

Thermodynamic Formalism for Suspension Flows over Countable Markov Shifts

Tom Kempton

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Abstract

We introduce a new notion of topological pressure for suspension flows over countable Markov shifts. We show that our definition, which is a natural analogue of Gurevich pressure for a Markov shift, is equivalent to previous notions of topological pressure which were defined on a restricted class of flows. We also widen the domain of definition of these previous notions, making them suitable for application to some important examples.

1 Introduction

The notion of topological pressure is an extremely powerful tool in the study of dynamical systems. In the case of suspension flows over finite alphabet Markov shifts, topological pressure was studied by Sinai, Ruelle, Bowen and others, and this research lead to much progress in the study of systems modeled by suspension flows, such as geodesic flows on surfaces of constant negative curvature and more general axiom A flows.

More recently, several definitions of topological pressure for suspension flows over countable Markov shifts have been studied. In [12], a definition of entropy was given for suspension flows with roof function locally constant on each cylinder of length one. In [2], a definition of topological pressure was given for suspension flows with Hölder continuous roof functions which do not approach zero. They were shown to be equivalent when they are both defined.

The purpose of this article is twofold. Firstly we wish to remove some of the technical assumptions in the work of [2] and [12], allowing their techniques to be applied to a wider class of systems including billiard flows with cusps. Secondly we wish to introduce a new equivalent definition of topological pressure for suspension flows, analogous to Gurevich pressure for countable Markov shifts. This new notion has the advantages that it is independent of the coding of the flow and that it lends itself well to computation.

Suspension flows over countable Markov shifts serve as useful models for a wide variety of flows on non-compact spaces, such as the geodesic flow on the modular surface and the Teichmüller flow, and so having an effective notion of topological pressure for such suspension flows will allow the techniques of thermodynamic formalism to be applied to these related systems. Indeed, there is already a large volume of recent work applying the definitions of [2] and [12] to the study of systems modeled by suspension flows over countable Markov shifts, see for example [4, 5, 6, 7].

We now state our main result. All of the definitions will be made precise in the next section.

Theorem 1.1. *Let (Σ, σ) be a topologically mixing countable state Markov shift and $f : \Sigma \rightarrow \mathbb{R}^+$ a roof function with summable variation giving rise to a suspension flow ϕ on space Σ_f . For any function $g : \Sigma_f \rightarrow \mathbb{R}$ for which $\Delta_g(x) := \int_0^{f(x)} g(x, s) ds$ has summable variation, the following notions of topological pressure are equivalent.*

$$P_\phi(g) := \lim_{t \rightarrow \infty} \frac{1}{t} \log \left(\sum_{\phi_s(\underline{x}, 0) = (\underline{x}, 0), 0 < s \leq t} \exp \left(\int_0^s g(\phi_k(\underline{x}, 0)) dk \right) \chi_{[a]}(\underline{x}) \right) \quad (1)$$

$$= \sup_{K \in \mathcal{K}_{\Sigma_f}} P_\phi(g|_K) \quad (2)$$

$$= \inf \{t \in \mathbb{R} : P_\sigma(\Delta_g - tf) \leq 0\} = \sup \{t \in \mathbb{R} : P_\sigma(\Delta_g - tf) \geq 0\} \quad (3)$$

$$= \sup \{h_\nu(\phi) + \int g d\nu : \nu \in \mathcal{E}_\phi, \int g d\nu > -\infty\} \quad (4)$$

where \mathcal{E}_ϕ is the set of flow invariant ergodic probability measures on Σ_f , \mathcal{K}_{Σ_f} is the set of compact flow invariant subsets of Σ_f and P_σ is Gurevich pressure on Σ .

The function P_ϕ takes values in $(-\infty, \infty]$. We have stated our regularity condition on g in terms of the summable variation of Δ_g , this is to avoid having to define a metric on Σ_f . In [12], it was shown that (2) and (4) are equivalent if f is locally constant and $g = 0$. In [2], it was shown that (2), (3) and (4) are equivalent in the case that f is bounded away from zero and both f and Δ_g are weakly Hölder continuous. The definition (1) and the equivalence of the four definitions in our more general setting is new. A similar approach is taken in an article by Jaerisch, Kesseböhmer and Lamei, [8], which was completed at the same time as ours.

In section 3 we prove that the definition (1) of pressure is well defined. In section 4 we show that (1) = (2). In section 5 we recall a lemma from [2] giving that (2) = (3). Finally, in section 6 we show that (3) = (4).

