ and $A \in C$.

Let $A = \{a, b, c\}$

$B = \{\{a, b, c\}, \{d, e\}, f\}$

$C = \{\{\{a, b, c\}, \{d, e\}, f\}, \{a, b, c\}, g\}$

Here $A \in B$, $B \in C$, $A \in C$.

(f) Give an example, one each for sets A, B and C such that $A \subseteq B$; $B \subseteq C$, and $A \subseteq C$.

Let $A = \{a\}$, $B = \{a, \{b, c\}, d\}$ and $C = \{a, \{b, c\}, d, \{e\}, \{\{f\}, g\}, h\}$

Here $A \subseteq B$, $B \subseteq C$ and $A \subseteq C$.

(g) Determine which of the following are true and which are false. Justify.

1. $3 \in \{1, 3, 5\} \rightarrow$ <u>TRUE</u>.

   It is very clear that element 3 is a member of $\{1, 3, 5\}$

2. Let $A = \{\{a, b\}\}$

   (i) $a \in A \rightarrow$ <u>FALSE</u>.

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Here set $\{a, b\}$ is the only member of set A therefore $a \notin A$.

(ii) $A \in A \rightarrow$ FALSE

(iii) $\{a, b\} \in A \rightarrow$ TRUE.

Clearly set $\{a, b\} \in A$.

(iv) There are 2 elements in $A \rightarrow$ FALSE.

There is only one element $\{a, b\}$ in A.

3. $\{3\} \in \{1, 3, 5\} \rightarrow$ FALSE.

Element $3 \in \{1, 3, 5\}$ and set $\{3\}$ does not belong to $\{1, 3, 5\}$. The statement will be true if the set will be $\{1, \{3\}, \{5\}\}$ instead of $\{1, 3, 5\}$.

4. $\{3\} \subset \{1, 3, 5\} \rightarrow$ TRUE.

Set $\{3\}$ is a subset of $\{1, 3, 5\}$ and $\{3\}$ is not equal to set $\{1, 3, 5\}$. Therefore, $\{3\}$ is a proper subset of $\{1, 3, 5\}$.

5. $\{3, 5\} \subseteq \{1, 3, 5\} \rightarrow$ TRUE.

The set $\{3, 5\}$ is a subset of $\{1, 3, 5\}$ as elements 3, 5 both are present in $\{1, 3, 5\}$. Therefore, $\{3, 5\} \subseteq \{1, 3, 5\}$

6. $\{1, 3, 5\} \subset \{1, 3, 5\} \rightarrow$ FALSE.

$\{1, 3, 5\}$ is subset of $\{1, 3, 5\}$. But both the sets are equal. Therefore, $\{1, 3, 5\}$ is not proper subset of $\{1, 3, 5\}$. $\{1, 3, 5\} \subseteq \{1, 3, 5\}$ is true.

7. $\varnothing \subseteq \varnothing \rightarrow$ TRUE.

Here both sets are having no elements, i.e. $\varnothing = \varnothing$. Therefore, $\varnothing \subseteq \varnothing$ is true. Every set is always a subset (not proper subset) of itself.

8. $\varnothing \in \varnothing \rightarrow$ FALSE.

An empty set $\varnothing$ or { } does not contain any elements.

Therefore, $\varnothing \in \varnothing$ is false.

9. $\varnothing \subseteq \{\varnothing\} \rightarrow$ TRUE.

Above statement means $\{\} \subseteq \{\{\}\}$.

An empty set is subset of every set. Therefore, $\varnothing \subseteq \{\varnothing\}$ is true.

10. $\varnothing \in \{\varnothing\} \rightarrow$ TRUE.

$\varnothing$ is an element of $\{\varnothing\}$. Therefore, the statement is true.

11. $\{\varnothing\} \in \varnothing \rightarrow$ FALSE.

$\{\varnothing\}$ is a set containing $\varnothing$ as an element. Whereas, $\varnothing$ is a set containing no element. Therefore, the statement is false as $\varnothing$ is member of set $\{\varnothing\}$ but is not a member of set $\varnothing$.

12. $\{\varnothing\} \subseteq \{\varnothing\} \rightarrow$ TRUE.

Every set is a subset of itself. Therefore, $\{\varnothing\} \subseteq \{\varnothing\}$ is true.

13. $\{\varnothing\} \in \{\varnothing\} \rightarrow$ FALSE.

The statement is false.

14. $\{a, b\} \subseteq \{a, b, c, \{a, b, c\}\} \rightarrow$ TRUE.

The element a and b of set $\{a, b\}$ are members of set $\{a, b, c, \{a, b, c\}\}$. Therefore, $\{a, b\}$ is a subset of $\{a, b, c, \{a, b, c\}\}$.

15. $1 \in \{a + 2b \mid a, b \text{ even integer}\} \rightarrow$ FALSE.
If a and b are even integers then, $a + 2b$ will always even and 1 is not even number. Therefore, the statement is false.

16. $\{a, \varnothing\} \in \{a, \{a, \varnothing\}\} \rightarrow$ TRUE.

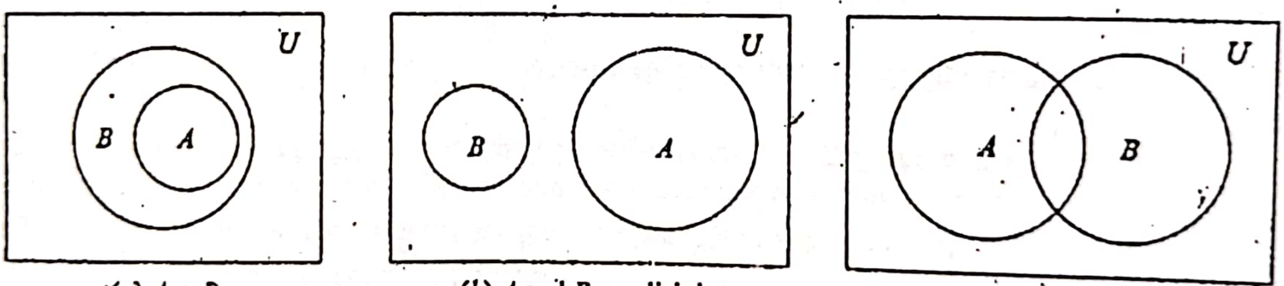
Theorem 1.1: (i) For any set A, we have $\varnothing \subseteq A \subseteq U$.
(ii) For any set A, we have $A \subseteq A$.
(iii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
(iv) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

1.5 VENN DIAGRAMS

A Venn diagram is a pictorial representation of sets in which sets are represented by enclosed areas in the plane.

The universal set U is represented by the interior of a rectangle, and the other sets are represented by disks lying within the rectangle. If $A \subseteq B$, then the disk representing A will be entirely within the disk representing B as in Fig. 1.1(a). If A and B are disjoint, i.e., if they have no elements in common, then the disk representing A will be separated from the disk representing B as in Fig. 1.1(b).

However, if A and B are two arbitrary sets, it is possible that some objects are in A but not in B, some are in B but not in A, some are in both A and B, and some are in neither A nor B; hence in general we represent A and B as in Fig. 1.1(c).



(a) $A \subseteq B$ (b) A and B are disjoint (c)

Fig. 1.1

Arguments and Venn Diagrams

Many verbal statements are essentially statements about sets and can therefore be described by Venn diagrams.

Hence Venn diagrams can sometimes be used to determine whether or not an argument is valid. Consider the following example.

Example 1.3

Show that the following argument (adapted from a book on logic by Lewis Carroll, the author of *Alice in Wonderland*) is valid:

S_1: My saucepans are the only things I have that are made of tin.
S_2: I find all your presents very useful.

23

S_3: None of my saucepans is of the slightest use.

S: Your presents to me are not made of tin.

(The statements S_1, S_2, and S_3 above the horizontal line denote the assumptions, and the statement S below the line denotes the conclusion. The argument is valid if the conclusion S follows logically from the assumptions S_1, S_2, and S_3.)

By S_1 the tin objects are contained in the set of saucepans and by S_3 the set of saucepans and the set of useful things are disjoint: hence draw the Venn diagram of Fig. 1.2.

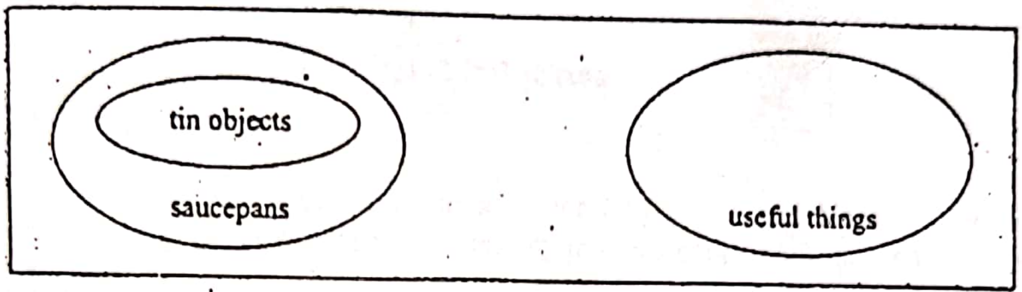


Fig. 1.2

By S_2 the set of "your presents" is a subset of the set of useful things; hence draw Fig. 1.3.

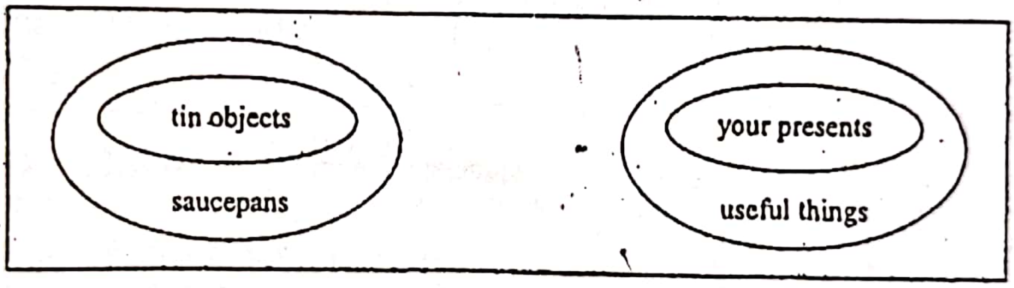


Fig. 1.3

The conclusion is clearly valid by the above Venn diagram because the set of "your presents" is disjoint from the set of tin objects.

1.6 SET OPERATIONS

This section introduces a number of important operations on sets.

Union and Intersection

The *union* of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to A or to B; that is,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

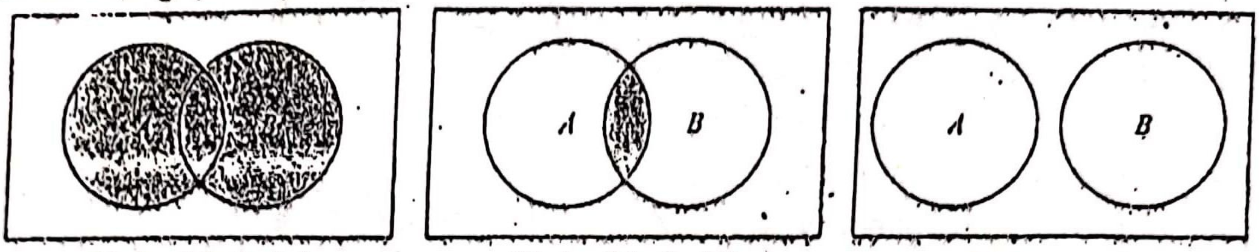
Here "or" is used in the sense of and/or. Figure 1.4(a) is a Venn diagram in which $A \cup B$ is shaded.

The *intersection* of two sets A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B; that is,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Figure 1.4(b) is a Venn diagram in which $A \cap B$ is shaded.

If $A \cap B = \emptyset$, that is, if A and B do not have any elements in common, then A and B are said to be *disjoint* or *nonintersecting*. (see Fig. 1.4 (c))



(a) $A \cup B$ is shaded (b) $A \cap B$ is shaded (c) $A \cap B = \emptyset$

Fig. 1.4

Example 1

1. (a) Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$, $C = \{2, 3, 5, 7\}$. Then

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\} \qquad A \cap B = \{3, 4\}$$

$$A \cup C = \{1, 2, 3, 4, 5, 7\} \qquad A \cap C = \{2, 3\}$$

(b) Let M denote the set of male students in a university C, and let F denote the set of female students in C. Then

$$M \cup F = C$$

since each student in C belongs to either M or F. On the other hand,

$$M \cap F = \emptyset$$

since no student belongs to both M and F.

2. Let A and B be sets such that $(A \cap B) \subseteq B$ and $B \not\subset A$. Draw corresponding Venn diagram.

Venn diagram of Fig. 1.5 shows that $(A \cap B) \subseteq B$. As $(A \cap B) \subseteq B$, it means that every member of A is also a member of B that is $A \subseteq B$. And $B \not\subset A$ means A and B are not equal. Hence $A \subseteq B$.

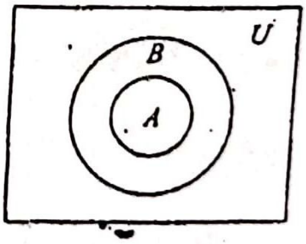


Fig. 1.5

3. Let A, B and C be sets such that $A \subseteq B$, $A \subseteq C$, $B \cap C \subseteq A$ and $A \subseteq (B \cap C)$. Draw Venn diagram.

Here $(B \cap C) \subseteq A$ and $A \subseteq (B \cap C)$. This means $A = (B \cap C)$.

Hence $A \subseteq B$ and $A \subseteq C$.

The Venn diagram (Fig. 1.6) shows this.

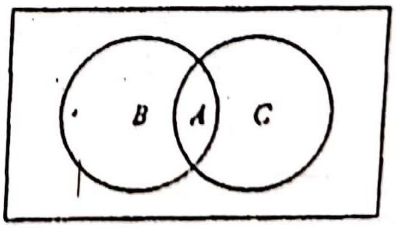


Fig. 1.6

24

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4. Let A, B, C be sets such that $(A \cap B \cap C) = \varnothing$, $A \cap B = \varnothing$, $B \cap C = \varnothing$, and $A \cap C = \varnothing$. This means A, B and C are all disjoint sets. This can be shown using Venn diagram (Fig. 1.7).

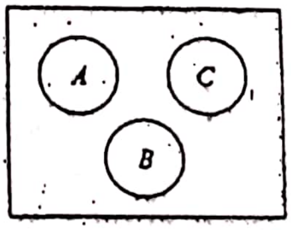


Fig. 1.7

5. For each of the following, what relation must hold between sets A and B? Draw Venn diagram.
   (a) $A \cap B = B$
   This relation between A and B is $B \subseteq A$ (Fig. 1.8).
   (b) $A \cup B = A$ and $A \neq B$
   For this the relation between A and B is $B \subseteq A$ (Fig. 1.9).

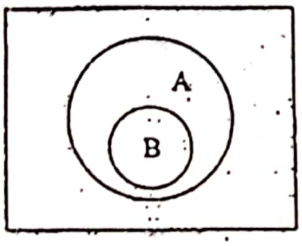


Fig. 1.8

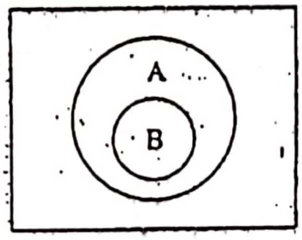


Fig. 1.9

The operation of set inclusion is closely related to the operations of union and intersection, as shown by the following theorem.

Theorem 1.2: The following are equivalent: $A \subseteq B$, $A \cap B = A$, and $A \cup B = B$.

Note: This theorem is proved in Problem 1.27. Other conditions equivalent to A Ō B are given in Problem 1.37.

Complements

Recall that all sets under consideration at a particular time are subsets of a fixed universal set U. The *absolute complement* or, simply, *complement* of a set A, denoted by A^c, is the set of elements which belong to U but which do not belong to A; that is,

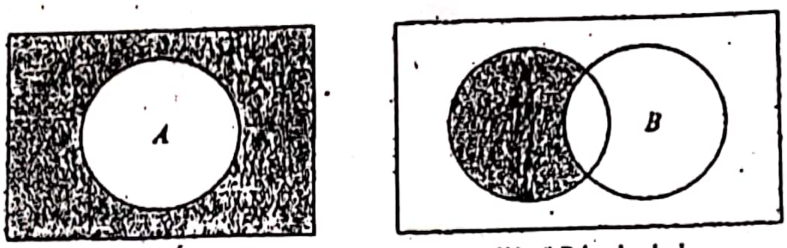
$$A^c = \{x : x \in U, x \notin A\}$$

Some texts denote the complement of A by A' or $\bar{A}$. Figure 1.10(a) is a Venn diagram in which A^c is shaded.

The *relative complement* of a set B with respect to a set A or, simply, the *difference* of A and B, denoted by $A \backslash B$, is the set of elements which belong to A but which do not belong to B; that is

$$A \backslash B = \{x : x \in A, x \notin B\}$$

The set $A \backslash B$ is read "A minus B". Many texts denote $A \backslash B$ by $A - B$ or $A \sim B$. Figure 1.10(b) is a Venn diagram in which $A \backslash B$ is shaded.



(a) A^c is shaded (b) $A\backslash B$ is shaded

Fig. 1.10

Example 1.5

Suppose $U = \mathbf{N} = \{1, 2, 3, \ldots\}$, the positive integers, is the universal set. Let

$$A = \{1, 2, 3, 4\}, \quad B = \{3, 4, 5, 6, 7\}, \quad C = \{6, 7, 8, 9\}$$

and let $E = \{2, 4, 6, 8, \ldots\}$, the even integers. Then

$$A^c = \{5, 6, 7, 8, \ldots\}, \quad B^c = \{1, 2, 8, 9, 10, \ldots\}, \quad C^c = \{1, 2, 3, 4, 5, 10, 11, \ldots\}$$

and

$$A\backslash B = \{1, 2\}, \quad B\backslash C = \{3, 4, 5\}, \quad B\backslash A = \{5, 6, 7\}, \quad C\backslash E = \{7, 9\}$$

Also, $E^c = \{1, 3, 5, \ldots\}$, the odd integers.

Fundamental Products

Consider n distinct sets $A_1, A_2, \ldots, A_n$. A *fundamental product* of the sets is a set of the form

$$A_1^* \cap A_2^* \cap \ldots \cap A_n^*$$

where A_i^* is either A_i or A_i^c. We note that (1) there are 2^n such fundamental products, (2) any two such fundamental products are disjoint, and (3) the universal set U is the union of all the fundamental products (Supplementary Problem 1.36). There is a geometrical description of these sets which is illustrated below in Example 1.6.

Symmetric Difference

The *symmetric difference* of sets A and B, denoted by $A \oplus B$, consists of those elements which belong to A or B but not to both; that is,

$$A \oplus B = (A \cup B)\backslash(A \cap B)$$

One can also show that

$$A \oplus B = (A\backslash B) \cup (B\backslash A)$$

For example, suppose $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{4, 5, 6, 7, 8, 9\}$. Then

$$A\backslash B = \{1, 2, 3\}, \quad B\backslash A = \{7, 8, 9\} \quad \text{and so} \quad A \oplus B = \{1, 2, 3, 7, 8, 9\}$$

Figure 1.12 is a Venn diagram in which $A \oplus B$ is shaded.

Example 1.6

1. Consider three sets A, B, and C. The following lists the eight fundamental products of the three sets:

$P_1 = A \cap B \quad P_2 = A^c \cap B \quad P_3 = A \cap B^c \quad P_4 = A^c \cap B^c$

$2^2 = 4$

$P_1 = A \cap B \cap C,$ $P_3 = A \cap B^c \cap C,$ $P_5 = A^c \cap B \cap C,$ $P_7 = A^c \cap B^c \cap C$

$P_2 = A \cap B \cap C^c,$ $P_4 = A \cap B^c \cap C^c,$ $P_6 = A^c \cap B \cap C^c,$ $P_8 = A^c \cap B^c \cap C^c$

These eight products correspond precisely to the eight disjoint regions in the Venn diagram of sets A, B, C in Fig. 1.11 as indicated by the labeling of the regions.

