Lecture 17

$E(T|T>20)>E(T)$

If memoryless, we would have $E(T|T>20)=20+E(T)$

Therom: If X is a positive continuous r.v. with memoryless property, then $X\sim Expo(\lambda)$ for some λ

Proof Let F be the CDF of X, $G(x)=P(X>x)=1-F(x)$

memoryless property is $G(s+t)=G(s)G(t)$ solve for G.

let $s=t$, $G(2t)=G(t)^2$, $G(3t)=G(t)^3$ ...$G(kt)=G(t)^k$

$G(t/2)=G(t)^{1/2}$ ...$G(t/k)=G(t)^{1/k}$

$G(\frac{m}{n}t)=G(t)^{\frac{m}{n}}$ So $G(xt)=G(t)^x$ for all real x >0

let $t=1$, $G(x)=G(1)^x$ $=e^{x\ln G(1)}$ $=e^{-\lambda x}$ $lnG(1)=-\lambda$

Moment Generating Function(MGF)

Definition

A r.v X has MGF $M(t)=E(e^{tx})$

as a function of t, if this is finite on some (-0, a), a>0

t is just a placeholder

Why moment "generating" ?

$$E(e^{tx})=E(\sum_{n=0}^{\infty}\frac{x^nt^n}{n!})=\sum_{n=0}^{\infty}\frac{E(x^n)t^n}{n!}\quad E(x^n)\text{ - nth moment}$$

Three reasons why MGF important:

Let X have MGF M(t)

1. The nth moment $E(x^n)$, is coef of $\frac{t^n}{n!}$ in Taylor series of M,

   and $M^{(n)}(0)=E(X^n)$
2. MGF determines the distribution. i.e. if X,Y have same MGF , then they have same CDF
3. If X has MGF M_x, Y has MGF M_y, X independent of Y, then MGF of X+Y is $E(e^{t(X+Y)})=E(e^{tX})+E(e^{tY})$

Example

$X\sim Bern(p)$, $M(t)=E(e^{tX})=pe^t+q, q=1-p$

$X\sim Bin(n,p) => M(t)=(pe^t+q)^n$

$Z \sim N(0,1)$ => $M(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tZ - \frac{Z^2}{2}} dz$

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Z-t)^2} dz = e^{t^2/2}$$

Laplace Rule of Succession

Given p, $X_1, X_2 \ldots$ i.i.d, $Bern(p)$

p unknown,

Laplace used the rule of succession to calculate the probability that the sun will rise tomorrow

Bayesian : treat p as a r.v.

Let $P \sim Unif(0,1)$ (prior) Let $S_n = X_1, \ldots X_n$

So $S_n|p \sim Bin(n,p)$, $p \sim Unif(0,1)$

Find Posterior $p|S_n$, and $P(X_{n+1} = 1|S_n = n)$

$$f(p|S_n = k) = \frac{P(S_n = k|p)f(p)}{P(S_n = k)}$$

f(p) - prior, 1; $P(S_n = k)$ does not depend on p

$P(S_n = k) = \int_0^1 P(S_n = k/p)f(p)dp$

$$\propto \frac{p^k(1-p)^{n-k}}{f(p|S_n = n) = (n+1)p^n}$$

$p(X_{n+1} = 1|S_n = n) = \int_0^1 (n+1)pp^m dp = \frac{n+1}{n+2}$