2 Set Up

Let \mathcal{A} be a countable set and M be a matrix of zeroes and ones indexed by $\mathcal{A} \times \mathcal{A}$. We define the two sided topological Markov shift (Σ, σ) to be the set

$$\Sigma := \{\underline{x} \in \mathcal{A}^{\mathbb{Z}} : M_{x_i x_{i+1}} = 1 \forall i \in \mathbb{Z}\}$$

coupled with the shift map $\sigma : \Sigma \rightarrow \Sigma$, $(\sigma(\underline{x}))_i = (\underline{x})_{i+1}$. We further define the cylinder set $[x_m \cdots x_n]$ to be the set of sequences $\underline{y} = (y_i)_{i=-\infty}^{\infty}$ in Σ such that $y_m \cdots y_n = x_m \cdots x_n$, and the n -th variation of a function $f : \Sigma \rightarrow \mathbb{R}$ by

$$\text{var}_n(f) = \sup\{|f(\underline{x}) - f(\underline{y})| : x_{-(n-1)} \cdots x_{n-1} = y_{-(n-1)} \cdots y_{n-1}\}.$$

A function $f : \Sigma \rightarrow \mathbb{R}$ is said to have summable variation if $\sum_{n=1}^{\infty} \text{var}_n(f) < \infty$ and be weakly Hölder continuous if there exist $c > 0$ and $\theta \in (0, 1)$ such that $\text{var}_n(f) < c\theta^n$ for all $n \geq 1$. We do not require functions of summable variation to be bounded.

A Markov shift is called topologically mixing if, for any $a, b \in \mathcal{A}$, there exists an $N \in \mathbb{N}$ such that $\sigma^{-n}[a] \cap [b]$ is non empty for all $n > N$. In [11], Sarig defined Gurevich pressure for topologically mixing countable alphabet Markov shifts as follows:

$$P_\sigma(g) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{\sigma^n(\underline{x}) = \underline{x}} \exp(g^n(\underline{x})) \chi_{[a]}(\underline{x}) \right)$$

where a is any element of \mathcal{A} and $g^n(\underline{x}) := \sum_{k=0}^{n-1} g(\sigma^k(\underline{x}))$. The definition is independent of the choice of $a \in \mathcal{A}$. It was proved in [11] that Gurevich pressure satisfies

$$\begin{aligned} P_\sigma(g) &= \sup_{K \in \mathcal{K}_\Sigma} P(g|_K) \\ &= \sup\{h_\mu(\sigma) + \int g d\mu < \infty : \mu \in \mathcal{M}_\sigma, \int g d\mu > -\infty\}. \end{aligned}$$

where \mathcal{K}_Σ is the set of compact shift invariant subsets of Σ , $h(\mu)$ denotes the metric entropy of μ and \mathcal{M}_σ the set of σ invariant Borel probability measures on Σ .

2.1 Suspension Flows

Given a Markov shift Σ and a function $f : \Sigma \rightarrow \mathbb{R}^+$ which we call the roof function we define the space $\Sigma_f := \{(x, t) : x \in \Sigma, 0 \leq t \leq f(x)\}$. We further define the suspension flow ϕ on Σ_f by $\phi_t(x, s) = (x, s + t)$ for $-s \leq t \leq f(x) - s$, and extend this to a flow for all time t using the identification $(x, f(x)) = (\sigma(x), 0)$. We let \mathcal{M}_ϕ denote the set of ϕ invariant probability measures on Σ_f .

In order to be a well defined flow we require that $\phi_t(x, s)$ is defined for all $t \in \mathbb{R}$, and hence that $\sum_{n=1}^{\infty} f(\sigma^n(x)) = \sum_{n=1}^{\infty} f(\sigma^{-n}(x)) = \infty$ for all $x \in \Sigma$.

In the case that \mathcal{A} is finite, topological pressure for a function g on Σ_f can be defined by

$$P_\phi(g) := \lim_{t \rightarrow \infty} \frac{1}{t} \log \left(\sum_{\phi_s(\underline{x}, 0) = (\underline{x}, 0), 0 < s \leq t} \exp \left(\int_0^s g(\phi_k(\underline{x}, 0)) dk \right) \right).$$