$2^3 = 8$

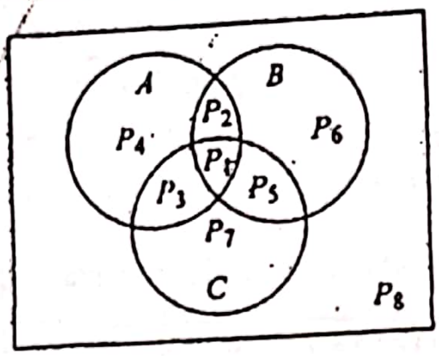
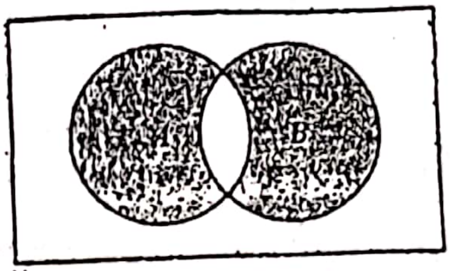
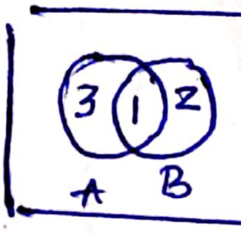


Fig. 1.11



$A \oplus B$ is shaded

Fig. 1.12



2. For each of the following conditions, what relation must be held between sets. Draw Venn diagram.

(a) $A^c \cap U = \varnothing$

Here A^c and $\cup$ are disjoint sets. This means A and $\cup$ are equal, as shown in Venn diagram drawn below (see Fig. 1.13).

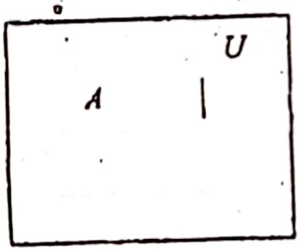


Fig. 1.13

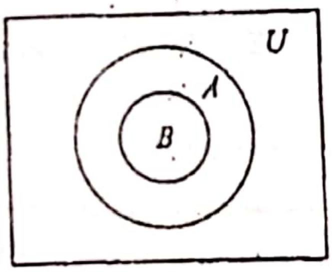


Fig. 1.14

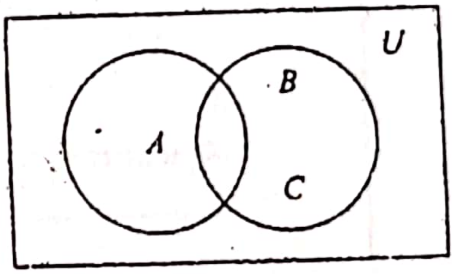


Fig. 1.15

(b) $(A \cap B)^c = B^c$

Here $(A \cap B)^c = A^c \cup B^c$

Hence as $A^c \cup B^c = B^c$, B must be a subset of A (see Fig. 1.14).

(c) $A \cap B = A \cap C$ and $A^c \cap B = A^c \cap C$

For above conditions, B and C should be equal sets (see Fig. 1.15).

1.7 ALGEBRA OF SETS AND DUALITY

Sets under the operations of union, intersection, and complement satisfy various laws or identities which are listed in Table 1.1. In fact, we formally state this:

Theorem 1.3: Sets satisfy the laws in Table 1.1.

There are two methods of proving equations involving set operations. One way is to use what it means for an object x to be an element of each side, and the other way is to use Venn diagrams. For example, consider the first of DeMorgan's laws.

$$(A \cup B)^c = A^c \cap B^c$$

Method 1: We first show that $(A \cup B)^c \subseteq A^c \cap B^c$. If $x \in (A \cup B)^c$, then $x \notin A \cup B$. Thus $x \notin A$ and $x \notin B$, and so $x \in A^c$ and $x \in B^c$. Hence $x \in A^c \cap B^c$.

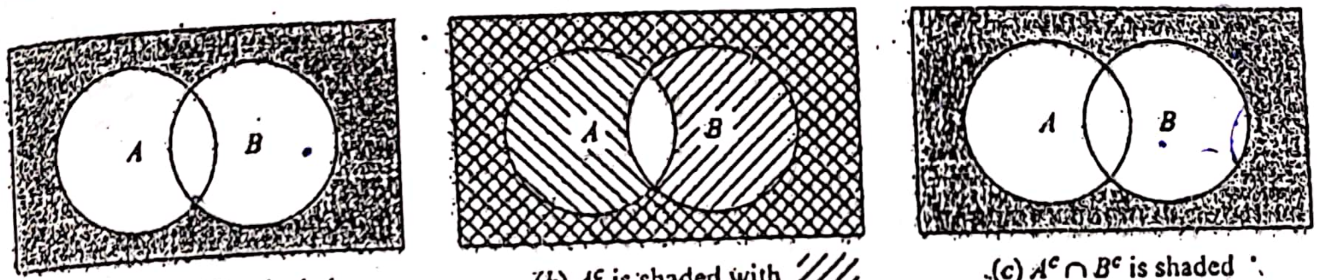
Next we show that $A^c \cap B^c \subseteq (A \cup B)^c$. Let $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$, so $x \notin A$ and $x \notin B$. Hence $x \notin A \cup B$, so $x \in (A \cup B)^c$.

We have proven that every element of $(A \cup B)^c$ belongs to $A^c \cap B^c$ and that every element of $A^c \cap B^c$ belongs to $(A \cup B)^c$. Together, these inclusions prove that the sets have the same elements, i.e., that $(A \cup B)^c = A^c \cap B^c$.

Method 2: From the Venn diagram for $A \cup B$ in Fig. 1.9 we see that $(A \cup B)^c$ is represented by the shaded area in Fig. 1.16(a). To find $A^c \cap B^c$, the area in both A^c and B^c, we shaded A^c with strokes in one direction and B^c with strokes in another direction as in Fig. 1.16(b). Then $A^c \cap B^c$ is represented by the crosshatched area, which is shaded in Fig. 1.16(c). Since $(A \cup B)^c$ and $A^c \cap B^c$ are represented by the same area, they are equal.

Table 1.1 *Laws of the algera of sets* Identities

			Idempotent laws
(1a)	$A \cup A = A$	(1b)	$A \cap A = A$
			Associative laws
(2a)	$(A \cup B) \cup C = A \cup (B \cup C)$	(2b)	$(A \cap B) \cap C = A \cap (B \cap C)$
			Commutative laws
(3a)	$A \cup B = B \cup A$	(3b)	$A \cap B = B \cap A$
			Distributive laws
(4a)	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	(4b)	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
			Identity laws
(5a)	$A \cup \varnothing = A$	(5b)	$A \cap U = A$
(6a)	$A \cup U = U$	(6b)	$A \cap \varnothing = \varnothing$
			Involution laws
(7)			$(A^c)^c = A$
			Complement laws
(8a)	$A \cup A^c = U$	(8b)	$A \cap A^c = \varnothing$
(9a)	$U^c = \varnothing$	(9b)	$\varnothing^c = U$
			DeMorgan's laws
(10a)	$(A \cup B)^c = A^c \cap B^c$	(10b)	$(A \cap B)^c = A^c \cup B^c$



(a) $(A \cup B)^c$ is shaded

(b) A^c is shaded with ////, B^c is shaded with \\\\\\\\

(c) $A^c \cap B^c$ is shaded

Fig. 1.16

Duality

Note that the identities in Table 1.1 are arranged in pairs, as, for example, (2a) and (2b). We now consider the principle behind this arrangement. Suppose E is an equation of set algebra. The *dual* E^* of E is the equation obtained by replacing each occurrence of $\cup$, $\cap$, U, and $\varnothing$ in E by $\cap$, $\cup$, $\varnothing$ and U, respectively. For example, the dual of

$$(U \cap A) \cup (B \cap A) = A \quad \text{is} \quad (\varnothing \cup A) \cap (B \cup A) = A$$

Observe that the pairs of laws in Table 1.1 are duals of each other. It is a fact of set algebra, called the *principle of duality*, that, if any equation E is an identity, then its dual E^* is also an identity.

Ordered Set

We defined set as an unordered collection of distinct objects. Let us study the notion of ordered set. When we enlist the set elements, the order in which we list them does not matter. The sequence has no relevance. Hence, the sets $\{a, b, c\}$ and $\{b, c, a\}$ both represent the same set. But while enlisting the elements, if the sequence has some relevance then the set is said to be an ordered set. The *ordered set is defined as ordered collection of distinct objects*. Let us consider an example of listing the students roll numberwise as $\{3, 6, 7, 8, 9\}$. Another example could be week days = { SUN, MON, TUE, WED, THU, FRI, SAT}. Here the first member of set is SUN, 2nd member is MON, and 7th member is SAT. One may refer them with this order. Enumerated data type set in many programming languages and list in LISP are examples of ordered sets. Ordered sets are also defined in terms of ordered pairs.

Ordered Pairs

An ordered pair of objects is a pair of objects arranged in some order. Thus in the set (a, b) of two objects a is first and b is second object of a pair. Thus, (a, b) and (b, a) are two different ordered pairs. Also the two objects in ordered pair need not be distinct. Thus, (a, a) is a well-defined ordered pair.

An ordered triple is ordered triple of objects (a, b, c) where a is first, b is second and c is the third element of triple. An ordered triple can also be written in terms of ordered pair as $((a, b), c)$.

Similarly, ordered quadruple is an ordered pair $(((a, b), c), d)$ with first element as ordered triple.

An ordered n-tuple is an ordered pair where the first component is an ordered $(n - 1)$ tuple. Let us redefine ordered set and ordered pair now. An ordered n-tuple is an ordered set with n elements. First component of this n-tuple is an ordered $(n - 1)$-tuple and nth element is second component.

26

An ordered set of n elements is ordered pair of $(n-1)$-tuple and an element. For example, an ordered set of 5 elements $\{a, b, c, d, e\}$ can be represented as $\{(((a, b), c), d), e\}$.

The Cartesian Product of Sets

There is yet one more way by which two or more sets can be combined to obtain another. If A and B are two sets, the cartesian product (or cross product or direct product) of A and B is the set,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

The cartesian product of $A \times A$ is denoted as A_2. More generally,

$$A_n = A \times A \times \ldots \times A \ (n \text{ times})$$

$$= \{(a_1, a_2, \ldots a_n) \mid a_i \in A, i = 1, 2, \ldots n\}$$

Let us redefine the cartesian product in terms of ordered pairs.

The Cartesian Product of A and B is denoted as $A \times B$, is set of all ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

Example 1.7

Let $A = \{a, b\}$ and $B = \{a, c, d\}$

$$A \times B = \{a, b\} \times \{a, c, d\}$$

$$= \{(a, a), (a, c), (a, d), (b, a), (b, c), (b, d)\}$$

and

$$B \times A = \{a, c, d\} \times \{a, b\}$$

$$= \{(a, a), (a, b), (c, a), (c, b), (d, a), (d, b)\}$$

The elements of $A \times B$ are ordered pairs of the order count, i.e. $(a, b) \neq (b, a)$ nless $a = b$.

From above example, we also note that $A \times B \neq B \times A$.

We shall study more details and use cartesian product of set in forthcoming chapters.

1.8 FINITE, INFINITE SETS AND COUNTING PRINCIPLE

1.8.1 Finite Sets

Sets are extensively used in counting problems for which we need to study about the size of the sets. Naturally, it is clear that by the size of the set we mean the number of distinct elements in the set. How do we count number of distinct elements in the set? Let us consider an example.

"Count number of distinct words in this statement." In practice, we usually point or pickup the object and say the number in sequence starting with one as one, two, three, ..., and so on, stopping when we reach the last one, counting each object exactly once.

Count	number	of	distinct	words	in	this	statement.
1	2	3	4	5	6	7	8

There are eight words in the statement.

For A, B and C, where A is $\{1, 2, 3, 4\}$, B is $\{5, \{6, 7\}, 8\}$ and C is $\{\ \}$, the sizes are 4, 3 and 0 respectively. Now what about size of set of natural numbers? What is size of set of nonnegative even numbers? How to count? We are in trouble. Indeed, we could discuss on the size of sets if we think of only size of "finite" sets. Let us learn about finite set and its definition.

Definition

A set is said to be finite if it contains exactly n distinct elements where n is a nonnegative integer. Here n is said to be "cardinality" of set. The cardinality of set is denoted by various notations in the text as $|A|$, $\#A$, card(A) or $n(A)$.

For example, cardinality of empty set, $\varnothing$, is 0 and is denoted as $|\varnothing| = 0$. $\varnothing$ and the set English alphabets are finite sets. There are many sets that are too big to be counted, the point and say process never stops.

For example, the set of even positive integers is not a finite set. What is a set that is not finite called as? A set, which is not finite, is a infinite set. Set of even positive integers is an infinite set. Let us try to get more precise definitions of finite set and infinite sets. When we talk about size of a set, clearly we talk about the number of elements in the set. Actually counting process is related with the mapping of the elements to a set of natural of numbers. One-to-one mapping is also termed as one-to-one correspondence. Given two sets A and B, we can say that one-to-one correspondence between the elements of set A and elements of set B exists if it is possible to pair off the elements in A and B such that every element of A is paired off with the distinct element in B. For example, one-to-one correspondence between the elements of sets $\{1, 2, 3, 4, 5\}$ and elements of set $\{a, b, c, d, e\}$ exits. Also a one-to-one correspondence between the elements in the set $\{a, b, \varnothing\}$ and $\{5, 6, 7\}$ exists. On the other hand, there exists no one-to-one correspondence between the elements in the set $\{1, 2, 3\}$ and elements in set $\{\{a\}\}$. Now let us redefine the terms, finite set and infinite set.

Finite Set

A set is called a finite set if there is one-to-one correspondence between the elements in the set and the elements in some set n, where n is a natural number and n is cardinality of the set. Finite sets are also termed as *numerable* sets. n is termed as cardinality of set or *cardinal number* of set.

1.8.2 Infinite Sets

A set, which is not finite, is called as infinite set. What about size of infinite set? Let us be more precise about the size of infinite set.

A set is said to be countably infinite if there is one-to-one correspondence between the elements in the set and elements in N. Equivalently if set is countably infinite, we can make a list of its members in such a way that each one corresponds uniquely to a natural number. A countably infinite set is also termed as *denumerable*. A set that is either finite or denumerable is called countable. A set, which is not countable, is called as uncountable. The set of nonnegative even integers is countably infinite.

1.8.3 Uncountably Infinite Set

A set, which is not countably infinite, is called uncountably infinite set or nondenumerable set or simply uncountable set.

The most common example of an uncountable set is the set R of all positive real numbers less than 1 that can be represented by the decimal of the form $0.a_1 a_2 a_3 \ldots$, where a_i is an integer such that $0 \le a_i \le 9$. To show that

27

the set is uncountable, we have to show that there exists no one-to-one correspondence between the elements of the given set and elements in N. We use an important proof method, known as the Cantor diagonalization argument. This proof technique is used extensively in mathematical logic and in the theory of computation. We shall take help of Cantor's diagonal arguments and the contradiction. We first assume that the set R is countably infinite and then prove that at least one real number exists, that is different from all those enumerated, showing that the enumeration is not exhaustive. Hence arriving at a contradiction. The argument is termed as 'diagonal argument', as to obtain a nonlisted member, we move along the diagonal of the list.

Let us show that the set of real numbers between 0 and 1 is an uncountable set. To show that the set is uncountable, we assume that the set is countable and arrive at a contradiction. As per our assumption if the set of real numbers between 0 and 1 is countably infinite, then there is a one-to-one correspondence between these real numbers and the elements in N. Now we could exhaustively list them as:

$$
\begin{array}{llllll}
0. & a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\
0. & a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\
0. & a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\
 & \vdots & & & \vdots & \\
0. & a_{i1} & a_{i2} & a_{i3} & a_{i4} & \cdots
\end{array}
$$

Here a_{ij} is the jth digit of the ith number. Each of these real numbers must appear somewhere on the list. We shall establish a contradiction by constructing a decimal that is not in the above list. Now let us construct a real number X as follows.

$$0.\quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad \cdots$$

where, $\quad x_i = 9 - a_{ii}$ for all i. ($i = 1, 2, 3, 4, \ldots$)

For example if $d_{ii} = 0.23795208\ldots$, then we have X as 0.76204791

This process results in number, which differ from each number in the list by at least one digit. It differs from the first number by the first digit, the second number by second digit, and so on. Thus X is not in the list. Consequently we can say that the list above (no matter how the list is constructed) is not an exhaustive listing of the set of all real between 0 and 1, a contradiction to our assumption. Hence the set of real numbers between 0 and 1 is uncountable.

Lemma 1.4: If A and B are disjoint finite sets, then $A \cup B$ is finite and

$$n(A \cup B) = n(A) + n(B)$$

Proof. In counting the elements of $A \cup B$, first count those that are in A. There are $n(A)$ of these. The only other elements of $A \cup B$ are those that are in B but not in A. But since A and B are disjoint, no element of B is in A, so there are $n(B)$ elements that are in B but not in A. Therefore, $n(A \cup B) = n(A) + n(B)$.

We also have a formula for $n(A \cup B)$ even when they are not disjoint. This is proved in Solved Problem 1.28.

1.9 THE INCLUSION–EXCLUSION PRINCIPLE

Let A and B be any finite sets. Then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

2

In other words, to find the number $n(A \cup B)$ of elements in the union $A \cup B$, we add $n(A)$ and $n(B)$ and then we subtract $n(A \cap B)$; that is, "include" $n(A)$ and $n(B)$, and we "exclude" $n(A \cap B)$. This follows from the fact that, when we add $n(A)$ and $n(B)$, we have counted the elements of $A \cap B$ twice. This principle holds for any number of sets. We first state it for three sets.

Theorem 1.5: For any finite sets A, B, C we have

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

That is, we "include" $n(A)$, $n(B)$, $n(C)$, we "exclude" $n(A \cap B)$, $n(A \cap C)$, $n(B \cap C)$, and we include $n(A \cap B \cap C)$.

Example 1.8

Find the number of mathematics students at a college taking at least one of the languages French, German, and Russian given the following data:

65 study French	20 study French and German
45 study German	25 study French and Russian
42 study Russian	15 study German and Russian
	8 study all three languages

We want to find $n(F \cup G \cup R)$ where, F, G, and R denote the sets of students studying French, German, and Russian, respectively.

By the inclusion–exclusion principle,

$$n(F \cup G \cup R) = n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R)$$
$$+ n(F \cap G \cap R)$$
$$= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100$$

Thus 100 students study at least one of the languages.

Now, suppose we have any finite number of finite sets, say, $A_1, A_2, \ldots, A_m$. Let s_k be the sum of the cardinalities

$$n(A_{i1} \cap A_{i2} \cap \ldots \cap A_{ik})$$

of all possible k-tuple intersections of the given m sets. Then we have the following general inclusion–exclusion principle.

Theorem 1.6: $n(A_1 \cup A_2 \cup \ldots A_m) = s_1 - s_2 + s_3 - \ldots + (-1)^{m-1} s_m$.

Theorem 1.7: If A and B are finite sets, then $A \cup B$ and $A \cap B$ are finite and

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

We can apply this result to obtain a similar formula for three sets:

Corollary 1.8: If A, B, and C are finite sets, then so is $A \cup B \cup C$, and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

Mathematical induction (Section 1.10) may be used to further generalize this result to any finite number of sets.

Example 1.9

Consider the following data for 120 mathematics students at a college concerning the languages French, German, and Russian:

65 study French
45 study German
42 study Russian
20 study French and German
25 study French and Russian
15 study German and Russian
8 study all three languages.

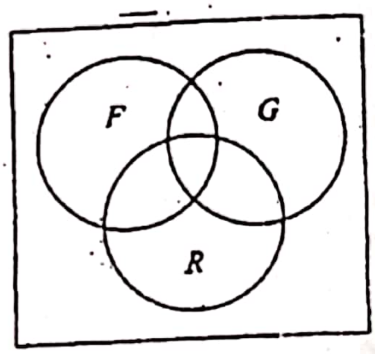


Fig. 1.17

Let F, G, and R denote the sets of students studying French, German and Russian, respectively. We wish to find the number of students who study at least one of the three languages, and to fill in the correct number of students in each of the eight regions of the Venn diagram shown in Fig. 1.17.

By Corollary 1.8

$$(F \cup G \cup R) = n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) + n(F \cap G \cap R)$$
$$= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100$$

That is $n(F \cup G \cup R) = 100$ students study at least one of the three languages.
We now use this result to fill in the Venn diagram. We have:

8 study all three languages,
$20 - 8 = 12$ study French and German but not Russian
$25 - 8 = 17$ study French and Russian but not German
$15 - 8 = 7$ study German and Russian but not French
$65 - 12 - 8 - 17 = 28$ study only French
$45 - 12 - 8 - 7 = 18$ study only German
$42 - 17 - 8 - 7 = 10$ study only Russian
$120 - 100 = 20$ do not study any of the languages

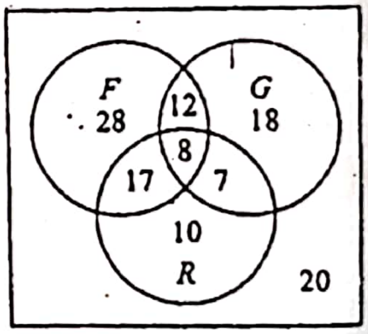


Fig. 1.18

Accordingly, the completed diagram appears in Fig. 1.18. Observe that $28 + 18 + 10 = 56$ students study only one of the languages.

