Lecture 20

Example

Find $E|Z_1 - Z_2|$, with Z_1, Z_2 i.i.d $N(0,1)$

Therom

$X \sim N(\mu_1, \sigma_1^2), Y \sim N(\mu, \sigma_2^2)$ indep

Then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Proof

Use MGF, MGF of $X + Y$ is

$$e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2}$$

Note $Z_1, Z_2 \sim N(0,2)$

$E|Z_1 - Z_2| = E|\sqrt{2}Z| \;\; Z \sim N(0,1)$

$= \sqrt{2}E|Z| = \sqrt{2/\pi}$

Multinomial多项分布

generalization of binomial

Defn/story of $Mult(n, \vec{p})$,

$\vec{p} = (p_1, \dots p_k)$ probability vector$p_j \geq 0, \sum pj = 1$

$\vec{X} \sim Mult(n,p), X = (X_1, \dots X_k)$

story: have n objects independent putting into k categories

$P_j = P(category\ j)$ X_j = #objects in category j

Joint PMF $P(X_1 = n_1, .. X_k = n_k) = \frac{n!}{n_1! n_2! \dots n_k!} P_1^{n_1} P_2^{n_2} \dots P_k^{n_k}$

if$n_1 + .. + n_k = 1$; 0 otherwise

$\vec{X} \sim Mult(n,p)$ Find marginal dist of X_j Then $X_j \sim Bin(n, p_j)$

(each of objects either in this category j or it isn't)

$E(X_j) = np_j, Var(X_j) = np_j(1 - p_j)$

Lumping Property

Merge category together

$\vec{X} = (X_1, \dots X_{10}) \sim Mult(n, (p_1, \dots p_{10}))$

Story: ten political parties, take n people , ask people which party they in

$\vec{Y} = (X_1, X_2, X_3 + .. + X_{10})$ Then $Y \sim Mult(n, (p_1, p_2, p_3 + .. + p_{10}))$

(wouldn't work if one can be in more than one category)

$\vec{X} \sim Mult(n, p)$, Then give $X_1 = n_1$, PMF

$(X_2, \ldots X_k) \sim Mult_{k-1}(n - n_1, (p_2', \ldots p_k'))$

(we know how many people in the first catgory , don't know rest)

with p_2' = P(being in category 2| not in category 1)

= $\frac{p_2}{1-p_1}$

$$p_j' = \frac{p_j}{p_2 + \ldots p_k}$$

Cauchy Interview Problem

The Cauchy is dist. of $T = X/Y$ with X, Y i.i.d $N(0,1)$

Find PDF of T

(doesn't have a mean and variance)

average of million cauchy is still cauchy

$P(\frac{X}{Y} \leq t) = P(\frac{X}{|Y|} \leq t)$ symmetry of $N(0,1)$

$= P(X \leq t|Y|) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} e^{y^2/2} \int_{-\infty}^{t|y|} \frac{1}{\sqrt{2\pi}} e^{x^2/2} dxdy$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{y^2/2} \Phi(t|y|) dy$

$= \sqrt{2/\pi} \int_{0}^{\infty} e^{y^2/2} \Phi(ty) dy$

PDF: $F'(t) = 1/\pi(1 + t^2)$

$P(X \leq t|Y|) = \int P(X \leq t|Y||Y = y)\varphi(y)dy$

$= \int \Phi(t|y|)\varphi(y)dy$