The relationship between the pressure functions P_σ and P_ϕ in this case was studied by Bowen and Ruelle, see [3] for proofs of the following facts. We have that

$$P_\sigma(\Delta_g - P_\phi(g).f) = 0$$

where $\Delta_g(\underline{x}) := \int_0^{f(\underline{x})} g(\underline{x}, s) ds$. This allows the study of thermodynamic formalism on suspension flows over finite Markov shifts to be reduced to the study of thermodynamic formalism on the base. Given a measure μ on Σ for which $\int_{\Sigma} f d\mu$ is finite, we can lift the measure to Σ_f by defining

$$\mu_f := \mathcal{L}(\mu) := \frac{(\mu \times m)|_{\Sigma_f}}{\int_{\Sigma} f d\mu}$$

where m is lebesgue measure. In the case that there exist $c_1, c_2 > 0$ with $c_1 < f < c_2$, $\mathcal{L} : \mathcal{M}_{\sigma} \rightarrow \mathcal{M}_{\phi}$ is a bijection. However, in the case that Σ is a countable Markov shift, f may not be bounded away from zero, in which case members of \mathcal{M}_{ϕ} may be the lift of σ -finite infinite invariant measures on Σ . This fact weakens the relationship between the thermodynamic formalism of the shift and the thermodynamic formalism of the flow and makes our proof of the variational principle significantly more technical.

2.2 Entropy

Given a transformation $T : X \rightarrow X$ and a probability measure μ , we let the metric entropy $h_{\mu}(T)$ be the standard Kolmogorov-Sinai entropy (see [14]). This was extended in [9] to spaces (X, T, μ) for which $\mu(X)$ is conservative but need not be finite, by defining

$$h_{\mu}(T) = \sup\{h_{\mu|_E}(T|_E) : E \subset X, 0 < \mu(E) < \infty\}.$$

Here $\mu|_E(A) = \mu(E \cap A)$ and $T|_E : E \rightarrow E$ is the induced transformation

$$T|_E(x) := T^n(x),$$

where $n := \min\{m \geq 1 : T^m(x) \in E\}$.

Furthermore, we have that for any sweep out set E , $h_{\mu}(T) = h_{\mu|_E}(T|_E)$ (see [9, 15]), where a sweep out set is any set E to which almost every point of X returns infinitely often. If T is ergodic then every set of positive measure is a sweep out set.

For a flow $\phi : X \rightarrow X$ on a conservative measure space (X, μ) , we define the entropy $h_{\mu}(\phi)$ to be the entropy $h_{\mu}(\phi_1)$ of the time one transformation $\phi_1 : X \rightarrow X$.

Suppose now that we have a countable Markov shift Σ and continuous roof function f giving suspension flow ϕ on Σ_f , and measure μ on Σ with $0 < \int f d\mu < \infty$. Then we have that

$$h_{\mu_f}(\phi) = \frac{h_{\mu}(\sigma)}{\int f d\mu}.$$

This was proved in the case of finite measures and compact spaces by Abramov in [1], and generalised to our setting by Savchenko in [12].

Topological entropy for a transformation T can be defined as the supremum over all invariant probability measures μ of the metric entropy $h_{\mu}(T)$. Similarly for a flow ϕ topological entropy can be defined as the supremum of $h_{\mu}(\phi)$. Then putting $g = 0$ into

definition (2) of pressure gives topological entropy, and so as a corollary to theorem 1.1 we have that definitions (1), (2), (3) and (4) with $g = 0$ each give an equivalent notion of topological entropy for suspension flows over a countable Markov shift.

2.3 Flows and Semiflows

The results that we have stated apply both to (invertible) suspension flows over two sided Markov shifts and to (non-invertible) semiflows over one sided Markov shifts.

In [13], Sinai proved that each function f of summable variation supported on a countable two-sided Markov shift Σ is cohomologous to a function \tilde{f} on Σ , also of summable variation, depending only positive coordinates. Furthermore, since \tilde{f} depends only on positive coordinates we can consider the related function \tilde{f}^+ defined on the one sided Markov shift Σ^+ with the same incidence matrix as Σ . Since none of the definitions (1) – (4) are affected by the addition of coboundaries, or by the passing from one sided to two sided shifts if f and g depend only on positive coordinates, we can assume without loss of generality that Σ is a two sided shift and that the functions f and Δ_g depend only on positive coordinates of Σ .

A good modern introduction to the result of Sinai and its consequences for the thermodynamic formalism of one sided and two sided shifts is given in [10].

3 An analogue of Gurevich pressure for suspension flows

We define

$$P_\phi(g) := \lim_{t \rightarrow \infty} \frac{1}{t} \log \left(\sum_{\phi_s(\underline{x}, 0) = (\underline{x}, 0), 0 < s \leq t} \exp \left(\int_0^s g(\phi_k(\underline{x}, 0)) dk \right) \chi_{[a]}(\underline{x}) \right).$$

As in the work of Sarig [11], we are interested in measuring a weighted growth rate of the number of periodic orbits of length less than or equal to t . By restricting to orbits which pass through some cylinder $[a]$ we are able to gain an effective notion of this growth rate while avoiding complications which may arise when there are infinitely many periodic points of certain length.