1.10 CLASSES OF SETS, POWER SETS, PARTITIONS

Given a set S, we might wish to talk about some of its subsets. Thus we would be considering a set of sets. Whenever such a situation occurs, to avoid confusion we will speak of a *class* of sets of *collection* of sets rather than a set of sets. If we wish to consider some of the sets in a given class of sets, then we speak of a *subclass* or *subcollection*.

Example 1.10

Suppose $S = \{1, 2, 3, 4\}$. Let A be the class of subsets of S which contain exactly three elements of S. Then

$$A = [\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}]$$

The elements of A are the sets $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, and $\{2, 3, 4\}$.
Let B be the class of subsets of S which contain 2 and two other elements of S. Then

$$B = [\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}]$$

The elements of B are the sets $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{2, 3, 4\}$. Thus B is a subclass of A, since every element of B is also an element of A. (To avoid confusion, we will sometimes enclose the sets of a class in brackes instead of braces.)

Power Sets

For a given set S, we may speak of the class of all subsets of S. This class is called the *power set* of S, and will be denoted by Power(S). If S is finite, then so is Power(S). In fact, the number of elements in Power(S) is 2 raised to the cardinality of S; that is,

$$n(\text{Power}(S)) = 2^{n(S)}$$

(For this reason, the power set of S is sometimes denoted by 2^S.)

Example 1.11

Suppose $S = \{1,2,3\}$. Then

$$\text{Power}(S) = [\varnothing\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, S]$$

Note that the empty set $\varnothing$ belongs to Power(S) since $\varnothing$ is a subset of S. Similarly, S belongs to Power(S). As expected from the above remark, Power(S) has $2^3 = 8$ elements.

Partitions

Let S be a nonempty set. A partition of S is a subdivision of S into nonoverlapping, nonempty subsets. Precisely, a *partition* of S is a collection $\{A_i\}$ of nonempty subsets of S such that:

(i) Each a in S belongs to one of the A_i.
(ii) The sets of $\{A_i\}$ are mutually disjoint; that is, if

$$A_i \neq A_j \quad \text{then} \quad A_i \cap A_j = \varnothing$$

The subsets in a partition are called *cells*. Figure 1.19 is a Venn diagram of a partition of the rectangular set S of points into five cells, A_1, A_2, A_3, A_4, and A_5.

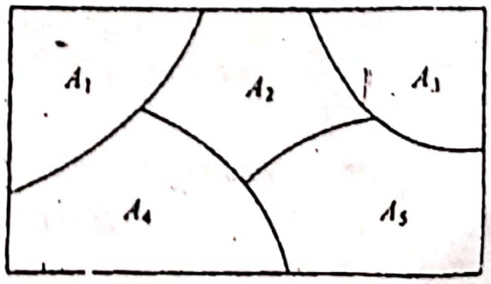


Fig. 1.19

tiset. *Multiset* is an unordered collection of elements where an element can occur as a member more than once. Hence, multiset is a set in which elements are not necessarily distinct. {1, 1, 1, 2, 2, 3}, {2, 2, 2}, {X, Y, Z} and { } are examples of multiset.

Let $A = \{1, 1, 1, 2, 2, 3\}$. Here 1 appears three times, 2 appears twice and three appears once in set. The notation used to represent multiset is as

$$S = \{ n_1.a_1, n_2.a_2, \ldots, n_i.a_i, \ldots\}$$

This denotes that a_1 occurs n_1 times, a_2 occurs n_2 times, a_i occurs n_i times, and so on.

The number n_i, $i = 1, 2\ldots$ are called the *multiplicities* of the elements a_i.

Set A can be written as {3.1, 2.2, 1.3}.

The *multiplicity* of an element in a multiset is defined to be the number of times the element appears in the multiset. Note that the set is special case of multiset in which multiplicity of each element is either 0 or 1. Cardinality of set $A = \{1, 1, 1, 2, 2, 3\}$ is $|A| = 3$. The cardinality of a multiset is the cardinality of the set in correspondence to, assuming that the elements in the multiset are distinct.

Let A and B be multisets. The *union* of the multisets A and B is the multiset where the multiplicity of an element is the maximum of its multiplicities in A and B.

Let
$$A = \{1, 1, 1, 2, 2, 3\} \text{ and}$$
$$B = \{1, 1, 4, 3, 3\}$$

Now
$$A \cup B = \{ 1, 1, 1, 4, 2, 2, 3, 3\}$$

The *intersection* of A and B is the multiset where the multiplicity of an element is the minimum of its multiplicities in A and B.

$$A \cap B = \{1, 1, 3\}$$

The *difference* of A and B is the multiset where the multiplicity of an element is the multiplicity of element in A less its multiplicity in B unless this difference is negative, in which case the multiplicity is zero.

Let
$$A = \{1, 1, 1, 2, 2, 3, 4, 4, 5\}$$
$$B = \{1, 1, 2, 2, 2, 3, 3, 4, 4, 6\}$$

Now
$$A - B = \{1, 5\}$$

The *sum* of A and B is the multiset where the multiplicity of an element is sum of multiplicities in set A and set B, denoted by $A + B$.

Let
$$A = \{1, 1, 2, 3, 3\} \quad \text{and} \quad B = \{1, 2, 2, 4\}, \text{ we have } A + B \text{ as}$$
$$A + B = \{1, 1, 1, 2, 2, 2, 3, 3, 4\}$$

Example 1.19

Let A and B be multisets as $A = \{3.a, 2.b, 1.c\}$ and $B = \{2.a, 3.b, 4.d\}$

Find

(a) $A \cup B = \{3.a, 3.b, 1.c, 4.d\}$
(b) $A \cap B = \{2.a, 2.b\}$
(c) $A \setminus B = \{1.a, 1.c\}$
(d) $B - A = \{1.b, 4.d\}$
(e) $A + B = \{5.a, 5.b, 1.c, 4.d\}$

SOLVED PROBLEMS

SETS AND SUBSETS

(r, t, s) (s, t, r, s) (t, s, ~~t~~, ~~r~~) (s, r, s, t)

1.1. Which of these sets are equal: $\{r, t, s\}$, $\{s, t, r, s\}$, $\{t, s, t, r\}$, $\{s, r, s, t\}$? [illegible]

They are all equal. Order and repetition do not change a set.

1.2. List the elements of the following sets; here $N = \{1, 2, 3, \ldots\}$.
(a) $A = \{x : x \in N, 3 < x < 12\}$ [illegible]
(b) $B = \{x : x \in N, x \text{ is even}, x < 15\}$ [illegible]
(c) $C = \{x : x \in N, 4 + x = 3\}$

N = (1, 2, 3 —)
$A = \{x : x \in N; 3 < x < 12\}$
$B = \{x : x \in N, x \text{ is even}, x < 15\}$
$C = \{x : x \in N, 4 + x = 3\}$

(a) A consists of the positive integers between 3 and 12; hence
$$A = \{4, 5, 6, 7, 8, 9, 10, 11\}$$
(b) B consists of the even positive integers less than 15; hence
$$B = \{2, 4, 6, 8, 10, 12, 14\}$$
(c) There are no positive integers which satisfy the condition $4 + x = 3$; hence C contains no elements. In other wods, $C = \varnothing$, the empty set.

1.3. Consider the following sets: [illegible]
$\varnothing$, $A = \{1\}$, $B = \{1, 3\}$, $C = \{1, 5, 9\}$, $D = \{1, 2, 3, 4, 5\}$, $E = \{1, 3, 5, 7, 9\}$, $U = \{1, 2, \ldots, 8, 9\}$
Insert the correct symbol $\subseteq$ or $\not\subseteq$ between each pair of sets:
(a) $\varnothing, A$ (b) A, B (c) B, C (d) B, E (e) C, D (f) C, E (g) D, E (h) D, U

(a) $\varnothing \subseteq A$ because $\varnothing$ is a subset of every set.
(b) $A \subseteq B$ because 1 is the only element of A and it belongs to B.
(c) $B \not\subseteq C$ because $3 \in B$ but $3 \notin C$.
(d) $B \not\subseteq E$ because the elements of B also belong to E.
(e) $C \not\subseteq D$ because $9 \in C$ but $9 \notin D$.

$\subseteq$ $\not\subset$

A = {1}
B = {1, 3}
C = {1, 5, 9}
D = {1 – 5}
E = {1, 3, 5, 7, 9}
U = {1 — 9}

(a) $\varnothing$ A (b) A B (c) B C
(d) B E (e) C D (f) C E
(g) D E (h) D U

33

(f) $C \subseteq E$ because the elements of C also belong to E.

(g) $D \not\subseteq E$ because $2 \in D$ but $2 \notin E$.

(h) $D \subseteq U$ because the elemets of D also belong to U.

1.4 Show that $A = \{2, 3, 4, 5\}$ is not a subset of $B = \{x : x \in \mathbf{N}, x \text{ is even}\}$.

It is necessary to show that at least one element in A does not belong to B. Now $3 \in A$ and, since B consists of even numbers, $3 \notin B$; hence A is not a subset of B.

1.5 Show that $A = \{2, 3, 4, 5\}$ is a proper subset of $C = \{1, 2, 3, ..., 8, 9\}$.

Each element of A belongs to C so $A \subseteq C$. On the other hand, $1 \in C$ but $1 \notin A$. Hence $A \neq C$. Therefore A is a proper subset of C.

SET OPERATIONS

Solved Problems 1.6 to 1.8 refer to the universal set $U = \{1, 2, ..., 9\}$ and the sets

$$A = \{1, 2, 3, 4, 5\}, \quad C = \{5, 6, 7, 8, 9\}, \quad E = \{2, 4, 6, 8\}$$
$$B = \{4, 5, 6, 7\}, \quad D = \{1, 3, 5, 7, 9\}, \quad F = \{1, 5, 9\}$$

1.6 Find:

(a) $A \cup B$ and $A \cap B$ (b) $B \cup D$ and $B \cap D$ (c) $A \cup C$ and $A \cap C$
(d) $D \cup E$ and $D \cap E$ (e) $E \cup E$ and $E \cap E$ (f) $D \cup F$ and $D \cap F$

Recall that the union $X \cup Y$ consists of those elements in either X or Y (or both) and that the intersection $X \cap Y$ consists of those elements in both X and Y.

(a) $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$ $\qquad A \cap B = \{4, 5\}$

(b) $B \cup D = \{1, 3, 4, 5, 6, \cancel{7}, 9\}$ $\qquad B \cap D = \{5, 7\}$

(c) $A \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} = U$ $\qquad A \cap C = \{5\}$

(d) $D \cup E = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} = U$ $\qquad D \cap E = \varnothing$

(e) $E \cup E = \{2, 4, 6, 8\} = E$ $\qquad E \cap E = \{2, 4, 6, 8\} = E$

(f) $D \cup F = \{1, 3, 5, 7, 9\} = D$ $\qquad D \cap F = \{1, 5, 9\} = F$

Observe that $F \subseteq D$; so by Theorem 1.2 we must hav $D \cup F = D$ and $D \cap F = F$.

1.7 Find: (a) A^c, B^c, D^c, E^c; (b) $A \setminus B, B \setminus A, D \setminus E, F \setminus D$; (c) $A \oplus B, C \oplus D, E \oplus F$.

Recall that:

(1) The complement X^c consists of those elements in the universal set U which do not belong to X.

(2) The difference $X \setminus Y$ consist of the elements in X which do not belong to Y.

(3) the symmetric difference $X \oplus Y$ consists of the elements in X or in Y but not in both X and Y.

Therefore:

(a) $A^c = \{6, 7, 8, 9\}$; $B^c = \{1, 2, 3, 8, 9\}$; $D^c = \{2, 4, 6, 8\} = E$; $E^c = \{1, 3, 5, 7, 9\} = D$.

(b) $A \backslash B = \{1, 2, 3\}$; $B \backslash A = \{6, 7\}$; $D \backslash E = \{1, 3, 5, 7, 9\} = D$; $F \backslash D = \varnothing$.

(c) $A \oplus B = \{1, 2, 3, 6, 7\}$; $C \oplus D = \{1, 3, 8, 9\}$; $E \oplus F = \{2, 4, 6, 8, 1, 5, 9\} = E \cup F$.

1.8 Find: (a) $A \cap (B \cup E)$; (b) $(A \backslash E)^c$; (c) $(A \cap D) \backslash B$; (d) $(B \cap F) \cup (C \cap E)$.

(a) First compute $B \cup E = \{2, 4, 5, 6, 7, 8\}$. Then $A \cap (B \cup E) = \{2, 4, 5\}$.

(b) $A \backslash E = \{1, 3, 5\}$. Then $(A \backslash E)^c = \{2, 4, 6, 7, 8, 9\}$.

(c) $A \cap D = \{1, 3, 5\}$. Now $(A \cap D) \backslash B = \{1, 3\}$.

(d) $B \cap F = \{5\}$ and $C \cap E = \{6, 8\}$. So $(B \cap F) \cup (C \cap E) = \{5, 6, 8\}$.

1.9 Show that we can have $A \cap B = A \cap C$ without $B = C$.

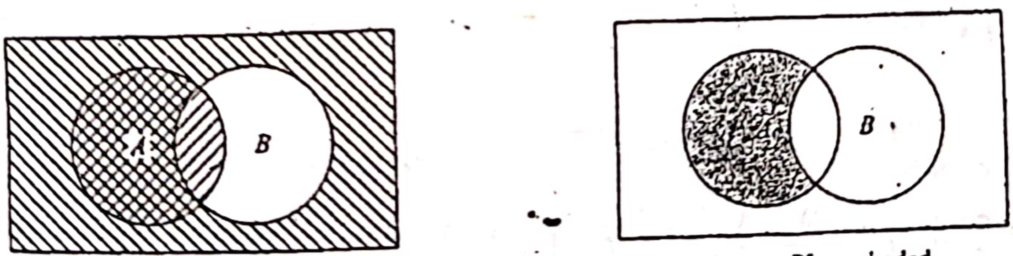
Let $A = \{1, 2\}$, $B = \{2, 3\}$, and $C = \{2, 4\}$. Then $A \cap B = \{2\}$ and $A \cap C = \{2\}$. Thus $A \cap B = A \cap C$ but $B \neq C$.

VENN DIAGRAMS

1.10 Consider the Venn diagram of two arbitrary sets A and B in Fig. 1.1(c). Shade the sets: (a) $A \cap B^c$; (b) $(B \backslash A)^c$.

(a) First shade the area represented by A with strokes in one direction (///), and then shade the area represented by B^c (the area outside B), with strokes in another direction (\\\\\\). This is shown in Fig. 1.20(a). The cross-hatched area is the intersection of these two sets and represents $A \cap B^c$ and this is shown in Fig. 1.20(b). Observe that $A \cap B^c = A \backslash B$. In fact, $A \backslash B$ is sometmes defined to be $A \cap B^c$.

$A \cap B^c$



(a) A and B^c are shaded

(b) $A \cap B^c$ are shaded

Fig. 1.20

(b) First shade the area represented by $B \backslash A$ (the area of B which does not lie in A) as in Fig. 1.21(a). Then the area outside this shaded region, which is shown in Fig. 1.21(b), represents $(B \backslash A)^c$.

$(B/A)^c$

31

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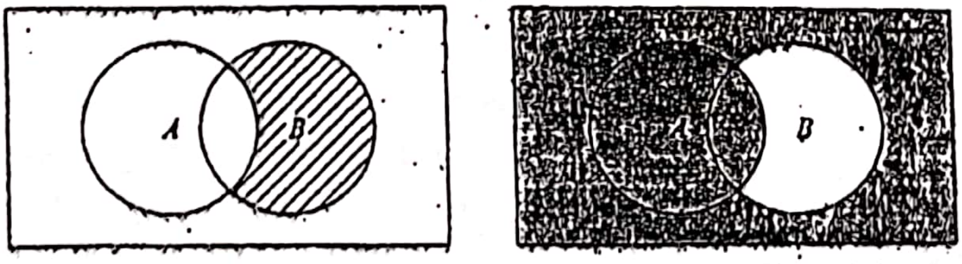


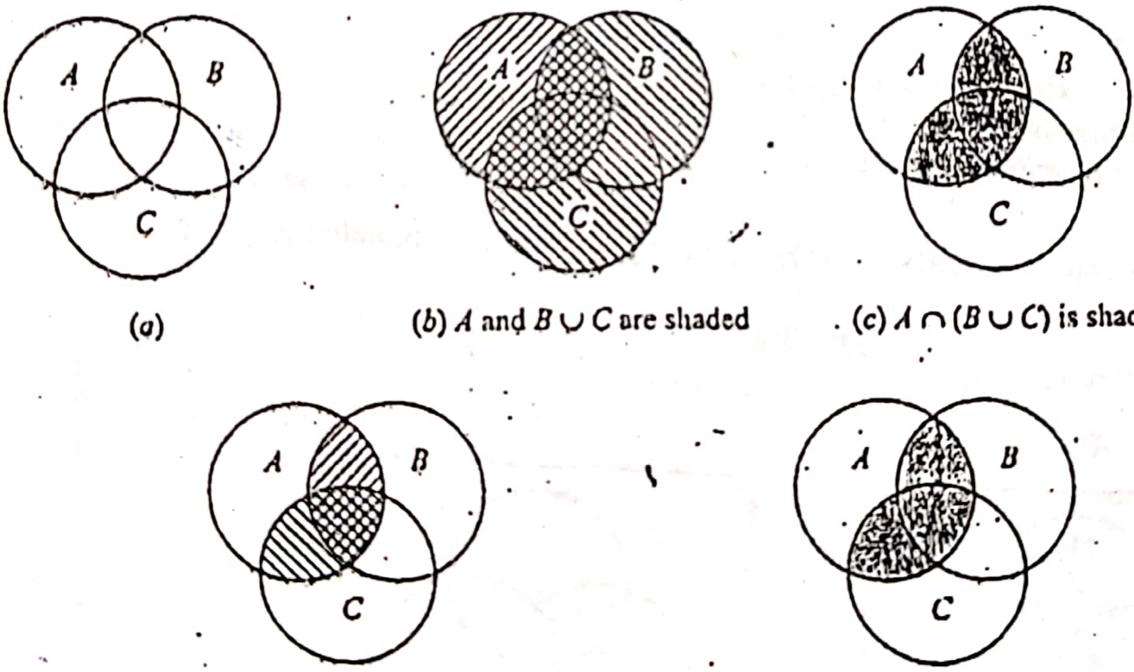
(a) $B \setminus A$ is shaded (b) $(B \setminus A)^c$ is shaded

Fig. 1.21

11. Illustrate the distributive law $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ with Venn diagrams.

Draw three intersecting circles labeled A, B, C, as in Fig. 1.22(a). Now, as in Fig. 1.22(b) shade A with strokes in one direction and shade $B \cup C$ with strokes in another direction; the crosshatched area is $A \cap (B \cap C)$, as in Fig. 1.22(c). Next shade $A \cap B$ and then $A \cap C$, as in Fig. 1.22(d); the total area shaded is $(A \cap B) \cup (A \cap C)$, as in Fig. 1.22($e$).

As expected by the distributive law, $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ are both represented by the same set of points.



(a) (b) A and $B \cup C$ are shaded (c) $A \cap (B \cup C)$ is shaded

(d) $A \cap B$ and $A \cap C$ are shaded (e) $(A \cap B) \cup (A \cap C)$ is shaded

Fig. 1.22

12. Determine the validity of the following argument:

S_1: All my friends are musicians.

S_2: John is my friend.

S_3: None of my neighbors are musicians.

S: John is not my neighbor.

The premises S_1 and S_3 lead to the Venn diagram in Fig. 1.23. By S_2, John belongs to the set of friends which is disjoint from the set of neighbors. Thus S is a valid conclusion and so the argument is valid.

