Lecture 18

MGF

Expo MGF

$X \sim Expo(1)$, find MGF, moment

$M(t) = E(e^{tx}) = \int_0^\infty e^{-tx}e^{-x}dx = \int_0^\infty e^{-x(1-t)}dx = \frac{1}{1-t}, t < 1$

$M'(0) = E(X), M''(0) = E(X^2), M'''(0) = E(X^3)\ldots$

$|t| < 1, \frac{1}{1-t} = \sum_{n=0}^\infty t^n = \sum_{n=0}^\infty n!\frac{t^n}{n!} \quad E(X^n) = n!$

$Y \sim Expo(\lambda)$, let $X = \lambda Y \sim Expo(1)$, so $Y^n = \frac{X^n}{\lambda^n}$

$E(Y^n) = \frac{E(X^n)}{\lambda^n} = \frac{n!}{\lambda^n}$

Normal MGF

Let $Z \sim N(0,1)$,find all its moment

$E(Z^n) = 0$ for n, odd by symmetry

MGF $M(t) = e^{t^2/2} = \sum_{n=0}^{\infty} \frac{(t^2/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(2n)!t^{2n}}{z^n n!(2n)!}$

=>$E(Z^{2n}) = \frac{(2n)!}{2^n n!}$

Poisson MGF

$X \sim Pois(\lambda) \quad E(e^{tx}) = \sum_{k=0}^\infty e^{tx}e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda e^t}$

let $Y \sim Pois(\mu)$ **indep** of X ,find distribution of $X+Y$,

Multiply MGFs, $e^{\lambda(e^t-1)}e^{\mu(e^t-1)} = e^{(\lambda+\mu)(e^t-1)}$ => $X+Y \sim Pois(\lambda+\mu)$

sum of independent Poisson is still Poisson

Counter-example if X, Y dependent: $X = Y$ => $X+Y = 2X$ is not Poisson since:

$E(X+Y) = E(2X) = 2\lambda, Var(2X) = 4\lambda$

Joint Distribution

X, Y Bernouli

Example. 2D

	Y=0	Y=1	
X=0	2/6	1/6	3/6
X=1	2/6	1/6	3/6
	4/6	2/6	

They are independent

X, Y r.vs

Joint CDF

$F(x, y) = P(X \leq x, Y \leq y)$

Joint PMF (discrete case)

$P(X = x, Y = y)$

Marginal CDF

$P(X \leq x)$ is marginal dist. of X

Joint PDF (cont.)

$f(x, y)$ such that

$P((X, Y) \in B) = \iint_B f(x, y) dx dy$

Independence

X, Y independent if and only if $F(x, y) = F_X(x) F_Y(y)$

Equiv.

$P(X = x, Y = y) = P(X = x) P(Y = y)$

$f(x, y) = f_X(x) f_Y(y)$ for all $x, y > 0$

Getting marginals

$P(X = x, Y = y) = \sum_y P(X = x, Y = y)$ discrete

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx$

Example

1. Uniform on $square\{(x, y) : x, y \in [0, 1]\}$

   joint PDF const. on the square, 0 outside

   integral is area c = 1/area = 1

   marginal: X,Y are independent Unif(0,1)
2. Unif in disc $x^2 = y^2 \leq 1$

joint PDF: $1/\pi$ inside the circle; 0 otherwise

X,Y dependent.

Given X=x , $|y| \leq \sqrt{1-x^2}$