In this section we show that the sum exists for any choice of $a \in \mathcal{A}$, and further that it is independent of any such choice, making P_ϕ well defined. We begin with a classical lemma on sequences.

Lemma 3.1. *If (a_t) is a sequence for which there exist constants c_1, c_2 such that*

$$a_{s+t+c_2} + c_1 \geq a_s + a_t \tag{5}$$

for all $s, t \in \mathbb{R}^+$, then $\lim_{t \rightarrow \infty} \frac{a_t}{t}$ exists, taking a value in $(-\infty, \infty]$.

Furthermore, for any $\epsilon, \delta > 0$ there exists $T > 0$ depending only on c_1, c_2, ϵ and δ such that for all $t > T$,

$$\frac{a_t}{t} \leq \frac{1}{1 - \delta} \left(\lim_{t \rightarrow \infty} \frac{a_t}{t} \right) + \epsilon.$$

The proof of this lemma is standard, requiring only a minor modification from the ‘almost subadditivity’ lemma of [11], and can be found for example in the author’s thesis. We now show that the limit in our definition of pressure is well defined for any $a \in \mathcal{A}$.

Lemma 3.2. *Given $a \in \mathcal{A}$, the limit*

$$P_{\phi,a}(g) := \lim_{t \rightarrow \infty} \frac{1}{t} \log \left(\sum_{\phi_s(\underline{x}, 0) = (\underline{x}, 0), 0 < s \leq t} \exp \left(\int_0^s g(\phi_k(\underline{x}, 0)) dk \right) \chi_{[a]}(\underline{x}) \right)$$

exists, taking a value in $(-\infty, \infty]$.

Proof. We consider the following sequence.

$$a_t := \log \left(\sum_{\phi_s(\underline{x}, 0) = (\underline{x}, 0), 0 < s \leq t} \exp \left(\int_0^s g(\phi_k(\underline{x}, 0)) dk \right) \chi_{[a]}(\underline{x}) \right).$$

If there exists some value of t for which $a_t = \infty$ then $P_{\phi,a}(g)$ will be infinite. Given a word $x_1 \cdots x_n \in \Sigma$ such that $x_n x_1 \in \Sigma$, we let $\overline{(x_1 \cdots x_n)}$ denote the sequence $(y_i)_{i=-\infty}^{\infty} \in \Sigma$ where $y_i = x_{i \pmod n}$.

Now suppose that γ_1 is the periodic orbit of ϕ based at $(\overline{(x_1 \cdots x_n)}, 0)$, and that γ_2 is the periodic orbit of ϕ based at $(\overline{(y_1 \cdots y_m)}, 0)$, where $x_1 = y_1 = a$. Then γ_1 has period $t = f^n(\overline{(x_1 \cdots x_n)})$, γ_2 has period $s = f^m(\overline{(y_1 \cdots y_m)})$, and the periodic orbit $\gamma_1 \gamma_2$ based at $(\overline{(x_1 \cdots x_n y_1 \cdots y_m)}, 0)$ has period

$$\begin{aligned} f^{n+m}(\overline{(x_1 \cdots x_n y_1 \cdots y_m)}) &= f^n(\overline{(x_1 \cdots x_n y_1 \cdots y_m)}) + f^m(\overline{(y_1 \cdots y_m x_1 \cdots x_n)}) \\ &\leq f^n(\overline{(x_1 \cdots x_n)}) + \sum_{k=1}^n \text{var}_k(f) + f^m(\overline{(y_1 \cdots y_m)}) + \sum_{k=1}^m \text{var}_k(f) \\ &\leq s + t + 2 \sum_{k=1}^{\infty} \text{var}_k(f). \end{aligned}$$

We define $c_2 := 2 \sum_{k=1}^{\infty} \text{var}_k(f) < \infty$, and observe that it is independent of the lengths n and m . Thus any two periodic orbits γ_1 and γ_2 sharing a common base point can be interwoven to give the periodic orbit $\gamma_1 \gamma_2$ of period less than or equal to $l(\gamma_1) + l(\gamma_2) + c_2$.