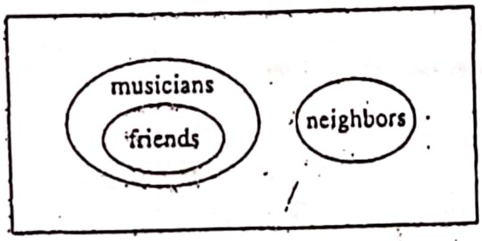


Fig. 1.23

FINITE SETS AND THE COUNTING PRINCIPLE

1.13 Determine which of the following sets are finite.

(a) A = {seasons in the year}
(b) B = {states in the Union}
(c) C = {positive integers less than 1}
(d) D = {odd integers}
(e) E = {positive integral divisors of 12}
(f) F = {cats living in the United States}

(a) A is finite since there are four seasons in the year, i.e. $n(A) = 4$.
(b) B is finite because there are 50 states in the Union, i.e. $n(B) = 50$.
(c) There are no positive integers less than 1; hence C is empty. Thus C is finite and $n(C) = 0$.
(d) D is infinite.
(e) The positive integer divisors of 12 are 1, 2, 3, 4, 6, and 12. Hence E is finite and $n(E) = 6$.
(f) Although it may be difficult to find the number of cats living in the United States, there is still a finite number of them at any point i time. Hence F is finite.

1.14 In a survey of 60 people, it was found that:

25 read *Newsweek* magazine
26 read *Time*
26 read *Fortune*
9 read both *Newsweek* and *Fortune*
11 read both *Newsweek* and *Time*
8 read both *Time* and *Fortune*
3 read all three magazines

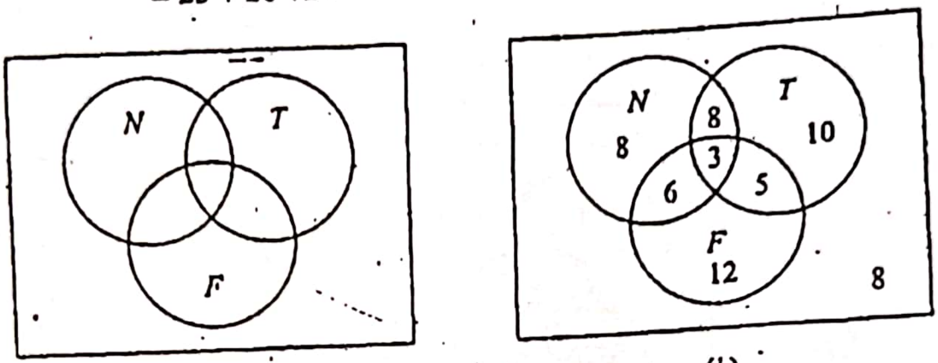
(a) Find the number of people who read at least one of the three magazines.
(b) Fill in the correct number of people in each of the eight regions of the Venn diagram in

Fig. 1.24(*a*), where *N*, *T*, and *F* denote the set of people who read *Newsweek*, *Time* and *Fortune*, respectively.

(c) Find the number of people who read exactly one magazine.

(a) We want $n(N \cup T \cup F)$. By Corollary 1.6,

$$n(N \cup T \cup F) = n(N) + n(T) + n(F) - n(N \cap T) - n(N \cap F) - n(T \cap F) + n(N \cap T \cap F)$$
$$= 25 + 26 + 26 - 11 - 9 - 8 + 3 = 52.$$



(*a*) (*b*)

Fig. 1.24

(b) The required Venn diagram in Fig. 1.24(*b*) is obtained as follows:

3 read all three magazines
$11 - 3 = 8$ read *Newsweek* and *Time* but not all three magazines
$9 - 3 = 6$ read *Newsweek* and *Fortune* but not all three magazines
$8 - 3 = 5$ read *Time* and *Fortune* but not all three magazines
$25 - 8 - 6 - 3 = 8$ read only *Newsweek*
$26 - 8 - 5 - 3 = 10$ read only *Time*
$26 - 6 - 5 - 3 = 12$ read only *Fortune*
$60 - 52 = 8$ read no magazine at all

(c) $8 + 10 + 12 = 30$ read only one magazine.

1.15. Show that the set of even positive integers is a countable set.

To show that the set of even positive integers is a countable set, we derive a one-to-one correspondence between this set and the set of natural numbers, *N*.

1	2	3	4	5	6	7	8	
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	....
2	4	6	8	10	12	14	16	

To confirm one-to-one correspondence, we pair of every *n* of *N* with 2*n* of set of even positive integers. By pairing of each *n* in *N* with 2*n* in the said set, we get pairs as (*n* and 2*n*) which is unique pair of each number in set of even positive integers. Hence set of positive integer is countably infinite.

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1.16 Give example to show that the intersection of two countably infinite sets can either be finite or countably infinite.

The intersection of two countably infinite sets can be either finite or countably infinite. Consider the following example. For all examples, A and B are countably infinite.

1. Let A be set of odd natural number and let B be set of even natural number. Now $A \cap B = \varnothing$ and $|\varnothing| = 0$ is a finite set.
2. Let A be a set of odd natural numbers. Now $A \cap N = A$ which is countably infinite.
3. Let A be a set of prime integers. Let B be a set of odd integers. Now $A \cap B$ is countably infinite.

1.17 Give examples to show that the intersection of two uncountably infinite set can be finite, or countably infinite or uncountable. The example of intersection of two uncountably infinite set yielding either finite or countably finite or uncountably infinite are as follows.

1. The intersection of 2^N with the set of all real numbers is empty set, and thus the finite set.
2. The intersection of $2^N \cup N$ and $(2^N \times 2^N) \cup N$ results in N, i.e.

$$(2^N \cup N) \cap ((2^N \times 2^N) \cup N) = N$$

   and N is countably infinite.
3. The intersection of 2^N wth itself is uncountable.

1.18 Show that the set of positive rational number is countable. Let us list the positive rational numbers as sequence $r_1, r_2, r_3, \ldots$ Note that every positive rational number is quotient p/q of positive integers.

Let us arrange the positive rational numbers by listing those with denominator $q = 1$ in the first row, those with denominator $q = 2$ in the second row, those with denominator $q = i$ in ith row, and so on.

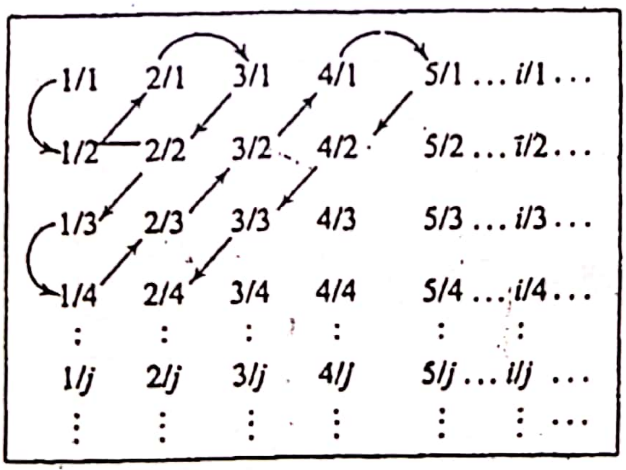


Fig. 1.25 *The set of positive rational numbers is countable*

Here the positive rational numbers are listed such that first list the positive rational numbers p/q with $p + q = 2$, followed by those with $p + q = 3$, followed by those with $p + q = 4$, and so on.

Following the zig-zag path as shown in above Fig. 1.25. If we encounter a number p/q which is already listed, we do not list the same again. For example, we come across $2/2 = 1$, we do not list it as we already listed 1/1=1, we can list first few numbers in the list of positive rational number as 1, 1/2, 2, 3, 1/3, 1/4, 2/3, 3/2, 4, 5, 1/5 ....

Such a exhausting listing includes all the positive rational numbers. One-to-one mapping with N can be formed as

$$\begin{array}{cccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots \\ 1 & 1/2 & 2 & 3 & 1/3 & 1/4 & 2/3 & 3/2 & 4 & 5 & 1/5 & 1/6 & \end{array}$$

We can verify that the set is countable, as it forms one-to-one correspondence with N. Hence set of positive rational numbers is countable.

1.19 Determine whether finite, countably infinite or uncountable, exhibit a one-to-one correspondence between the set of N and the set if exists.

(a) The set of even integers

Countably infinite, as we can form one-to-one correspondence with N as,

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7\ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 0 & -2 & 2 & -4 & 4 & -6 & 6\ldots \end{array}$$

(b) The set of negative integers

Countably infinite, as we can form one-to-one correspondence with N as,

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7\ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ -1 & -2 & -3 & -4 & -5 & -6 & -7\ldots \end{array}$$

(c) The set of integers that of multiples of 5

Countably infinite, as the elements of said set has one-to-one correspondence with elements in set N.

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 0 & 5 & 10 & 15 & 20 & 25 & 30 & 35\ldots \end{array}$$

(d) The set of real number between 0 and 0.6. The set of real numbers between 0 and 0.6 is uncountable.

(e) The set of all trees on earth.

Finite, even though the nmber is large, is finite.

1.20 Determine Cardinal number of the following sets.

(a) $A = \{n^7 \mid n \text{ is positive integer}\}$

$$A = \{1^7, 2^7, 3^7, \ldots, i^7, \ldots\}$$

Elements of set A and set N has one-to-one correspondence, hence A is countably infinite set.

(b) $B = \{n^{100} \mid n$ is positive integer$\}$

The set of $B = \{1^{100}, 2^{100}, 3^{100}, 4^{100} \ldots\}$

Each n^{100} has one-to-one correspondence with $n \in N$. Hence B s countably infinite set.

21 Show that at the most countably infinite number of books can be written in English.

Let us assume that till this moment some finite number, N of books are written in English. Each next book written in English will be mapped with successor of N, and so on. This means there exists one-to-one correspondence between elements in N and elements in set of books, which will be ever written in English. Hence we say that, at the most countably infinite number of books will be ever written in English.

22 Find Cardinality of the set $A \cup B$ where A and B are any arbitrary sets.

Cardinality of $A \cup B$ depends upon Cardinalities of A and B both. Let us consider all possibilities as:

(a) A and B are both finite then,
$|A \cup B|$ will be finite.

(b) A finite and B countably infinite then,
$|A \cup B|$ will be countably infinite

(c) A countably infinite and B finite then,
$|A \cup B|$ is countably infinite.

(d) A and B both countably infinite then,
$|A \cup B|$ will be countably infinite.

(e) One of the A or B is uncountably infinite and one is countably infinite then,
$|A \cup B|$ will be uncountably infinite.

(f) Both A and B uncountably infinite then,
$|A \cup B|$ will be uncountably infinite.

23 State whether the following sets are finite, countably finite or uncountably finite.

(a) Class of all programs that can ever be written in programming language, 'C'.
Countably infinite

(b) All songs sung by the great Indian singer Lata Mangeshkar
Finite

(c) Number of fish in Pacific ocean
Finite, even though the count is too large, is finite.

(d) Set of all prime numbers
Countably infinite

(e) Set of real numbers between 0 and 1
Uncountably infinite

1.24 Give example such that intersection of countably infinite number of infinite sets may be finite.

Let

$$A = \text{set of all integers less than or equal to 2.}$$
$$A = \{2, 1, 0, -1, -2, -3 \ldots\}$$
$$N = \{\text{set of natural numbers}\}$$
$$= \{1, 2, 3, \ldots\}$$

Now $A \cap N = \{1, 2\}$ this is finite

Hence intersection of two countahly infiite sets could be finite.

MORE PROBLEMS ON PRINCIPLE OF INCLUSION AND EXCLUSION

1.25 In a class of 80 students, 50 students know English, 55 know French and 46 know German language. 37 students know English and French, 28 students know French and German, 7 students know none of the languages. Find out:

(a) How many students know all the 3 languages?

(b) How many students know exactly 2 languages?

(c) How many know only one language?

Here $|U| = 80$, total students in a class.

Let $A \rightarrow$ set of students that know English.

$$|A| = 50.$$

Let $B \rightarrow$ set of students that know French.

$$|B| = 55$$

Let $C \rightarrow$ set of students that know German.

$$|C| = 46 \text{ and}$$

7 students know none of the languages, that is, $|\overline{A \cup B \cup C}| = 7$. This can also be written as,

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = 7$$

Also, $|A \cap B| = 37$ and $|B \cap C|$ 28 and $|A \cap C| = 25$

As $|\overline{A \cup B \cup C}| = 7$ we get, $|U| - |A \cup B \cup C| = 7$

$\therefore 80 - |A \cup B \cup C| = 7$

Hence, $|A \cup B \cup C| = 80 - 7 = 73$. Now let us solve sub-question ⓐ, ⓑ and ⓒ.

(a) Number of students who know all the three languages is $|A \cap B \cap C|$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$$73 = 50 + 55 + 46 - 37 - 28 - 25 + |A \cap B \cap C|$$

$$|A \cap B \cap C| = 73 - 50 - 55 - 46 + 37 + 28 + 25$$

$$= 12$$

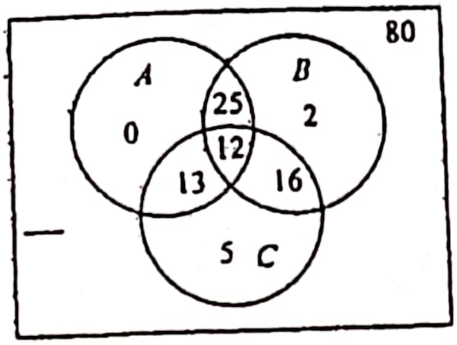


Fig. 1.26

There are 12 students who know all the three languages.

(b) Number of students who know exactly 2 languages

From the Venn diagram, students that know exacdly 2 languages are

$$= |A \cap B| + |B \cap C| + |A \cap C| - 3 \times |A \cap B \cap C|$$

$$= 37 + 28 + 25 - 36$$

$$= 54.$$

(c) How many students know only one language?

$= |U| -$ |students that know none of the languages|

$-$ |students that know all the 3 languages| $-$

| students knowing exactly 2 languages |

$$= 80 - 7 - 12 - 54$$

$$= 80 - 73$$

$= 7$ students know exactly one language.

All results can be verified from Venn diagram in Fig. 1.26.

1.26 : Among integers 1 to 1000,

(a) How many of them are not divisible by 3 nor by 5 nor by 7?

(b) How many are not divisible by 5 or 7 but divisible by 3?

Let A be set of numbers between 1 to 1000 that are divisible by 3.

$B \rightarrow$ Set of numbers between 1 to 1000 that are divisible by 5

$C \rightarrow$ Set of numbers between 1 to 1000 that are divisible by 7

$$|A| = |1000/3| = 333$$

$$|B| = |1000/5| = 200$$

and $|C| = |1000/7| = 142.$

$$|A \cap B \cap C| = \lfloor 1000/3 \times 5 \times 7 \rfloor = 9$$
$$|A \cap B| = \lfloor 1000/3 \times 5 \rfloor = 66$$
$$|B \cap C| = \lfloor 1000/5 \times 7 \rfloor = 28$$
and $|A \cap C| = \lfloor 1000/3 \times 7 \rfloor = 47$

(a) Cardinality of set of numbers that are not divisible by 3 nor by 5 nor by 7 is

$$= |\bar{A} \cap \bar{B} \cap \bar{C}|$$
$$= |\overline{A \cup B \cup C}|$$
$$= U - |A \cup B \cup C|$$
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$= 333 + 200 + 142 - 66 - 47 - 28 + 9$$
$$= 543$$

Therefore, $|\bar{A} \cap \bar{B} \cap \bar{C}| = 1000 - 612 = 457$

Therefore, 457 numbers among 1 to 1000 are not divisible by 3 nor by 5 nor by 7.

(b) Cardinality of set of numbers that are not divisible by 5 and 7 but are divisible by 3 is

$$= A \cap \bar{B} \cap \bar{C}$$
$$= |A \cap \overline{B \cup C}|$$

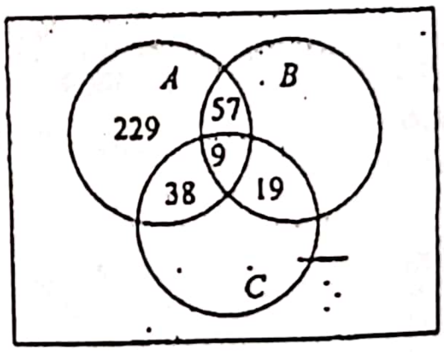


Fig. 1.27

From Venn diaram,

$$|A \cap \bar{B} \cap \bar{C}| = |A - (\overline{B \cup C})|$$
$$= 333 - (57 + 9 + 38)$$
$$= 229$$

Therefore, 229 integers between 1 to 1000 are not divisible by 5 or 7 but divisible by 3.

1.27 Among integers 1 to 300, how many of them are divisible neither by 3, nor by 5, nor by 7? How many of them are divisible by 3 but not by 5, nor by 7?

Total integers are = 300

Out of them, numbers divisible by 3 is set $A1$

Out of them, numbers divisible by 5 is set $A2$

Out of them, numbers divisible by 7 is set $A3$

$$|A1| = \lfloor 300/3 \rfloor = 100$$
$$|A2| = \lfloor 300/5 \rfloor = 60$$

$$|A3| = \lfloor 300/7 \rfloor = 42$$
$$|A1 \cap A2| = \lfloor 300/3 \times 5 \rfloor = 20$$
$$|A1 \cap A3| = \lfloor 300/3 \times 7 \rfloor = 14$$
$$|A2 \cap A3| = \lfloor 300/3 \times 5 \times 7 \rfloor = 2$$
$|A1 \cup A2 \cup A3|$ = Numbers divisible either by 3, 5, or 7

$300 - |A1 \cup A2 \cup A3|$ = Numbers not divisible by 3 nor 5 nor 7

(a) Numbers not divisible by 3 nor 5 nor 7

$= 300 - [\,|A1| + |A2| + |A3| - |A1 \cap A2| - |A2 \cap A3| - |A1 \cap A3| + |A1 \cap A2 \cap A3|$

$= 300 - [|A1| + |A2| + |A3| - |A1 \cap A2| - |A2 \cap A3| - |A1 \cap A3| + |A1 \cap A2 \cap A3|$

$= 300 - [162] = 138$

(b) Number divisible by 3, but not by 5 nor 7 $= 162 - |A2 \cup A3|$

$= 162 - [60 + 42 - 8]$

$= \underline{68}$

1.28 A survey among 1000 people, 595 are democrats, 595 wear glasses and 550 like ice-cream. 395 of them are democrats who wear glasses, 350 of them are democrats who like ice-cream, 400 of them wear glasses and like ice-cream and 250 all the three.

(a) How many of them are not Democrats, do not wear glasses and do not like ice-cream?

(b) How many of them are democrats who do not wear glasses and do not like ice-cream?

Let $|D| = 595$, $|G| = 595$, $|I| = 550$, $|D \cap G| = 395$, $|D \cap I| = 350$, $|G \cap I| = 400$, and $|D \cap G \cap I| = 250$

$$|D \cup G \cup I| = 595 + 595 + 550 - 395 - 350 - 400 + 250$$
$$= 845$$

(a) |people neither democrats nor wearing glasses, nor liking ice-cream| =

$$= 1000 - 845 = 155$$

(b) |Democrats do not like ice-cream and do not wear glasses| =

$$= 845 - |G \cup I|$$
$$= 845 - [|G| + |I| - |G \cap I|]$$
$$= 845 - [595 + 550 - 400]$$
$$= 845 - [745]$$
$$= 100$$

1.29 It is known that in university 60% of professors play tennis, 50% of them play bridge, 70% jog, 20% play tennis and bridge, 40% play bridge and jog and 30% play tennis and jog. If someone claimed that 20% professors jog and play tennis and bridge, would you believe this claim? Why?

% of professors playing tennis $= |T| = 60$

% of professors playing bridge $|B| = 50$

% of professors jogging $|J| = 70$

$$|T \cap B| = 20, |B \cap J| = 40, |T \cap J| = 30.$$

Let us find how many of them jog and play tennis and bridge

$$|T \cup B \cup J| = |T| + |B| + |J| - |T \cap B| - |B \cap J| - |T \cap J| + |B \cap T \cap J|$$

$$|B \cap T \cap J| = 100 - (60 + 50 + 70 - 20 - 40 - 30)$$

$$= 100 - (90)$$

$$= 10$$

Therefore, 10% of professors jog and play tennis and bridge.
Therefore, if someone claims that 20% of them jog and play tennis and bridge, *what he claims is wrong.*

130 The 60000 fans who attended the homecoming football game brought up all the paraphernalia for their cars. Altogether 20000 bumper stickers, 36000 windows decals and 12000 key rings were sold. We know that 52000 fans bought at least 1 item and no one bought more than 1 of the given items. Also 6000 fans bought both decals and key rings, 9000 bought both decals and bumper stickers and 5000 bought both key rings and bumper stickers.

(a) How many fans bought all 3 items?
(b) How many fans bought exactly 1 item?
(c) Someone questioned the accuracy of total no. of purchasers is 52000. He claimed purchasers to be either 60000 or 44000. Dispel the claim.

Given: Total fans visited = 60000

$$|B| = \text{No. of bumper stickers sold} = 20000$$

$$|D| = \text{No. of window decals sold} = 36000$$

$$|K| = \text{No. of key rings sold} = 12000$$

$$|D \cap K| = 6000, |D \cap B| = 9000, |K \cap B| = 5000$$

(a) People who bought at least one item but no one bought more than one of the given items are 52000

$$28000 - 3x = 52000$$

$$3x = 24000$$

$$x = 800$$

$$52000 = |D| + |K| + |B| - |D \cap B| - |K \cap B| + x$$

(b) $x = 4000$

(c) $x = 40000$

131 Among 75 children who went to an amusement park, where they could ride on merry-go-round, roller coaster and ferris wheel. It is known that 20 of them had taken all three rides and 55 had taken at least two of the 3 rides. Each ride costs Rs. 0.50 and total receipt of park is Rs. 70. Determine the number of children who did not try any of the rides?

Total children = 75

$\therefore$ Total Receipt = Rs. 70 (Rs. 0.50/ride)

$\therefore$ Total rides = $70 \times 2 = 140$

20 children had taken all the 3 rides

$\therefore$ 55 had taken at least 2 rides (2 or 3 rides).

So, $55 - 20 = 35$ had taken exactly 2 rides.

Let us draw a Venn diagram for the same, as shown in Fig. 1.28.

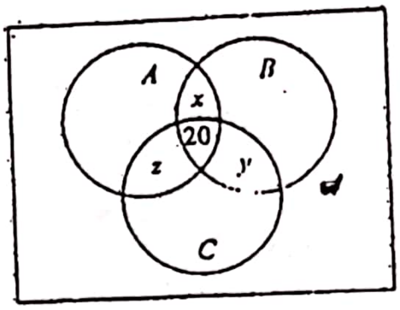


Fig. 1.28

Let $\qquad x + y + z = 35$

Children who had taken exactly one ride

$$\text{Total single rides} = 140 - (35 \times 2 + 20 \times 3)$$
$$= 140 - (70 + 60)$$
$$= 10$$

So, total no. of students who took exactly single ride = 10

$$\text{Children who took no ride} = 75 - (35 + 20 + 10)$$
$$= 75 - (65).$$
$$= 10$$

1.32 If number of students who got Grade A in first examination is equal to that of in second examination. If total number of students who got Grade A in exactly one examination is 40, and 4 students did not get Grade A in either examinations, determine the no. of students who got Grade A in first exam only, who got Grade A in second exam only and who got Grade A in both the exams?

Let us assume that,

$E1$ denote a set of students who got Grade A in first exam.

$E2$ denote a set of students who got Grade A in second exam.

Let $\qquad |E1| = |E2| = x$ and $|E1 \cap E2| = y$

$|E1 \cup E2| = 50 - 4 = 46$, as 4 students did not get Grade A in either examinations

$$\therefore 46 = |E1| + |E2| - |E1 \cap E2|$$
$$46 = 2x - |E1 \cap E2|$$
$$46 = 2x - y \qquad (1)$$

Also students who got A in exactly one exam = 40

Therefore, $\qquad E1 \oplus E2 = 40$

$$|E1| + |E2| - 2\,|E1 \cap E2| = 40$$
$$x + x - 2y = 40$$
$$2x - 2y = 40 \qquad (2)$$

Solving Eqs (1) and (2) we would get values of x and y.

(subtract (2) from (1)) we get

$$\begin{array}{r} - \quad 2x - y = 46 \\ 2x - 2y = 40 \\ \hline y = 6 \end{array}$$

Substituting $y = 6$ in Eq. (1), we get

$$2x - y = 46$$
$$\therefore 2x - 6 = 46$$
$$\therefore 2x = 52$$
$$\therefore x = 26.$$
$$|E1| = |E2| = 26.$$

Therefore,

1. Students who got A in first exam only $= |E1| - |E1 \cap E2|$
$= 26 - 6 = 20.$

2. Students who got A in second exam only $= |E2| - |E1 \cap E2|$
$= 26 - 6 = 20.$

3. Students who got A in both he exams is $|E1 \cap E2| = 6$

1.33 Among 50 students in a class, 26 got an A in the first examination and 21 got an A in the second examination. If 17 students did not get an A in either examination, how many students got A in both the examinations?

Total students in class = 50

Let $E1$ denote set of students who got A grade in first examination, $|E1| = 26$

Let $E2$ denote set of students who got A grade in second examination $|E2| = 21$

There are 17 students did not get A in either of the examinations

Hence $|E1 \cup E2| = 50 - 17 = 33$

$$33 = |E1| + |E2| - |E1 \cap E2|$$

Hence
$$|E1 \cap E2| = 26 + 21 - 33$$
$$= 47 - 33$$
$$= 14.$$

Therefore, 14 students got Ain both the examinations.

1.34 Describe the following Venn diagram shown in Fig. 1.29 in terms of A, B, C and U.

Sets A and B are neither disjoint nor subsets of one another, sets A and C are neither disjoints nor subsets of one another, sets B and C are neither disjoint nor subsets of one another. Set A, B and C have some elements in common. And they do not make up the entire universal set.

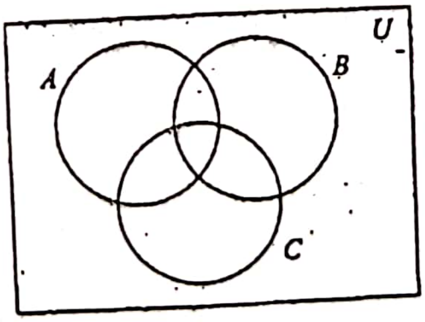


Fig. 1.29

1.35 Draw a Venn diagram and test the validity of the following argument: All guilty people will be arrested. All thieves are guilty people. Therefore, all thieves will be arrested. Venn diagram of Fig. 1.30 satisfies these arguments.

The argument is valid since every Venn diagram drawn will satisfy the conditions of the both premises and conclusions using the rules of logic. In this case thee is no couter example.

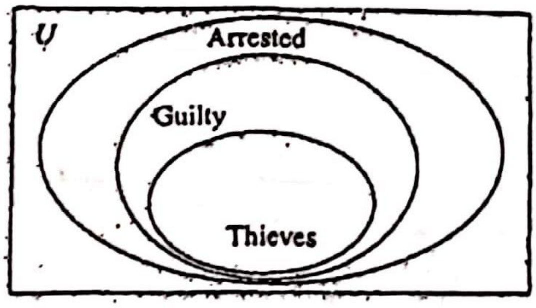


Fig. 1.30

1.36 Draw Venn diagrams showing

(a) $(A \cup B) \in (A \cup C)$ but $B \not\subset C$

(b) $(A \cap B) \in (A \cap C)$ but $B \not\subset C$

(c) $(A \cup B) = (A \cup C)$ but $B \neq C$

(d) $(A \cap B) = (A \cap C)$ but $B \neq C$

(a) $(A \cup B) \in (A \cup C)$ but B *not a subset of* C. Let $x \in A$ and $y \in C$, also $y \notin B$, $x \notin B$, and x is *not a member of* C hence B is not subset of C but $(A \cup B) \in (A \cup C)$ (Fig. 1.31)

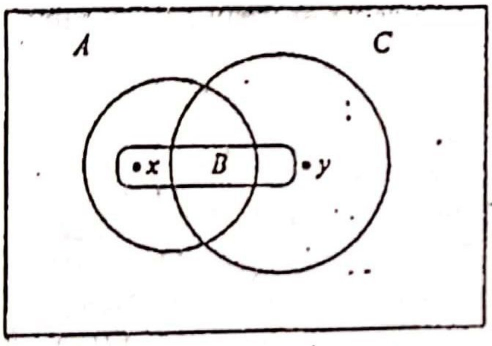


Fig. 1.31

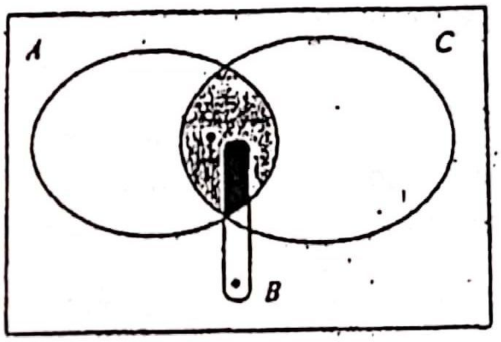
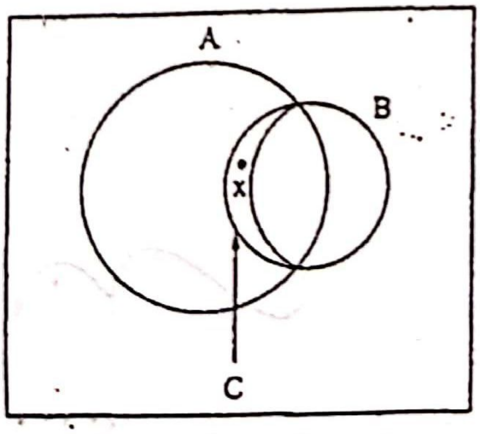


Fig. 1.32

(b) $(A \cap B) \in (A \cap C)$ but B not a subset of C. Here B not a subset of C, but $(A \cap B) \in (A \cap C)$ (Fig. 1.32)

(c) $(A \cup B) = (A \cup C)$ but $B \neq C$
$B \cup (x) = C$ where $x \notin A - B$. Here $A \cup B = A \cup C$ but $B \neq C$. (Fig. 1.33)



Here $x \notin B, x \in C, x \in A$

Fig. 1.33

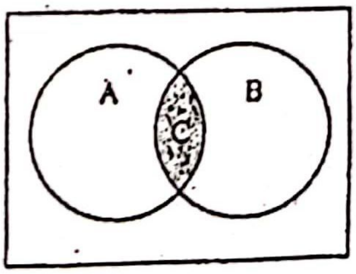


Fig. 1.34

(d) $(A \cap B) = (A \cap C)$ but $B \neq C$. For $A \cap B = A \cap C$ but $B \neq C$, refer Fig. 1.34.

Chapter Two

Relations

2.1 INTRODUCTION

The reader is familiar with many relations which are used in mathematics and computer in mathematics and computer science, e.g., "less than", "is parallel to", "is a subset of", and so on. In a certain sense, these relations consider the existence or nonexistence of a certain connection between pairs of objects taken in a definite order. Formally, we define a relation in terms of these "ordered pairs".

There are three kinds of relations which play a major role in our theory: (i) equivalence relations, (ii) order relations, (iii) functions. Equivalence relations are mainly covered in this chapter. Order relations are introduced here, but will also be discussed in Chapter 14. Functions are covered in the next chapter.

Relations, as noted above, will be defined in terms of ordered pairs (a, b) of elements, where a is designated as the first element and b as the second element. In particular,

$$(a, b) = (c, d)$$

if and only if $a = c$ and $b = d$. The $(a, b) \neq (b, a)$ unless $a = b$. This contrasts with sets studied in Chapter 1, where the order of elements is irrelevant; for example $\{3, 5\} = \{5, 3\}$.

Although matrices will be covered in Chapter 5, we have included their connection with relations here for completeness. These sections, however, can be ignored at a first reading by those with no previous knowledge of matrix theory.

2.2 PRODUCT SETS

Consider two arbitrary sets A and B. The set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the *product*, or *Cartesian product*, of A and B. A short designation of this product is $A \times B$, which is read "A cross B". By definition,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

One frequently writes A^2 instead of $A \times A$.

Example 2.1

$\mathbf{R}$ denotes the set of real numbers and so $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ is the set of ordered pairs of real numbers. The reader is familiar with the geometrical representation of $\mathbf{R}^2$ as points in the plane as in Fig. 2.1. Here each point P represents an ordered pair (a, b) of real numbers and vice versa; the vertical line through P meets the x axis at a, and the horizontal line through P meets the y axis at b. $\mathbf{R}^2$ is frequently called the *Cartesian plane*.

Example 2.2

Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$$

Also $$A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

There are two things worth noting in the above example. First of all $A \times B \neq B \times A$. The Cartesian product deals with ordered pairs, so naturally the order in which the sets are considered is important. Secondly, using $n(S)$ for the number of elements in a set S, we have

$$n(A \times B) = 6 = 2 \cdot 3 = n(A) \cdot n(B)$$

In fact, $n(A \times B) = n(A) \cdot n(B)$ for any finite sets A and B. This follows from the observation that, for an ordered pair (a, b) in $A \times B$, there are $n(A)$ possibilities for a, and for each of these there are $n(B)$ possibilities for b.

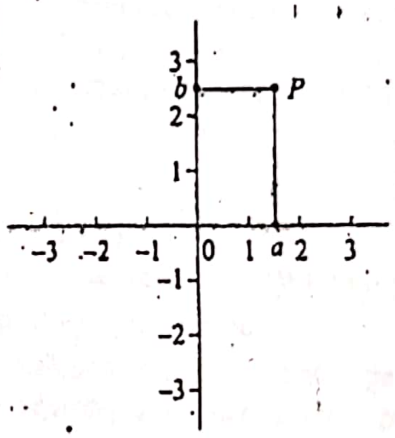


Fig. 2.1

The idea of a product of sets can be extended to any finite number of sets. For any sets $A_1, A_2, \ldots, A_n$, the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$ is called the product of the sets $A_1, \ldots, A_n$ and is denoted by

$$A_1 \times A_2 \times \cdots \times A_n \quad \text{or} \quad \prod_{i=1}^{n} A_i$$

Just as we write A^2 instead of $A \times A$, so we write A^n instead of $A \times A \times \cdots \times A$, where there are n factors all equal to A. For example, $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$ denotes the usual three-dimensional space.

2.3 RELATIONS

We begin with a definition.

Definition

Let A and B be sets. A *binary relation* or, simply, *relation* from A to B is a subset of $A \times B$.

Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes

bi-2/two

from A and each second element comes from B. That is, for each pair $a \in A$ and $b \in B$, exactly one of the following is true:

(i) $(a, b) \in R$; we then say "a is R-related to b", written aRb.

(ii) $(a, b) \notin R$; we then say "a is not R-related to b", written $a \not R b$.

If R is a relation from a set A to itself, that is, if R is a subset of $A^2 = A \times A$, then we say that R is a relation on A.

The domain of a relation R is the set of all first elements of the ordered pairs which belong to R, and the range of R is the set of second elements.

Although n-ary relations, which involve ordered n-tuples, are introduced in Section 2.12, the term relation shall mean binary relation unless otherwise stated or implied.

Example 2.3

(a) Let $A = (1, 2, 3)$ and $B = \{x, y, z\}$, and let $R = \{(1, y), (1, z), (3, y)\}$. Then R is a relation from A to B since R is a subset of $A \times B$. With respect to this relation.

$$1Ry, \quad 1Rz, \quad 3Ry, \quad \text{but} \quad 1\not Rx, \quad 2\not Rx, \quad 2\not Ry, \quad 2\not Rz, \quad 3\not Rx, \quad 3\not Rz,$$

The domain of R is $\{1, 3\}$ and the range is $\{y, z\}$.

(b) Let A = {eggs, milk, corn} and B = {cows, goats, hens}. We can define a relation R from A to B by $(a, b) \in R$ if a is produced by b. In other words,

R = {(eggs, hens), (milk, cows), (milk goats)}

With respect to this relation.

eggs R hens, milk R cows, .etc.

(c) Suppose we say that two countries are *adjacent* if they have some part of their boundaries in common. Then "is adjacent to" is a relation R on the countries of the earth. Thus

(Italy, Switzerland) $\in R$ but (Canada, Mexico) $\notin R$

(d) Set inclusion $\subseteq$ is a relation on any collection of sets. For, given any pair of sets A and B, either $A \subseteq B$ or $A \subseteq B$.

(e) A familiar relation on the set Z of integers in "m divides n". A common notation for this relation is to write $m|n$ when m divides n. Thus $6|30$ but $7\nmid 25$.

(f) Consider the set L of lines in the plane. Perpendicularity, written $\perp$, is a relation on L. That is, given any pair of lines a and b, either $a \perp b$ or $a \not\perp b$. Similarly, "is parallel to", written $||$, is a relation on L since either $a \,||\, b$ or $a \not\parallel b$.

(g) Let A be any set. An important relation on A is that of *equality*.

$$\{(a, a) : a \in A\}$$

which is usually denoted by "=". This relation is also called the *identity* or *diagonal relation* on A and it will also be denoted by Δ_A or simply Δ.

(h) Let A be any set. Then $A \times A$ and $\varnothing$ are subsets of $A \times A$ and hence are relations on A called the *universal relation* and *empty relation*, respectively.

(i) If a set has n elements, how many relations are there on the set?

Let A be the set with $|A| = n$.

Relation on set A is subset of $A \times A$ and $|A \times A| = n^2$. Also set of all possible subsets of $A \times A$ is power set of $A \times A$ i.e. $P(A \times A)$.

If $|A \times A| = n^2$ then $|P(A \times A)| = 2^{n^2}$ i.e. if set has n element then there are 2^{n^2} relations on it.

(j) Let A be the set $\{1, 2, 3, 4\}$, enlist elements of relation

$R = \{(a, b) \mid a \text{ divisible by } b\}$

$R = \{(1,1), (2,1), (3,1), (4,1), (4,2), (2,2), (3,3), (4,4)\}$

(k) Let A be the set of programmers and B denote the set of computer languages, what possible interpretation can be given to

(i) $A \times B$: all possible set of ordered pairs (a, b) where $a \in A$ and $b \in B$ which means we get all possible pairs of programmers and computer languages.

(ii) Relation from A to B as $A \times B$.

Relation R from $A \times B$, represents all the relations such as

$R = \{(a, b) \mid a \in A, b \in B$ such that a can work on b language based project$\}$ or

$S = \{(a, b) \mid a \in A, b \in B$ such that a is undergoing training on programming language $b\}$ and so on.

(l) Relations from sets, say A to B, are subsets of the Cartesian product as $A \times B$, two relations on sets, say from A to B can be combined in any way two sets are combined. Solve following.

1. Let $A = \{a, b, c\}$ and $B = \{a, b, c, d\}$. The relations

$R = \{(a, a), (b, b), (c, c)\}$ and $S = \{(a, a), (a, b), (a, c), (a, d)\}$

Now $R \cup S = \{(a, a), (a, b), (a, c), (a, d), (b, b), (c, c)\}$

$R \cap S = \{(a, a)\}$

$R - S = \{(b, b), (c, c)\}$

2. Let A be the set of computer science students and B be the set of computer languages. Let R be the relation consisting of all ordered pairs (a, b) where student a is required to learn language b in a course. Let S be the relation consisting of all ordered pairs (a, b) such that student a has learnt language b. Describe the ordered pairs in each of the following relations.

1. $R \cup S$

$R \cup S = \{(a, b) \mid$ student a is required to learn or has learnt language $b\}$

2. $R \cap S$

$R \cap S = \{(a, b) \mid a$ is required to learn and has learnt language $b\}$

3. $R \oplus S$

$R \oplus S = \{(a, b) \mid$ either a is required to learn b but has not learnt it or a has learnt b but is not required to$\}$

4. $R - S$

$R - S = \{(a, b) \mid a$ is required to learn b but has not learnt it$\}$

5. $S - R$

$S - R = \{(a, b) \mid a$ has learnt language b but is not required to$\}$

Inverse Relation

Let R be any relation from a set A to a set B. The *inverse* of R, denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs which, when reversed, belong to R; that is,

$$R^{-1} = \{(b, a): (a, b) \in R\}$$

For example, the inverse of the relation $R = \{(1, y), (1, z), (3, y)\}$ from $A = \{1, 2, 3\}$ to $B = \{x, y, z\}$ follows:

$$R^{-1} = \{(y, 1), (z, 1), (y, 3)\}$$

Clearly, if R is any relation, then $(R^{-1})^{-1} = R$. Also, the domain and range of R^{-1} are equal, respectively, to the range and domain of R. Moreover, if R is a relation on A, then R^{-1} is also a relation on A.

→ 2 dimensional

2.4 PICTORIAL REPRESENTATIONS OF RELATIONS

First we consider a relation S on the set R of real numbers; that is, S is a subset of $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$. Since $\mathbf{R}^2$ can be represented by the set of points in the plane, we can picture S by emphasizing those points in the plane which belong to S. The pictorial representation of the relation is sometimes called the *graph* of the relation.