For a periodic orbit γ of period t passing through point $(\overline{(x_1 \cdots x_n)}, 0)$ we write

$$\int_{\gamma} g := \int_0^t g(\phi_k(\overline{(x_1 \cdots x_n)}, 0)) dk = \sum_{k=1}^n \Delta_g(\sigma^k(\overline{(x_1 \cdots x_n)})).$$

Now Δ_g has summable variation, and so letting $c_1 := 2 \sum_{n=1}^{\infty} \text{var}_k(\Delta_g) < \infty$ and using the same arguments given above for f , we have

$$\int_{\gamma_1 \gamma_2} g + c_1 \geq \int_{\gamma_1} g + \int_{\gamma_2} g.$$

So for any γ_1, γ_2 in the summations for a_s and a_t , their concatenation $\gamma_1 \gamma_2$ is in the summation for a_{s+t+c_2} , and the evaluation of g over this orbit differs by at most c_1 . We may have extra orbits in the summation for a_{s+t+c_2} but these cannot contribute negatively. Thus we get the inequality,

$$a_{s+t+c_2} + c_1 \geq a_s + a_t.$$

Then lemma 3.1 gives that $\lim_{t \rightarrow \infty} \frac{a_t}{t}$ exists and hence that $P_{\phi,a}(g)$ is well defined. □

Lemma 3.3. *$P_{\phi,a}(g)$ is independent of a , and hence $P_{\phi}(g)$ is well defined.*

Proof. Let $a, b \in \mathcal{A}$. We let a_t be defined as in the previous lemma and let b_t be defined analogously:

$$b_t := \log \left(\sum_{\phi_s(\underline{x}, 0) = (\underline{x}, 0), 0 < s \leq t} \exp \left(\int_0^s g(\phi_k(\underline{x}, 0)) dk \right) \chi_{[b]}(\underline{x}) \right).$$

We choose and fix finite words $x_1 \cdots x_m$ and $y_1 \cdots y_n$, where $x_1 = a$, $y_1 = b$, and where $x_m b$ and $y_n a$ are admissible words. These should be thought of as paths in Σ linking a to b and b to a respectively. We define

$$\begin{aligned} T_{a,b} &= \sup\{f^m(x) : x \in [x_1 \cdots x_m b]\} + \sup\{f^n(y) : y \in [y_1 \cdots y_n a]\} \\ G_{a,b} &= \inf\{\Delta_g^m(x) : x \in [x_1 \cdots x_m b]\} + \inf\{\Delta_g^n(y) : y \in [y_1 \cdots y_n a]\}. \end{aligned}$$

These are finite since f and Δ_g have summable variation, even though f and Δ_g may be unbounded.

Then any periodic orbit γ_1 of length t based at $(\overline{z_1 \cdots z_p}, 0)$ with $z_1 = a$ can be extended to a periodic orbit γ_2 based at $(\overline{y_1 \cdots y_n z_1 \cdots z_p x_1 \cdots x_m}, 0)$. This orbit passes through $([b], 0)$ and so is included in the summation for b_t .

We see that

$$\begin{aligned} l(\gamma_2) &= f^{n+p+m}(\overline{y_1 \cdots y_n z_1 \cdots z_p x_1 \cdots x_m}) \\ &\leq \sup\{f^n(y) : y \in [y_1 \cdots y_n a]\} + f^p(\overline{z_1 \cdots z_p}) + \sum_{n=1}^{\infty} \text{var}_n(f) \\ &\quad + \sup\{f^m(x) : x \in [x_1 \cdots x_m b]\} \\ &\leq t + c_2 + T_{a,b}. \end{aligned}$$

Similarly

$$\int_{\gamma_1} g - c_1 + G_{a,b} \leq \int_{\gamma_2} g.$$

So we see that

$$\begin{aligned} & \log \left(\sum_{\phi_s(\underline{x},0)=(\underline{x},0), 0 < s \leq t+T_{a,b}+c_2} \exp \left(\int_0^s g(\phi_k(\underline{x},0)) dk \right) \chi_{[b]}(\underline{x}) \right) \geq \\ & \log \left(\sum_{\phi_s(\underline{x},0)=(\underline{x},0), 0 < s \leq t} \exp \left(G_{a,b} - c_1 + \int_0^s g(\phi_k(\underline{x},0)) dk \right) \chi_{[a]}(\underline{x}) \right), \end{aligned}$$

i.e. that

$$b_{t+T_{a,b}+c_2} - G_{a,b} + c_1 \geq a_t.$$

Dividing by t , taking the limit as t tends to infinity we see that

$$\lim_{t \rightarrow \infty} \frac{b_t}{t} \geq \lim_{t \rightarrow \infty} \frac{a_t}{t}.$$

But since $a, b \in \mathcal{A}$ were arbitrary, this gives us that $\lim_{t \rightarrow \infty} \frac{b_t}{t} = \lim_{t \rightarrow \infty} \frac{a_t}{t}$, and that our definition of pressure is independent of the choice of $a \in \mathcal{A}$. \square

4 Compact Invariant Subsets

We relate our notion of topological pressure to the classical notion of pressure on compact invariant subsets of Σ_f .