Frequently, the relation S consists of all ordered pairs of real numbers which satisfy some given equation

$$E(x, y) = 0$$

We usually identify the relation with the equation; that is, we speak of the relation $E(x, y) = 0$.

Example 2.4

Consider the relation S defined by the equation

$$x^2 + y^2 = 25$$

That is, S consists of all ordered pairs (x, y) which satisfy the given equation. The graph of the equation is a circle having its center at the origin and radius 5. See Fig. 2.2.

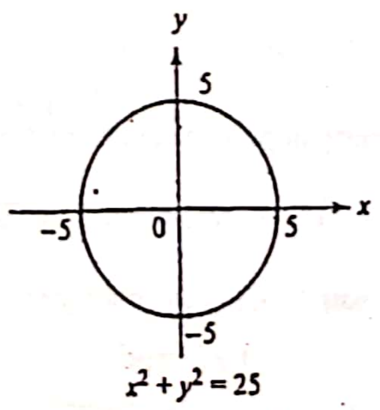


Fig. 2.2

Representations of Relations on Finite Sets

Suppose A and B are finite sets. The following are two ways of picturing a relation R from A to B.

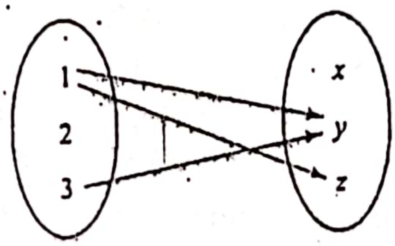
(i) From a rectangular array whose rows are labeled by the elements of A and whose columns are labeled by the elements of B. Put a 1 or 0 in each position of the array according as $a \in A$ is or is not related to $b \in B$. This array is called the *matrix of the relation.*

(ii) Write down the elements of A and the elements of B in two disjoint disks, and then draw an arrow from $a \in A$ to $b \in B$ whenever a is related to b. The picture will be called the *arrow diagram* of the relation.

Figure 2.3 pictures the first relations in Example 2.3 by the above two ways.

	x	y	z
1	0	1	1
2	0	0	0
3	0	1	0

(i)



(ii)

$R = \{(1, y), (1, z), (3, y)\}$

Fig. 2.3

Directed Graphs of Relations on Sets

There is another way of picturing a relation R when R is a relation from a finite set to itself. First we write down the elements of the set, and then we draw an arrow from each element x to each element y whenever x is related to y. This diagram is called the *directed graph* of the relation. Figure 2.4, for example, shows the directed graph of the following relation R on the set $A = \{1, 2, 3, 4\}$:

$$R = \{(1, 2), (2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)\}$$

Observe that there is an arrow from 2 to itself, since 2 is related to 2 under R.

These directed graphs will be studied in detail as a separate subject in Chapter 8. We mention it here mainly for completeness.

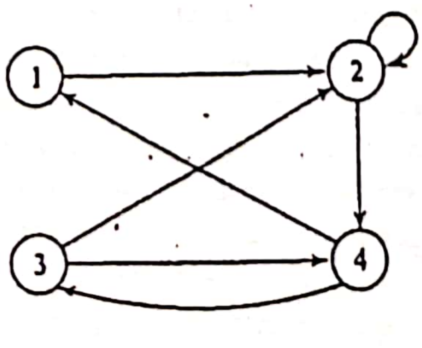


Fig. 2.4

2.5 COMPOSITION OF RELATIONS

Let A, B, and C be sets, and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of $A \times B$ and S is a subset of $B \times C$. Then R and S give rise to a relation from A to C denoted by $R \circ S$ and defined by

$$a(R \circ S)c \text{ if for some } b \in B \text{ we have } aRb \text{ and } bSc$$

That is,

$$R \circ S = \{(a, c): \text{there exists } b \in B \text{ for which } (a, b) \in R \text{ and } (b, c) \in S\}$$

The relation $R \circ S$ is called the *composition* of R and S; it is sometimes denoted simply by RS.

Suppose R is a relation on a set A, that is, R is a relation from a set A to itself. Then $R \circ R$, the composition of R with itself is always defined, and $R \circ R$ is sometimes denoted by R^2. Similarly, $R^3 = R^2 \circ R = R \circ R \circ R$, and so on. Thus R^n is defined for all positive n.

Warning: Many texts denote the composition of relations R and S by $S \circ R$ rather than $R \circ S$. This is done in order to conform with the usual use of $g \circ f$ to denote the composition of f and g where f and g are functions. Thus the reader may have to adjust his notation when using this text as a supplement with another text. However, when a relation R is composed with itself, then the meaning of $R \circ R$ is unambiguous.

The arrow diagrams of relations give us a geometrical interpretation of the composition $R \circ S$ as seen in the following example.

Example 2.5

Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $C = \{x, y, z\}$ and let

$$R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\} \quad \text{and} \quad S = \{(b, x), (b, z), (c, y), (d, z)\}$$

Consider the arrow diagrams of R and S as in Fig. 2.5. Observe that there is an arrow from 2 to d which is followed by an arrow from d to z. We can view these two arrows as a "path" which "connects" the element $2 \in A$ to the element $z \in C$. Thus

$$2(R \circ S)z \quad \text{since} \quad 2Rd \text{ and } dSz$$

Similarly there is a path from 3 to x and a path from 3 to z. Hence

$$3(R \circ S)x \quad \text{and} \quad 3(R \circ S)z$$

No other element of A is connected to an element of C. Accordingly,

$$R \circ S = \{(2, z), (3, x), (3, z)\}$$

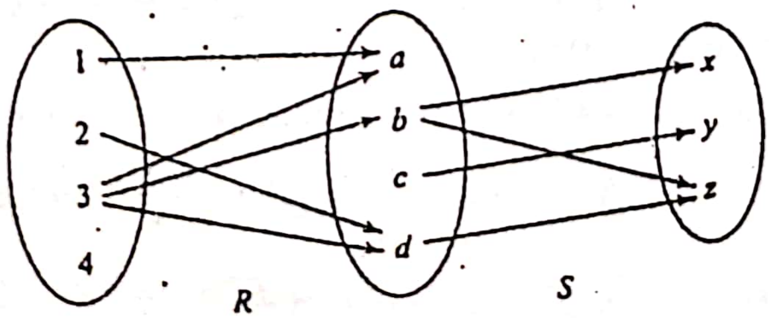


Fig.2.5

reservation system contains records of passenger's reservations, flight schedules, flight numbers, etc. Such an n-ary relation is represented using a table with n-columns. These n-ary relations are used to represent computer databases. A database is a collection of records that are manipulated by a computer. Modern large-scale computing systems efficiently handle huge data such as the employment history, student's information at university, supermarket customers data, etc. For effective handling of such a huge data, we must organize it in suitable format, so as to efficiently process the frequent operations such as insert, delete, update, search and retrieve on it. These representations help us to answer queries about the information stored in databases. One general way to view the organization of such data is with a 'relational data model' in which an n-ary relation among sets $S_1, S_2 \ldots S_n$ represented using n-column table. Database management systems are programs that help users access the information in databases. The relational database model, invented by E.F. Codd in 1970, is based on the concept of n-ary relation. We shall briefly introduce some of the basic concepts of theory of relational model. We begin with terminology. A database consists of *records*, which are n-tuples, made up of '*fields*'. The fields are the data entries of n-tuples. Each column of the table corresponds to an attribute of the database. Consider employees database with attribute of the database. Consider employee's database with attributes emp-id, name, designation, and date of joining.

The *domain* of an attribute is a set to which all the elements in that attribute belong. A domain of an n-ary relations is called *primary* key when the value of the n-tuple from this domain determines the n-tuple. That is, a single attribute or a combination of attributes for a relation is a primary key, if the values of the attributes uniquely define an n-tuple. For example, emp-id is a unique identification number and is a primary key, as shown in Table 2.1.

Table 2.1

Employee Information			
emp-id	*Name*	*Designation*	*Date of Joining*
101	Abolee	Programmer	03/05/2005
222	Saurabh	Manager	01/04/2002
010	Megha	CIO	25/02/1999
319	Varsha	CIO	10/06/1995
236	Arvind	Manager	01/07/1990
103	Hemant	CEO	04/11/1991

A database management system responds to user's request called as queries. A *query* is request for information from database such as "find all employees whose designation is CIO", "find all employees who have joined before 01/01/2000", etc. There are variety of operations on n-ary relations. Applied together, these operations can answer user queries. Some of such operations are 'SELECT', 'PROJECT' and 'JOIN'.

43

ORDERED PAIRS AND PRODUCT SETS

2.1 Given $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Find (a) $A \times B$, (b) $B \times A$, (c) $B \times B$.

(a) $A \times B$ consists of all ordered pairs (x, y) where $x \in A$ and $y \in B$. Hence

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

(b) $B \times A$ consists of all ordered pairs (y, x) where $y \in B$ and $x \in A$. Hence

$$B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

(c) $B \times B$ consists of all ordered pairs (x, y) where $x, y \in B$. Hence

$$B \times B = \{(a, a), (a, b), (b, a), (b, b)\}$$

As expected, the number of elements in the product set is equal to the product of the numbers of the elements in each set.

2.2 Given $A = \{1, 2\}$, $B = \{x, y, z\}$, and $C = \{3, 4\}$. Find: $A \times B \times C$.

$A \times B \times C$ consists of all ordered triplets (a, b, c) where $a \in A$, $b \in B$, $c \in C$. These elements of $A \times B \times C$ can be systematically obtained by a so-called tree diagram (Fig. 2.7). The elements of $A \times B \times C$ are precisely the 12 ordered triplets to the right of the tree diagram.

Observe that $n(A) = n(B) = 3$, and $n(C) = 2$ and, as expected,

$$n(A \times B \times C) = 12 = n(A) \cdot n(B) \cdot n(C)$$

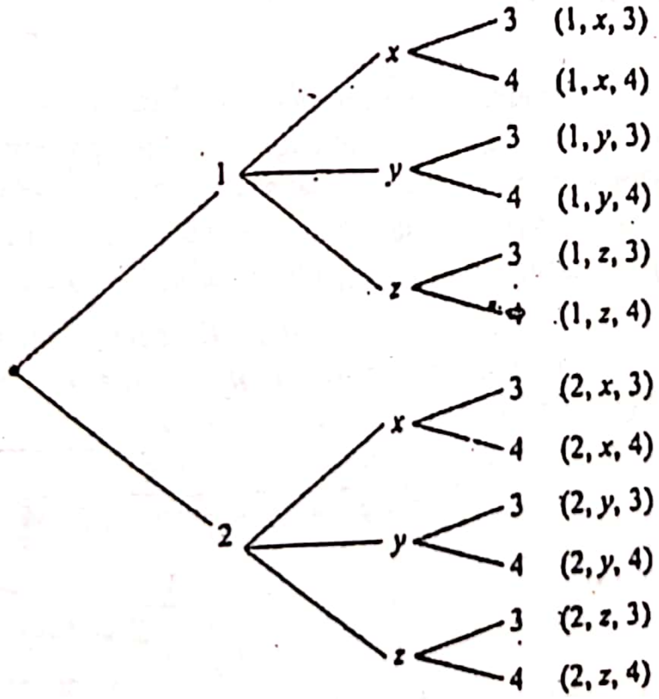


Fig. 2.7

2.3 Let $A = \{1, 2\}$, $B = \{a, b, c\}$ and $C = \{c, d\}$. Find $(A \times B) \cap (A \times C)$ and $(B \cap C)$.

We have

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$
$$A \times C = \{(1, c), (1, d), (2, c), (2, d)\}$$

Hence

$$(A \times B) \cap (A \times C) = \{(1, c), (2, c)\}$$

Since $B \cap C = \{c\}$,

$$A \times (B \cap C) = \{(1, c), (2, c)\}$$

Observe that $(A \times B) \cap (A \times C) = A \times (B \cap C)$. This is true for any sets A, B and C (see Problem 2.4).

2.4 Prove $(A \times B) \cap (A \times C) = A \times (B \cap C)$.

$$\begin{aligned}(A \times B) \cap (A \times C) &= \{(x, y): (x, y) \in A \times B \text{ and } (x, y) \in A \times C\} \\ &= \{(x, y): x \in A, y \in B \text{ and } x \in A, y \in C\} \\ &= \{(x, y): x \in A, y \in B \cap C\} = A \times (B \cap C)\end{aligned}$$

2.5 Find x and y given $(2x, x + y) = (6, 2)$.

Two ordered pairs are equal if and only if the corresponding components are equal. Hence we obtain the equations

$$2x = 6 \quad \text{and} \quad x + y = 2$$

from which we derive the answers $x = 3$ and $y = -1$.

2.6 Let A be a set show that $\phi \times A = A \times \phi = \phi$.

Here $\quad \phi \times A = \{(x, y) \mid x \in \phi \text{ and } y \in A\}$

$= \phi$, As $x \in \phi$ is not true, which means no element of ϕ is paired off with element of A, resulting in zero elements, that is ϕ.

Also, $\quad A \times \phi = \{(x, y) \mid x \in A \text{ and } y \in \phi\}$

$= \phi$, As $y \in \phi$ is not true, that is no element if A is paired off with element of A, resulting in zero elements, that is ϕ.

Hence, $\quad \phi \times A = A \times \phi = \phi$.

2.7 Suppose that $A \times B = \phi$, where A and B are sets. What can you conclude?

As $A \times B = \phi$, either A is ϕ or B is ϕ or both A and B are null sets.

2.8 Let A be set of airlines. Let B and C be set of all cities in United States. Interpret the Cartesian product $A \times B \times C$.

$A \times B \times C$ represents set of all airlines and cities such that

$A \times B \times C = \{(a, (b, c)) \mid a \in A, b \in B, c \in C$ where airline a connects cities b and $c\}$

2.9 Find the number of relations from $A = \{a, b, c\}$ to $B = \{1, 2\}$.

There are $3(2) = 6$ elements in $A \times B$, and hence there are $m = 2^6 = 64$ subsets of $A \times B$. Thus there are $m = 64$ relations from A to B.

2.10 Given $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$. Let R be the following relation from A to B:

$$R = \{(1, y), (1, z), (3, y), (4, x), (4, z)\}$$

(a) Determine the matrix of the relation.
(b) Draw the arrow diagram of R.
(c) Find the inverse relation R^{-1} of R.
(d) Determine the domain and range of R.

(a) See Fig. 2.7(a). Observe that the rows of the matrix are labeled by the elements of A and the columns by the elements of B. Also observe that the entry in the matrix corresponding to $a \in A$ and $b \in B$ is 1 if a is related to b and 0 otherwise.

(b) See Fig. 2.7(b). Observe that there is an arrow from $a \in A$ to $b \in B$ iff a is related to b, i.e., iff $(a, b) \in R$.

(c) Reverse the ordered pairs of R to obtain R^{-1}:

$$R^{-1} = \{(y, 1), (z, 1), (y, 3), (x, 4), (z, 4)\}$$

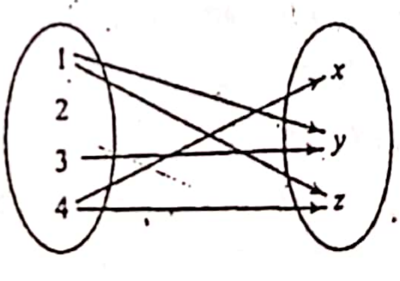
Observe that by reversing the arrows in Fig. 2.7(b) we obtain the arrow diagram of R^{-1}.

(d) The domain of R, Dom(R), consists of the first elements of the ordered pairs of R, and the range of R. Ran(R), consists of the second elements. Thus,

$$\text{Dom}(R) = \{1, 3, 4\} \quad \text{and} \quad \text{Ran}(R) = \{x, y, z\}$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}$$

(a)



(b)

Fig. 2.8

2.11 Let $A = \{1, 2, 3, 4, 6\}$, and let R be the relation on A defined by "x divides y", written $x|y$. (Note $x|y$ iff there exists an integer z such that $xz = y$.)

(a) Write R as a set of ordered pairs.
(b) Draw its directed graph.
(c) Find the inverse relation R^{-1} of R. Can R^{-1} be described in words?

(a) Find those numbers in A divisible by 1, 2, 3, 4, and then 6. These are:

$$1|1, \quad 1|2, \quad 1|3, \quad 1|4, \quad 1|6, \quad 2|2, \quad 2|4, \quad 2|6, \quad 3|3, \quad 3|6, \quad 4|4, \quad 6|6$$

Hence

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3)\ (3, 6), (4, 4), (6, 6)\}$

(b) See Fig. 2.9.

(c). Reverse the ordered pairs of R to obtain R^{-1}:

$R^{-1} = \{(1, 1), (2, 1), (3, 1), (4, 1), (6, 1), (2, 2), (4, 2), (6, 2), (3, 3), (6, 3), (4, 4), (6, 6)\}$

R^{-1} can be described by the statement "x is a multiple of y".

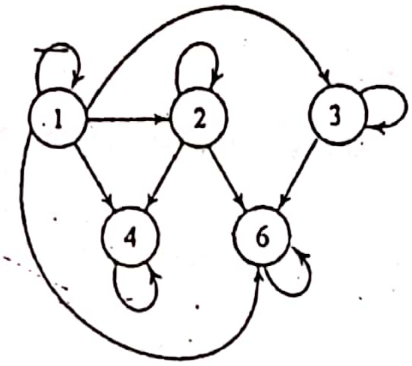


Fig. 2.9

2.12 Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$ and $C = \{x, y, z\}$. Consider the following relations R and S from A to B and from B to C respectively:

$$R = \{(1, b), (2, a), (2, c)\} \quad \text{and} \quad S = \{(a, y), (b, x), (c, y), (c, z)\}$$

(a) Find the composition relation $R \circ S$.

(b) Find the matrices M_R, M_S and $M_{R \circ S}$ of the respective relations R, S and $R \circ S$, and compare $M_{R \circ S}$ to the product $M_R M_S$.

(a) Draw the arrow diagram of the relations R and S as in Fig. 2.10. Observe that 1 in A is "connected" to x in C by the path $1 \to b \to x$; hence $(1, x)$ belongs to $R \circ S$. Similarly, $(2, y)$ and $(2, z)$ belong to $R \circ S$. We have

$$R \circ S = \{(1, x), (2, y), (2, z)\}$$

(See Example 2.5.)

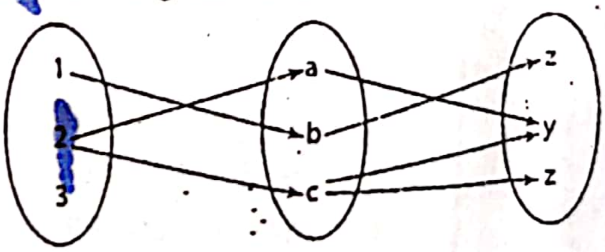


Fig. 2.10

(b) The matrices of M_R, M_S, and $M_{R \circ S}$ follow:

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \qquad M_S = \begin{array}{c} \\ a \\ b \\ c \end{array}\!\!\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array} \qquad M_{R \circ S} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

Multiplying M_R and M_S we obtain

$$M_R M_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Observe that $M_{R \circ S}$ and $M_R M_S$ have the same zero entries.

2.13 Let R and S be the following relations on $A = \{1, 2, 3\}$:

$$R = \{(1, 1), (1, 2), (2, 3), (3, 1), (3, 3)\}, \quad S = \{(1, 2), (1, 3), (2, 1), (3, 3)\}$$

Find (a) $R \cap S$, $R \cup S$, R^c; (b) $R \circ S$; (c) $S^2 = S \circ S$.

(a) Treat R and S simply as sets, and take the usual intersection and union. For R^c, use the fact that $A \times A$ is the universal relation on A.

$$R \cap S = \{(1, 2), (3, 3)\}$$

$$R \cup S = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1) (3, 3)\}$$

$$R^c = \{(1, 3), (2, 1), (2, 2), (3, 2)\}$$

(b) For each pair $(a, b) \in R$, find all pairs $(b, c) \in S$. Then $(a, c) \in R \circ S$. For example, $(1, 1) \in R$ and $(1, 2), (1, 3) \in S$; hence $(1, 2)$ and $(1, 3)$ belong to $R \circ S$. Thus,

$$R \circ S = \{(1, 2), (1, 3), (1, 1), (2, 3), (3, 2), (3, 3)\}$$

(c) Following the algorithm in (b), we get

$$S^2 = S \circ S = \{(1, 1), (1, 3), (2, 2), (2, 3), (3, 3)\}$$

2.14 Prove Theorem 2.1: Let A, B, C and D be sets. Suppose R is a relation from A to B, S is a relation from B to C and T is a relation from C to D. Then $(R \circ S) \circ T = R \circ (S \circ T)$.

We need to show that each ordered pair in $(R \circ S) \circ T$ belongs to $R \circ (S \circ T)$, and vice versa.

Suppose (a, d) belongs to $(R \circ S) \circ T$. Then there exists a c in C such that $(a, c) \in R \circ S$ and $(c\ d) \in T$. Since $(a, c) \in R \circ S$, there exists a b in B such that $(a, b) \in R$ and $(b, c) \in S$. Since $(b, c) \in S$ and $(c, d) \in T$, we have $(b, d) \in S \circ T$; and since $(a, b) \in R$ and $(b, d) \in S \circ T$, we have $(a, d) \in R \circ (S \circ T)$. Therefore, $(R \circ S) \circ T \subseteq R \circ (S \circ T)$. Similarly $R \circ (S \circ T) \subseteq (R \circ S) \circ T$. Both inclusion relations prove $(R \circ S) \circ T = R \circ (S \circ T)$.

TYPES OF RELATIONS AND CLOSURE PROPERTIES

2.15 Consider the following five relations on the set $A = \{1, 2, 3\}$:

$R = \{(1, 1), (1, 2), (1, 3), (3, 3)\}$, $\emptyset$ = empty relation

$S = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$, $A \times A$ = universal relation

$T = \{(1, 1), (1, 2), (2, 2), (2, 3)\}$

Determine whether or not each of the above relations on A is: (a) reflexive; (b) symmetric; (c) transitive; (d) antisymmetric.

(a) R is not reflexive since $2 \in A$ but $(2, 2) \notin R$. T is not reflexive since $(3, 3) \notin T$ and, similarly, $\varnothing$ is not reflexive. S and $A \times A$ are reflexive.

(b) R is not symmetric since $(1, 2) \in R$ but $(2, 1) \notin R$, and similarly T is not symmetric. S, $\varnothing$, and $A \times A$ are symmetric.

(c) T is not transitive since $(1, 2)$ and $(2, 3)$ belong to T, but $(1, 3)$ does not belong to T. The other four relations are transitive.

(d) S is not antisymmetric since $1 \neq 2$, and $(1, 2)$ and $(2, 1)$ both belong to S. Similarly, $A \times A$ is not antisymmetric. The other three relations are antisymmetric.

2.16 Given: $A = \{1, 2, 3, 4\}$. Consider the following relation in A:

$$R = \{(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)\}$$

(a) Draw its directed graph.
(b) Is R (i) reflexive, (ii) symmetric, (iii) transitive, or (iv) antisymmetric?
(c) Find $R^2 = R \circ R$.

(a) See Fig. 2.11.

(b) (i) R is not reflexive because $3 \in A$ but $3 \not R 3$, i.e. $(3, 3) \notin R$.

(ii) R is not symmetric because $4R2$ but $2 \not R 4$, i.e. $(4, 2) \in R$ but $(2, 4) \notin R$.

(iii) R is not transitive because $4R2$ and $2R3$ but $4 \not R 3$, i.e. $(4, 2) \in R$ and $(2, 3) \in R$ but $(4, 3) \notin R$.

(iv) R is not antisymmetric because $2R3$ and $3R2$ but $2 \neq 3$.

(c) For each pair $(a, b) \in R$, find all $(b, c) \in R$. Since $(a, c) \in R^2$,

$$R^2 = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 2), (4, 3), (4, 4)\}$$

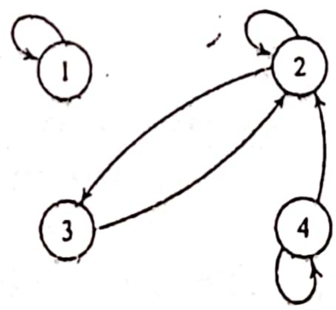


Fig. 2.11

2.17 Give examples of relations R on $A = \{1, 2, 3\}$ having the stated property.
(a) R is both symmetric and antisymmetric.
(b) R is neither symmetric nor antisymmetric.
(c) R is transitive but $R \cup R^{-1}$ is not transitive.

There are several possible examples for each answer. One possible set of examples follows:

(a) $R = \{(1, 1), (2, 2)\}$.
(b) $R = \{(1, 2), (2, 1), (2, 3)\}$.
(c) $R = \{(1, 2)\}$.

4.6

2.18 Suppose $\mathcal{C}$ is a collection of relations S on a set A and let T be the intersection of the relations S, that is, $T = \cap(S: S \in \mathcal{C})$. Prove:

(a) If every S is symmetric, then T is symmetric.

(b) If every S is transitive, then T is transitive.

(a) Suppose $(a, b) \in T$. Then $(a, b) \in S$ for every S. Since each S is symmetric, $(b, a) \in S$ for every S. Hence $(b, a) \in T$ and T is symmetric.

(b) Suppose (a, b) and (b, c) belong to T. Then (a, b) and (b, c) belong to S for every S. Since each S is transitive, (a, c) belongs to S for every S. Hence, $(a, c) \in T$ and T is transitive.

2.19 Let R be a relation on a set A and let P be a property of relations, such as symmetry and transitivity. Then P will be called R-closable if P satisfies the following two conditions:

1. There is a P-relation S containing R.
2. The intersection of P-relations is a P-relation.

(a) Show that symmetry and transitivity are R-closable for any relation R.

(b) Suppose P is R-closable. Then $P(R)$, the P-closure of R, is the intersection of all P-relations S containing R, that is,

$$P(R) = \cap(S: S \text{ is a } P\text{-relation and } R \subseteq S)$$

(a) The universal relation $A \times A$ is symmetric and transitive and $A \times A$ contains any relation R on A. Thus (1) is satisfied. By Problem 2.15, symmetry and transitivity satisfy (2). Thus symmetry and transitivity are R-closable for any relation R.

(b) Let $T = \cap\,(S: S$ is a P-relation and $R \subseteq S)$. Since P is R-closable, T is nonempty by (1) and T is a P-relation by (2). Since each relation S contains R, the intersection T contains R. Thus, T is a P-relation containing R. By definition, $P(R)$ is the smallest P-relation containing R; hence $P(R) \subseteq T$. On the other hand, $P(R)$ is one of the sets S defining T, that is, $P(R)$ is a P-relation and $R \subseteq P(R)$. Therefore, $T \subseteq P(R)$. Accordingly, $P(R) = T$.

2.20 Consider a set $A = \{a, b, c\}$ and the relation R on A defined by

$$R = \{(a, a), (a, b), (b, c), (c, c)\}$$

Find (a) reflexive(R), (b) symmetric(R), and (c) transitive(R).

(a) The reflexive closure on R is obtained by adding all diagonal pairs of $A \times A$ to R which are not currently in R. Hence,

$$\text{reflexive}(R) = R \circ \{(b, b)\} = \{(a, a), (a, b), (b, b), (b, c), (c, c)\}$$

(b) The symmetric closure on R is obtained by adding all the pairs in R^{-1} to R which are not currently in R. Hence,

$$\text{symmetric}(R) = R \cup \{(b, a), (c, b)\} = \{(a, a), (a, b), (b, a), (b, c), (c, b), (c, c)\}$$

(c) The transitive closure on R, since A has three elements, is obtained by taking the union of R with $R^2 = R \circ R$ and $R^3 = R \circ R \circ R$. Note that

$$R^2 = R \circ R = \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$$

$$R^3 = R \circ R \circ R = \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$$

Hence

$$\text{transitive}(R) = R \cup R^2 \cup R^3 = \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$$

(b) Domain = $\{1, 3\}$, range = $\{2, 3, 4\}$.
(c) $R^{-1} = \{(3, 1), (4, 1), (2, 3), (3, 3), (4, 3)\}$.
(d) See Fig. 2.17
(e) $R \circ R = \{(1, 2), (1, 3), (1, 4), (3, 2), (3, 3), (3, 4)\}$.

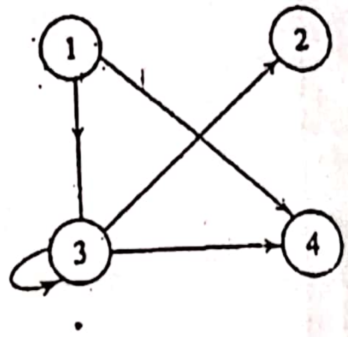


Fig. 2.17

2.6 (a) $R \circ S = \{(a, c), (a, d), (c, a), (d, a)\}$.
(b) $S \circ R = \{(b, a), (b, c), (c, b), (c, d), (d, a), (d, c)\}$.
(c) $R \circ R = \{(a, a), (a, b), (a, c), (a, d), (c, b)\}$.
(d) $S \circ S = \{(c, c), (c, a), (c, d)\}$.

2.7 (a) $\{(9, 1), (6, 2), (3, 3)\}$
(b) (i) $\{9, 6, 3\}$, (ii) $\{1, 2, 3\}$, (iii) $\{(1, 9), (2, 6), (3, 3)\}$
(c) $\{(3, 3)\}$

2.8 (a) None; (b) (2) and (3); (c) (1) and (4); (d) all except (3).

2.9 All are true except (e) $R = \{(1, 2)\}$, $S = \{(2, 3)\}$ and (f) $R = \{(1, 2)\}$, $S = \{(2, 1)\}$.

2.12 $[\{1, 6, 11, 16\}, \{2, 7, 12, 17\}, \{3, 8, 13, 18\}, \{4, 9, 14, 19\}, \{5, 10, 15, 20\}]$

2.13 (b) $\{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$

Chapter Three

Functions and Algorithms

3.1 INTRODUCTION

One of the most important concepts in mathematics is that of a function. The terms "map", "mapping", "transformation", and many others mean the same thing; the choice of which word to use in a given situation is usually determined by tradition and the mathematical background of the person using the term.

Related to the notion of a function is that of an algorithm. The notation for presenting an algorithm and a discussion of its complexity is also covered in this chapter.

3.2 FUNCTIONS

Suppose that to each element of a set A we assign a unique element of a set B; the collection of such assignments is called a *function* from A into B. The set A is called the *domain* of the function, and the set B is called the *codomain*.

Functions are ordinarily denoted by symbols. For example, let f denote a function from A into B. Then we write

$$f: A \rightarrow B$$

which is read: "f is a function from A into B", or "f takes (or; maps) A into B". If $a \in A$, then $f(a)$ (read "f of a") denotes the unique element of B which f assigns to a; it is called the *image* of a under f, or the *value* of f at a. The set of all image values is called the *range* or *image* of f. The image of $f: A \rightarrow B$ is donoted by Ran (f), Im (f) or $f(A)$.

Frequently, a function can be expressed by means of a mathematical formula. For example, consider the function which sends each real number into its square. We may describe this function by writing

$$f(x) = x^2 \quad \text{or} \quad x \mapsto x^2 \quad \text{or} \quad y = x^2$$

In the first notation, x is called a *variable* and the letter f denotes the function. In the second notation, the barred arrow $\mapsto$ is read "goes into". In the last notation, x is called the *independent variable* and y is called the *dependent variable* since the value of y will depend on the value of x.

Remark: Whenever a function is given by a formula in terms of a variable x, we assume, unless it is otherwise stated, that the domain of the function is **R** (or the largest subset of **R** for which the formula has meaning) and the codomain is **R**.

Example 3.1

(a) Consider the function $f(x) = x^3$, i.e. f assigns to each real number its cube. Then the image of 2 is 8, and so we may write $f(2) = 8$.

(b) Let f assign to each country in the world its capital city. Here the domain of f is the set of countries in the world; the codomain is the list of cities of the world. The image of France is Paris; or, in other words, $f(\text{France}) = \text{Paris}$.

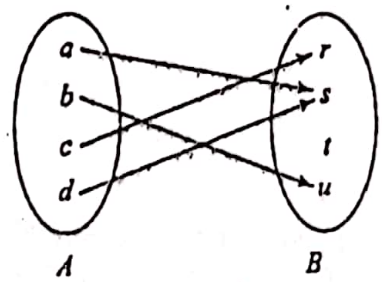


Fig. 3.1

(c) Figure 3.1 defines a function f from $A = \{a, b, c, d\}$ into $B = \{r, s, t, u\}$ in the obvious way. Here

$$f(a) = s, \quad f(b) = u, \quad f(c) = r, \quad f(d) = s$$

The image of f is the set of image values, $\{r, s, u\}$. Note that t does not belong to the image of f because t is not the image of any element under f.

(d) Let A be any set. The function from A into A which assigns to each element that element itself is called the *identity function* on A and is usually denoted by 1_A or simply 1. In other words,

$$1_A(a) = a$$

for every element a in A.

(e) Suppose S is a subset of A, that is, suppose $S \subseteq A$. The *inclusion map* or *embedding* of S into A, denoted by $i: S \to A$, is the function defined by

$$i(x) = x$$

for every $x \in S$; and the *restriction* to S of any function $f: A \to B$, denoted by $f|_S$, is the function from S to B defined by

$$f|_A(x) = f(x)$$

for every $x \in S$.

Functions as Relations

There is another point of view from which functions may be considered. First of all, every function $f: A \to B$ gives rise to a relation from A to B called the *graph of f* and defined by

Graph of $f = \{(a, b): a \in A, b = f(a)\}$

Two functions $f: A \to B$ and $g: A \to B$ are defined to be equal, written $f = g$, if $f(a) = g(a)$ for every $a \in A$; that is, if they have the same graph. Accordingly, we do not distinguish between a function and its graph. Now, such a graph relation has the property that each a in A belongs to a unique ordered pair (a, b) in the relation. On the other hand, any relation f from A to B that has this property gives rise to a function $f: A \to B$, where $f(a) = b$ for each (a, b) in f. Consequently, one may equivalently define a function as follows:

Definition: A function $f: A \to B$ is a relation from A to B (i.e. a subset of $A \times B$) such that each $a \in A$ belongs to a unique ordered pair (a, b) in f.

Although we do not distinguish between a function and its graph, we will still use the terminology "graph of f" when referring to f as a set of ordered pairs. Moreover, since the graph of f is a relation, we can draw its picture as was done for relations in general, and this pictorial representation is itself sometimes called the graph of f. Also, the defining condition of a function, that each $a \in A$ belongs to a unique pair (a, b) in f, is equivalent to the geometrical condition of each vertical line intersecting the graph in exactly one point.

Example 3.2

(a) Let $f: A \to B$ be the function defined in Example 3.1(c). Then the graph of f is the following set of ordered pairs:

$$\{(a, s), (b, u), (c, r), (d, s)\}$$

(b) Consider the following relations on the set $A = \{1, 2, 3\}$:

$$f = \{(1, 3), (2, 3), (3, 1)\}$$

$$g = \{(1, 2), (3, 1)\}$$

$$h = \{(1, 3), (2, 1), (1, 2), (3, 1)\}$$

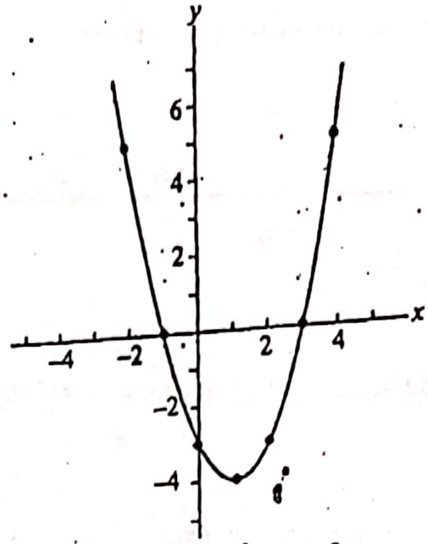
f is a function from A into A since each member of A appears as the first coordinate in exactly one ordered pair in f; here $f(1) = 3$, $f(2) = 3$ and $f(3) = 1$. g is not a function from A into A since $2 \in A$ is not the first coordinate of any pair in g and so g does not assign any image to 2. Also h is not a function from A into A since $1 \in A$ appears as the first coordinate of two distinct ordered pairs in h, (1, 3) and (1, 2). If h is to be a function it cannot assign both 3 and 2 to the element $1 \in A$.

(c) By a real polynomial function, we mean a function $f: \mathbf{R} \to \mathbf{R}$ of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where the a_i are real numbers. Since **R** is an infinite set, it would be impossible to plot each point of the graph. However, the graph of such a function can be approximated by first plotting some of its points and then drawing a smooth curve through these points. The points are usually obtained from a table where various values are assigned to x and the corresponding values of $f(x)$ are computed. Figure 3.2 illustrates this technique using the function $f(x) = x^2 - 2x - 3$.

x	$f(x)$
-2	-5
-1	0
0	-3
1	-4
2	-3
3	0
4	5



Graph of $f(x) = x^2 - 2x - 3$

Fig. 3.2.

Composition Function

Consider function $f: A \to B$ and $g: B \to C$; that is, where the codomain of f is the domain of g. Then we may define a new function from A to C, called the *composition* of f and g and written $g \circ f$, as follows:

$$(g \circ f)(a) \equiv g(f(a))$$

That is, we find the image of a under f and then find the image of $f(a)$ under g. This definition is not really new. If we view f and g as relations, then this function is the same as the composition of f and g as relations (see Section 2.6) except that here we use the functional notation $g \circ f$ for the composition of f and g instead of the notation $f \circ g$ which was used for relations.

Consider any function $f: A \to B$. Then

$$f \circ 1_A = f \quad \text{and} \quad 1_B \circ f = f$$

where 1_A and 1_B are the identity functions on A and B, respectively.

3.3 ONE-TO-ONE, ONTO, AND INVERTIBLE FUNCTIONS

A function $f: A \to B$ is said to be *one-to-one* (written 1–1) if different elements in the domain A have distinct images. Another way of saying the same thing is that f is *one-to-one* if $f(a) = f(a')$ implies $a = a'$.

A function $f: A \to B$ is said to be an *onto* function if each element of B is the image of some element of A. In other words, $f: A \to B$ is onto if the image of f is the entire codomain, i.e., if $f(A) = B$. In such a case we say that f is a function from A *onto* B or that f maps A *onto* B.

A function $f: A \to B$ is *invertible* if its inverse relation f^{-1} is a function from B to A. In general, the inverse relation f^{-1} may not be a function. The following theorem gives simple criteria which tells us when it is.

4

Theorem 3.1: A function $f: A \to B$ is invertible if and only if f is both one-to-one and onto.

If $f: A \to B$ is one-to-one and onto, then f is called a *one-to-one correspondence* between A and B. This terminology comes from the fact that each element of A will then correspond to a unique element of B and vice versa.

Some texts use the terms *injective* for a one-to-one function, *surjective* for an onto function, and *bijective* for a one-to-one correspondence.

Example 3.3

Consider the functions $f_1: A \to B$, $f_2: B \to C$, $f_3: C \to D$ and $f_4: D \to E$ defined by the diagram of Fig. 3.3. Now f_1 is one-to-one since no element of B is the image of more than one element of A. Similarly, f_2 is one-to-one. However, neither f_3 nor f_4 is one-to-one since $f_3(r) = f_3(u)$ and $f_4(v) = f_4(W)$.

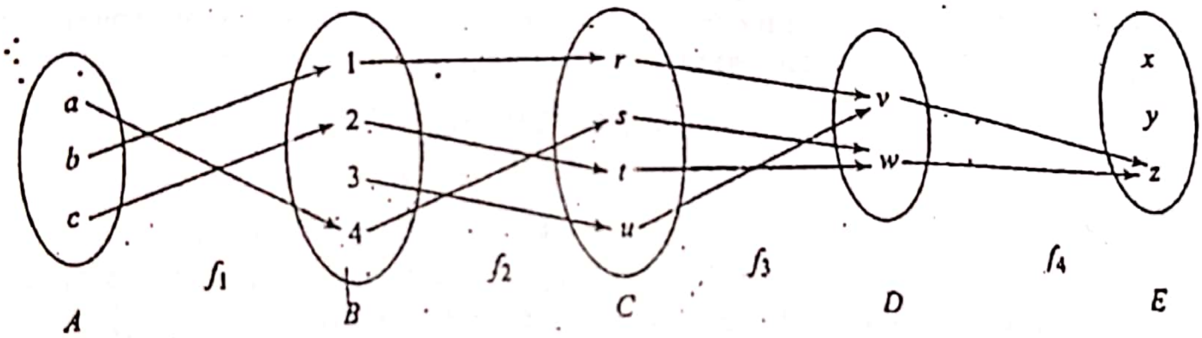


Fig. 3.3

As far as being onto is concerned, f_2 and f_3 are both onto functions since every element of C is the image under f_2 of some element of B and every element of D is the image under f_3 of some element of C, i.e., $f_2(B) = C$ and $f_3(C) = D$. On the other hand, f_1 is not onto since $3 \in B$ is not the image under f_1 of any element of A, and f_4 is not onto since $x \in E$ is not the image under f_4 of any element of D.