Lemma 4.1.

$$P_\phi(g) = \sup_{K_f \in \mathcal{K}_{\Sigma_f}} P_\phi(g|K_f)$$

We adapt our proof from the proof of a similar statement in [11].

Proof. We define

$$a_t(K_f) := \log \left(\sum_{(\underline{x},0) \in K_f: \phi_s(\underline{x},0)=(\underline{x},0), 0 < s \leq t} \exp \left(\int_0^s g(\phi_k(\underline{x},0)) dk \right) \chi_{[a]}(\underline{x}) \right).$$

where a is any member of the alphabet upon which K_f is supported. We recall that

$$P_\phi(g|K_f) := \lim_{t \rightarrow \infty} \frac{a_t(K_f)}{t}$$

and

$$P_\phi(g) := \lim_{t \rightarrow \infty} \frac{a_t}{t}.$$

Then since the summation in the definition of $a_t(K_f)$ is over a smaller set than the corresponding summation in the definition of a_t , we have that $a_t(K_f) \leq a_t$ and hence

$$P_\phi(g) \geq \sup\{P_\phi(g|K_f) : K_f \in \mathcal{K}_{\Sigma_f}\}.$$

We will prove the reverse inequality. Let us assume that $P_\phi(g) < \infty$, the infinite case is similar. Fix $\epsilon, \delta > 0$. We recall that lemma 3.2 gives that

$$a_{s+t+c_2}(K_f) + c_1 \geq a_s(K_f) + a_t(K_f)$$

where c_1 and c_2 do not depend on K_f , and hence by lemma 3.1 there exists $T > 0$ independent of K_f such that for all $t > T$,

$$(1 + \delta)P_\phi(g|K_f) + \epsilon \geq \frac{a_t(K_f)}{t}.$$

We choose and fix $t > T$ large enough so that

$$P_\phi(g) \leq \frac{1}{t}a_t + \epsilon.$$

Now the summation in the definition of a_t is a summation over the possibly countable set of periodic orbits of ϕ which pass through the point $([a], 0)$ and are of length less than or equal to t . Enumerating these periodic orbits arbitrarily, there exists some finite number N such that the restriction of the summation a_t to the first N orbits is greater than $a_t - \epsilon$, because countable summation is just the limit of finite summation. Now each loop can be coded using only finitely many members of \mathcal{A} , and hence the first N orbits are supported on some finite alphabet. Thus there exists a natural number M such that

$$\frac{1}{t}a_t \leq \frac{1}{t}a_t((\{1, \dots, M\}^{\mathbb{Z}} \cap \Sigma)_f) + \epsilon,$$

where by $(\{1, \dots, M\}^{\mathbb{Z}} \cap \Sigma)_f$ we mean the suspension flow over the restriction of Σ to the alphabet $\{1, \dots, M\}$. By adding a finite number of symbols we can extend $(\{1, \dots, M\}^{\mathbb{Z}} \cap \Sigma)_f$ to a space K_f which intersects $[a] \times \{0\}$, is still compact, and on which the shift transformation on the base is topologically mixing. We still have

$$\frac{1}{t}a_t \leq \frac{1}{t}a_t(K_f) + \epsilon.$$

We have argued that

$$\begin{aligned} P_\phi(g) &\leq \frac{a_t}{t} + \epsilon \\ \frac{a_t}{t} &\leq \frac{a_t(K_f)}{t} + \epsilon \text{ and} \\ \frac{a_t(K_f)}{t} &\leq (1 + \delta)P_\phi(g|K_f) + \epsilon. \end{aligned}$$

Then we have

$$P_\phi(g) \leq (1 + \delta)P_\phi(g|K_f) + 3\epsilon,$$

and since ϵ and δ were arbitrary this gives that

$$P_\phi(g) \leq \sup\{P_\phi(g|K_f) : K_f \in \mathcal{K}_{\Sigma_f}\}.$$

Combining with the reverse inequality given earlier, we have that $P_\phi(g)$ is indeed the supremum of the topological pressures of suspension flows over compact flow invariant subsets. \square

5 Equivalence with the definition of Barreira and Iommi

We recall that in [2] topological pressure was defined by the equation

$$P_{BI}(g) := \inf\{t \in \mathbb{R} : P_\sigma(\Delta_g - tf) \leq 0\} = \sup\{t \in \mathbb{R} : P_\sigma(\Delta_g - tf) \geq 0\}.$$

We use a lemma from [2] to prove that P_{BI} and P_ϕ are equivalent. It has been extended to deal with roof functions that approach zero without altering the proof.