Thus f_1 is one-to-one but not onto, f_3 is onto but not one-to-one and f_4 is neither one-to-one nor onto. However, f_2 is both one-to-one and onto, i.e., is a one-to-one correspondence between A and B. Hence f_2 is invertible and f_2^{-1} is a function from C to B.

Geometrical Characterization of One-to-One and Onto Functions

Since functions may be identified with their graphs, and since graphs may be plotted, we might wonder whether the concepts of being one-to-one and onto have geometrical meaning. We show that the answer is yes.

To say that a function $f: A \to B$ is one-to-one means that there are no two distinct pairs (a_1, b) and (a_2, b) in the graph of f; hence each horizontal line can intersect the graph of f in at most one point. On the other hand, to say that f is an onto function means that for every $b \in B$ there must be at least one $a \in A$ such that (a, b) belongs to the graph of f; hence each horizontal line must intersect the graph of f at least once. Accordingly, if f is both one-to-one and onto, i.e. invertible, then each horizontal line will intersect the graph of f in exactly one point.

Example 3.4

Consider the following four functions from R into R:

$$f_1(x) = x^2, \quad f_2(x) = 2^x, \quad f_3(x) = x^3 - 2x^2 - 5x + 6, \quad f_4(x) = x^3$$

The graphs of these functions appear in Fig. 3.4. Observe that there are horizontal lines which intersect the graph of f_1 twice and there are horizontal lines which do not intersect the graph of f_1 at all; hence f_1 is neither one-to-one nor onto. Similarly, f_2 is one-to-one but not onto, f_3 is onto but not one-to-one and f_4 is both one-to-one and onto. The inverse of f_4 is the cube root function, i.e., $f_4^{-1}(x) = \sqrt[3]{x}$.

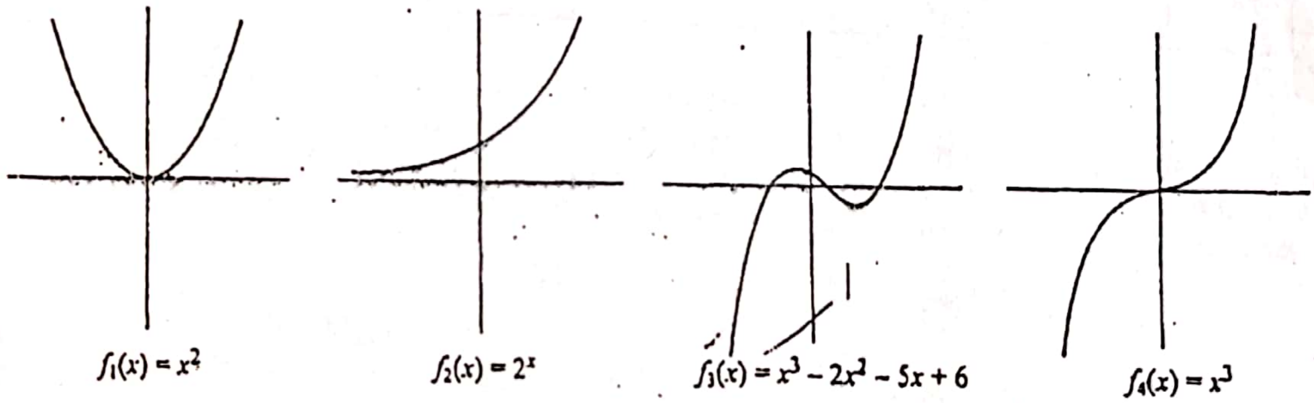


Fig. 3.4

Dirichlet Drawer Principle

Pigeon hole principle, one of the well known proof techniques, is also called as shoe box *arguments* or *Dirichlet Drawer principle*.

Let A and B be finite sets. If $|A| > |B|$, then for function f from A to B, there exists $a_1, a_2 \in D$ such that $f(a_1) = f(a_2)$. Some of the trivial applications of this principle are: if eight people are selected in any way from some group, at least two of them will have been born on the same week day. Also, if eleven shoes are picked up from ten pairs of shoes there must be a pair of matched shoe. Another common example is among 13 people; there are at least two, who are born in the same month. Let us term the 13 people as pigeons and 12 months as holes. And it can be said as if n pigeons are assigned to m pigeon holes, and $m < n$, then at least one pigeon hole contains two or more pigeons. Hence this principle is also called as pigeonhole principle or a shoebox argument.

The pigeonhole principle is called as *Dirichlet drawer principle*, after the nineteenth-century German mathematician Dirichlet, who often used this principle in his work. We shall study and use pigeonhole principle in Chapter 6 'Counting'.

3.4 MATHEMATICAL, EXPONENTIAL AND LOGARITHMIC FUNCTIONS

This section presents various mathematical functions which appear often in the analysis of algorithms, and in computer science in general, together with their notation. We also discuss the exponential and logarithmic functions, and their relationship.

Floor and Ceiling Functions

Let x be any real number. Then x lies between two integers called the floor and the ceiling of x. Specifically,

$\lfloor x \rfloor$, called the *floor* of x, denotes the greatest integer that does not exceed x.

$\lceil x \rceil$, called the *ceiling* of x, denotes the least integer that is not less than x.

If x is itself an integer, then $\lfloor x \rfloor = \lceil x \rceil$; otherwise $\lfloor x \rfloor + 1 = \lceil x \rceil$. For example,

$$\lfloor 3.14 \rfloor = 3, \quad \lceil \sqrt{5} \rceil = 2, \quad \lfloor -8.5 \rfloor = -9, \quad \lfloor 7 \rfloor = 7, \quad \lfloor -4 \rfloor = -4$$

$$\lceil 3.14 \rceil = 4, \quad \lceil \sqrt{5} \rceil = 3, \quad \lceil -8.5 \rceil = -8, \quad \lceil 7 \rceil = 7, \quad \lceil -4 \rceil = -4$$

Integer and Absolute Value Functions

Let x be any real number. The *integer value* of x, written INT(x), converts x into an integer by deleting (truncating) the fractional part of the number. Thus

$$\text{INT}(3.14) = 3, \quad \text{INT}(\sqrt{5}) = 2, \quad \text{INT}(-8.5) = -8, \quad \text{INT}(7) = 7$$

Observe that INT(x) $= \lfloor x \rfloor$ or INT(x) $= \lceil x \rceil$ according to whether x is positive or negative.

The *absolute value* of the real number x, written ABS(x) or $|x|$, is defined as the greater of x or $-x$. Hence ABS(0) = 0, and, for $x \neq 0$, ABS(x) = x or ABS(x) = $-x$, depending on whether x is positive or negative. Thus

$$|-15| = 15, \quad |7| = 7, \quad |-3.33| = 3.33, \quad |4.44| = 4.44, \quad |-0.075| = 0.075$$

We note that $|x| = |-x|$ and, for $x \neq 0$, $|x|$ is positive.

Remainder Function; Modular Arithmetic

Let k be any integer and let M be a positive integer. Then

$$k \pmod{M}$$

(read *k modulo M*) will denote the integer remainder when k is divided by M. More exactly, $k \pmod{M}$ is the unique integer r such that

$$k = Mq + r \quad \text{where} \quad 0 \le r < M$$

When k is positive, simply divide k by M to obtain the remainder r. Thus

$$25 \pmod 7 = 4, \quad 25 \pmod 5 = 0, \quad 35 \pmod{11} = 2, \quad 3 \pmod 8 = 3$$

$$= (1 + 2 + \cdots + n) \cdot \frac{1}{n}$$

$$= \frac{n(n+1)}{2} \cdot \frac{1}{2} = \frac{n+1}{2}$$

This agrees with our intuitive feeling that the average number of comparisons needed to find the location of ITEM is approximately equal to half the number of elements in the DATA list.

Remark: The complexity of the average case of an algorithm is usually much more complicated to analyze than that of the worst case. Moreover, the probabilistic distribution that one assumes for the average case may not actually apply to real situations. Accordingly, unless otherwise stated or implied, the complexity of an algorithm shall mean the function which gives the running time of the worst case in terms of the input size. This is not too strong an assumption, since the complexity of the average case for many algorithms is proportional to the worst case.

Rate of Growth; Big O Notation

Suppose M is an algorithm, and suppose n is the size of the input data: Clearly the complexity $f(n)$ of M increases as n increases. It is usually the rate of increase of $f(n)$ that we want to examine. This is usually done by comparing $f(n)$ with some standard function, such as

$$\log_2 n, \quad n, \quad n \log_2 n, \quad n^2, \quad n^3, \quad 2^n$$

The rates of growth for these standard functions are indicated in Fig. 3.6, which gives their approximate values for certain values of n. Observe that the functions are listed in the order of their rates of growth: the logarithmic function $\log_2 n$ grows most slowly, the exponential function 2^n grows most rapidly, and the polynomial functions n^c grow according to the exponent c.

n \ $g(n)$	$\log n$	n	$n \log n$	n^2	n^3	2^n
5	3	5	15	25	125	32
10	4	10	40	100	10^3	10^3
100	7	100	700	10^4	10^6	10^{30}
1000	10	10^3	10^4	10^6	10^9	10^{300}

Fig. 3.6 *Rate of growth of standard functions*

The way we compare our complexity function $f(n)$ with one of the standard functions is to use the functional "big O" notation which we formally define below.

Definition 3.4: Let $f(x)$ and $g(x)$ be arbitrary functions defined on $\mathbf{R}$ or a subset of $\mathbf{R}$. We say "$f(x)$ is of order $g(x)$", written

$$f(x) = O(g(x))$$

If there exists a real number k and a positive constant C such that, for all $x > k$, we have

$$|f(x)| \leq C\,|g(x)|$$

We also write

$$f(x) = h(x) + O(g(x)) \quad \text{when} \quad f(x) - h(x) = O(g(x))$$

(The above is called the "big O" notation since $f(x) = o(g(x))$ has an entirely different meaning.)

Consider now a polynomial $P(x)$ of degree m. Then we show in Solved Problem 3.27 that $P(x) = O(x^m)$. Thus, for example,

$$7x^2 - 9x + 4 = O(x^2) \quad \text{and} \quad 8x^3 - 576x^2 + 832x - 248 = O(x^3)$$

Complexity of Well-known Algorithms

Assuming $f(n)$ and $g(n)$ are functions defined on the positive integers, then

$$f(n) = O(g(n))$$

means that $f(n)$ is bounded by a constant multiple of $g(n)$ for almost all n.

To indicate the convenience of this notation, we give the complexity of certain well-known searching and sorting algorithms in computer science:

(a) Linear search: $O(n)$
(b) Binary search: $O(\log n)$
(c) Bubble sort: $O(n^2)$
(d) Merge-sort: $O(n \log n)$

SOLVED PROBLEMS

FUNCTIONS

3.1 State whether or not each diagram in Fig. 3-7 defines a function from $A = \{a, b, c\}$ into $B = \{x, y, z\}$.

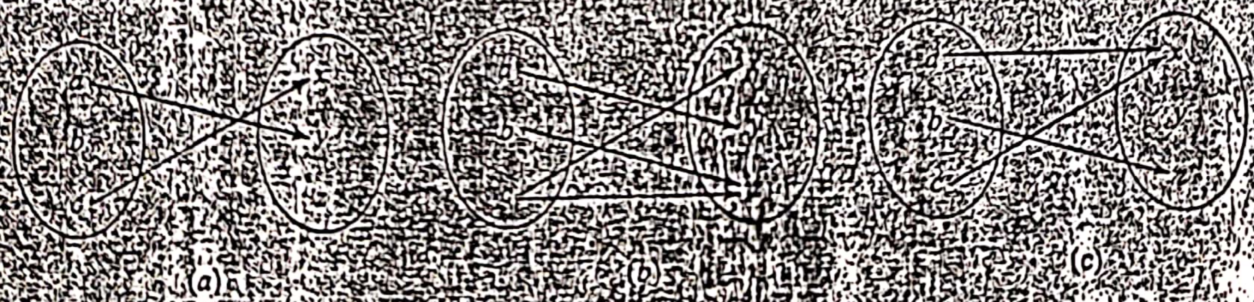


Fig. 3-7

(a) No. There is nothing assigned to the element $b \in A$.
(b) No. Two elements, x and z, are assigned to $c \in A$.
(c) Yes.

3.2 Let $X = \{1, 2, 3, 4\}$. Determine whether or not each relation below is a function from X into X.
(a) $f = \{(2, 3), (1, 4), (2, 1), (3, 2), (4, 4)\}$ (b) $g = \{(3, 1), (4, 2), (1, 1)\}$
(c) $h = \{(2, 1), (3, 4), (1, 4), (2, 1), (4, 4)\}$

Recall that a subset f of $X \times X$ is a function $f: X \to X$ if and only if each $a \in X$ appears as the first coordinate in exactly one ordered pair in f.

(a) No. Two different ordered pairs (2, 3) and (2, 1) in f have the same number 2 as their first coordinate.
(b) No. The element $2 \in X$ does not appear as the first coordinate in any ordered pair in g.
(c) Yes. Although $2 \in X$ appears as the first coordinate in two ordered pairs in h, these two ordered pairs are equal.

3.3 Let A be the set of students in a school. Determine which of the following assignments defines a function on A.
(a) To each student assign his age. (b) To each student assign his teacher.
(c) To each student assign his sex. (d) To each student assign his spouse.

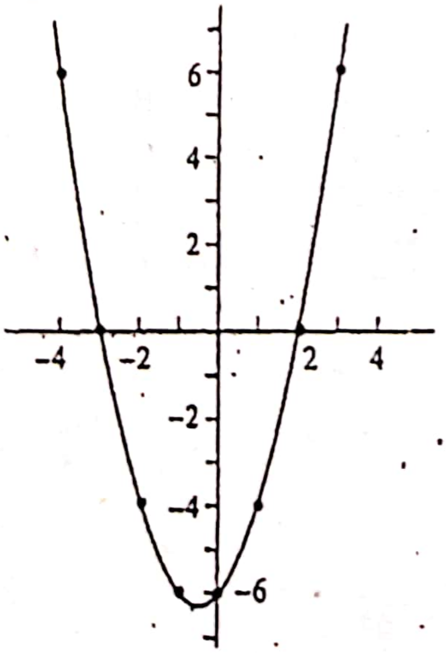
A collection of assignments is a function on A if and only if each element a in A is assigned exactly in one element. Thus:

(a) Yes, because each student has one and only one age.
(b) Yes, if each student has only one teacher; no, if any student has more than one teacher.
(c) Yes.
(d) No, if any student is not married; yes otherwise.

3.4 Sketch the graph of:
(a) $f(x) = x^2 + x - 6$ (b) $g(x) = x^3 - 3x^2 - x + 3$

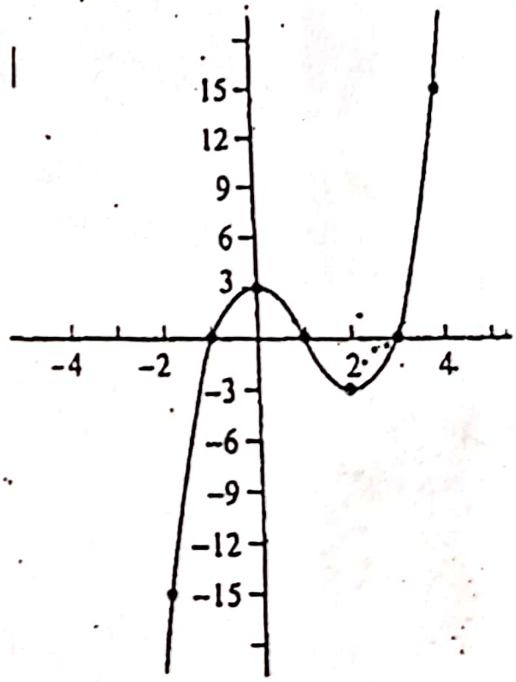
Set up a table of values for x and then find the corresponding values of the function. Since the functions are polynomials, plot the points in a coordinate diagram and then draw a smooth continuous curve through the points. See Fig 3.8.

x	$f(x)$
−4	6
−3	0
−2	−4
−1	−6
0	−6
1	−4
2	0
3	6



Graph of f

x	$g(x)$
−2	−15
−1	0
0	3
1	0
2	−3
3	0
4	15



Graph of g

Fig. 3.8

3.5 Let the functions $f: A \to B$ and $g: B \to C$ be defined by Fig. 3-9. Find the composition function $g \circ f: A \to C$.

Fig. 3-9

We use the definition of the composition function to compute:

$$(g \circ f)(a) = g(f(a)) = g(y) = t$$
$$(g \circ f)(b) = g(f(b)) = g(x) = s$$
$$(g \circ f)(c) = g(f(c)) = g(y) = t$$

Note that we arrive at the same answer if we "follow the arrows" in the diagram:

$$a \to y \to t, \qquad b \to x \to s, \qquad c \to y \to t$$

3.6 Let the functions f and g be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$. Find the formula defining the composition function $g \circ f$.

Compute $g \circ f$ as follows: $(g \circ f)(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2 - 2 = 4x^2 + 4x - 1$. Observe that the same answer can be found by writing

$$y = f(x) = 2x + 1 \qquad \text{and} \qquad z = g(y) = y^2 - 2$$

and then eliminating y from both equations:

$$z = y^2 - 2 = (2x + 1)^2 - 2 = 4x^2 + 4x - 1$$

ONE-TO-ONE, AND INVERTIBLE FUNCTIONS

3.7 Determine if each function is one-to-one.

(a) To each person on the earth assign the number which corresponds to his age.
(b) To each country in the world assign the latitude and longitude of its capital.
(c) To each book written by only one author assign the author.
(d) To each country in the world which has a prime minister assign its prime minister.

(a) No. Many people in the world have the same age.
(b) Yes.
(c) No. There are different books with the same author.
(d) Yes. Different countries in the world have different prime ministers.

3.8 Let the functions $f: A \to B$, $g: B \to C$, and $h: C \to D$ be defined by Fig. 3-10.

(a) Determine if each function is onto.
(b) Find the composition function $h \circ g \circ f$.

Fig 3.10

(a) The function $f: A \to B$ is not onto since $3 \in B$ is not the image of any element in A.
The function $g: B \to C$ is not onto since $z \in C$ is not the image of any element in B.
The function $h: C \to D$ is onto since each element in D is the image of some element of C.

(b) Now $a \to 2 \to x \to 4$, $b \to 1 \to y \to 6$, $c \to 2 \to x \to 4$.
Hence $h \circ g \circ f = \{(a, 4), (b, 6), (c, 4)\}$.

3.9 Consider functions $f: A \to B$ and $g: B \to C$. Prove the following:
(a) If f and g are one-to-one, then the composition function $g \circ f$ is one-to-one.
(b) If f and g are onto functions, then $g \circ f$ is an onto function.

(a) Suppose $(g \circ f)(x) = (g \circ f)(y)$; then $g(f(x)) = g(f(y))$. Hence $f(x) = f(y)$ because g is one-to-one. Furthermore, $x = y$ since f is one-to-one. Accordingly, $g \circ f$ is one-to-one.

(b) Let c be any arbitrary element of C. Since g is onto, there exists a $b \in B$ such that $g(b) = c$. Since f is onto, there exists an $a \in A$ such that $f(a) = b$. But then

$$(g \circ f)(a) = g(f(a)) = g(b) = c$$

Hence each $c \in C$ is the image of some element $a \in A$. Accordingly, $g \circ f$ is an onto function.

3.10 Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x - 3$. Now f is one-to-one and onto; hence f has an inverse function f^{-1}. Find a formula for f^{-1}.

Let y be the image of x under the function f:

$$y = f(x) = 2x - 3$$

Consequently, x will be the image of y under the inverse function f^{-1}. Solve for x in terms of y in the above equation:

$$x = (y + 3)/2$$

Then $f^{-1}(y) = (y + 3)/2$. Replace y by x to obtain

$$f^{-1}(x) = \frac{x + 3}{2}$$

which is the formula for f^{-1} using the usual independent variable x.

3.11 Prove the following generalization of DeMorgan's law: For any class of sets $\{A_i\}$, we have

$$(\cup A_i)^c = \cap A_i^c$$