Lemma 5.1. $P_{BI}(g) = \sup_{K \in \mathcal{K}_{\Sigma_f}} P_\phi(g|_K)$

Proof.

$$\begin{aligned} P_{BI}(g) &:= \inf\{t \in \mathbb{R} : P_\sigma(\Delta_g - tf) \leq 0\} \\ &= \inf\{t \in \mathbb{R} : P_\sigma((\Delta_g - tf)|_K) \leq 0 \ \forall K \in \mathcal{K}_\Sigma\} \\ &= \inf\{t \in \mathbb{R} : P_\phi(g|_K) \leq t \ \forall K \in \mathcal{K}_{\Sigma_f}\} \\ &= \sup_{K \in \mathcal{K}_{\Sigma_f}} P_\phi(g|_K) \end{aligned}$$

The second equality holds because Gurevich pressure can be approximated by pressure on compact invariant subsets, and because $g_1 \geq g_2 \implies P_\phi(g_1) \geq P_\phi(g_2)$. The third line uses the natural relation between members of \mathcal{K}_Σ and \mathcal{K}_{Σ_f} , and that $P_\sigma(\Delta_g - P_\phi(g).f) = 0$ on compact spaces. \square

Since we also have that $P_\phi(g) = \sup_{K \in \mathcal{K}_{\Sigma_f}} P_\phi(g|_K)$, we have the equivalence of P_ϕ and P_{BI} .

6 The Variational Principle

We now define some spaces of measures which allow us to state our variational principle. We recall that \mathcal{M}_ϕ was defined as the set of all flow invariant probability measures on the space Σ_f . We further define

$$\mathcal{E}_{\phi,g} := \{\nu \in \mathcal{M}_\phi : \int g d\nu > -\infty, \nu \text{ is ergodic}\}$$

and

$$\mathcal{E}_{\phi,g}^p := \{\nu \in \mathcal{E}_{\phi,g} : \nu = \mathcal{L}(\mu) \text{ for some } \mu \in \mathcal{M}_\sigma\}.$$

We stress that members of $\mathcal{E}_{\phi,g}^p$ must be the lift of *probability* measures on the base. Our variational principle is the following.

Theorem 6.1.

$$P_\phi(g) = \sup\{h(\nu) + \int g d\nu : \nu \in \mathcal{E}_{\phi,g}\}.$$

A first step in the proof of the variational principle is to restrict to those measures which are the lift of probability measures on the base, and then use the properties of Gurevich pressure on the base proved in [11]. The proof, which we include for completeness, follows that of similar statements in [3] and [2].

Lemma 6.1. $P_\phi(g) = \sup\{h_{\mu_f}(\phi) + \int g d\mu_f : \mu_f \in \mathcal{E}_{\phi,g}^p\}$

Proof. For K a compact subset of Σ , we have

$$P_\phi(g|_K) = \sup\{h_{\mu_f}(\phi|_K) + \int_K g d\mu_f : \mu_f \in \mathcal{E}_{\phi,g}^p\}.$$

This is just the statement of the variational principle for compact subsets of Σ , we note that the restriction to $\mathcal{E}_{\phi,g}^p$ is no problem here since Δ_g must be bounded below and $f > 0$ must be bounded away from zero on a compact set, and hence since $\int f d\mu < \infty$ we have $\mu(K) < \infty$. Using lemma 4.1 we take the supremum over compact subsets to get that

$$P_\phi(g) \leq \sup\left\{h_{\mu_f}(\phi) + \int g d\mu_f : \mu_f \in \mathcal{E}_{\phi,g}^p\right\}.$$

Then for $t > P_\phi(g) = \inf\{t : P_\sigma(\Delta_g - tf) \leq 0\}$ we have

$$\begin{aligned} 0 &\geq P_\sigma(\Delta_g - tf) \\ &\geq \sup\left\{h_\mu(\sigma) + \int_\Sigma \Delta_g d\mu - t \int_\Sigma f d\mu : \mu_f \in \mathcal{E}_{\phi,g}^p\right\} \\ &= \sup\left\{\int_\Sigma f d\mu \left(\frac{h_\mu(\sigma)}{\int_\Sigma f d\mu} + \frac{\int_\Sigma \Delta_g d\mu}{\int_\Sigma f d\mu} - t\right) : \mu_f \in \mathcal{E}_{\phi,g}^p\right\}. \end{aligned}$$

But since $\int_\Sigma f d\mu > 0$, we can divide by it to get

$$0 \geq \sup\left\{h_{\mu_f}(\phi) + \int g d\mu_f - t : \mu_f \in \mathcal{E}_{\phi,g}^p\right\},$$

giving

$$t \geq \sup\left\{h_{\mu_f}(\phi) + \int g d\mu_f : \mu_f \in \mathcal{E}_{\phi,g}^p\right\}.$$

Hence

$$\inf\{t : P_\sigma(\Delta_g - tf) \leq 0\} = \sup\left\{h_{\mu_f}(\phi) + \int g d\mu_f : \mu_f \in \mathcal{E}_{\phi,g}^p\right\}$$

as required. \square

Now theorem 6.1 follows directly from combining lemma 6.1 with the following:

Proposition 6.1.

$$\sup\{h_\nu(\phi) + \int g d\nu : \nu \in \mathcal{E}_{\phi,g}^p\} = \sup\{h_\nu(\phi) + \int g d\nu : \nu \in \mathcal{E}_{\phi,g}\}.$$

While the details of the proof of this proposition are slightly technical, the principle behind the proof here is simple.

We define

$$V(g) := \sup\{h_\nu(\phi) + \int g d\nu : \nu \in \mathcal{E}_{\phi,g}\}$$

and choose a measure μ on Σ such that μ_f is close to achieving the supremum $V(g)$. We then find a sequence of finite measures μ^n on Σ such that

$$\begin{aligned} \int f d\mu^n &\rightarrow \int f d\mu \\ \int \Delta_g d\mu^n &\rightarrow \int \Delta_g d\mu \\ h_{\mu^n}(\sigma) &\rightarrow h_\mu(\sigma). \end{aligned}$$

Then

$$h_{\mu_f^n}(\phi) + \int g d\mu_f^n = \frac{h_{\mu^n}(\sigma)}{\int f d\mu^n} + \frac{\int \Delta_g d\mu^n}{\int f d\mu^n} \rightarrow \frac{h_\mu(\sigma)}{\int f d\mu} + \frac{\int \Delta_g d\mu}{\int f d\mu} = h_{\mu_f}(\phi) + \int g d\mu_f.$$

Thus we have a sequence of measures $\mu_f^n \in \mathcal{E}_{\phi,g}^p$ which come arbitrarily close to achieving the supremum $V(g)$. We now fill in the details.

Proof. Given $\epsilon > 0$, we choose a measure $\mu_f \in \mathcal{E}_{\phi,g}$ such that

$$h_{\mu_f}(\phi) + \int g d\mu_f + \epsilon > V(g).$$

Then $\mu_f = \mathcal{L}(\mu)$ for some measure μ on Σ which may not be finite. Since μ_f is a probability measure it is conservative, and then since \mathcal{L} preserves ergodicity and conservativity we have that μ is ergodic and conservative.

Given A , we let the set $\{A_\infty\}$ be the set of sequences \underline{x} for which $\sigma^n(\underline{x})$ intersects A for infinitely many positive and negative values of n . Since μ is conservative and ergodic we have that for every set A such that $0 < \mu(A) < \infty$, $\mu(\Sigma \setminus \{A_\infty\}) = 0$ and $h_\mu(\sigma) = h_{\mu|_A}(\sigma|A)$ (see the earlier section on metric entropy). Given $\delta > 0$ we choose m such that $\sum_{k=m-1}^\infty \text{var}_k(f) < \delta$ and choose A to be a cylinder set $[a_1 \cdots a_m]$ for which there doesn't exist a $k < m$ such that $a_1 \cdots a_k = a_{m-k} \cdots a_m$. This technical restriction just avoids two occurrences of the word $a_1 \cdots a_m$ overlapping. Since multiplying μ by a constant has no effect on μ_f we suppose that $\mu[a_1 \cdots a_m] = 1$.

The set of all finite words in Σ in which $a_1 \cdots a_m$ appears at the start and end but nowhere else is countable. We number the elements arbitrarily $(\gamma_i)_{i=1}^\infty$, and say that

word γ_i has length $l(\gamma_i)$. The cylinder $[\gamma_i]$ is the set of sequences in Σ whose first $l(\gamma_i)$ coordinates coincide with those of γ_i . Then $(\Sigma \cap \{[a_1 \cdots a_m]_\infty\})$ is partitioned by the sets $\sigma^k[\gamma_i]$ for $0 \leq k \leq l(\gamma_i) - m$, $i \in \mathbb{N}$. Finally we let $q_n := \sum_{i=1}^n \mu[\gamma_i]$, and observe that q_n tends to $\mu[a_1 \cdots a_m] = 1$ as n tends to infinity.

Let μ^n be the σ -invariant measure on Σ such that

1. $\mu^n(\Sigma \setminus \{[a_1 \cdots a_m]_\infty\}) = 0$
2. $\mu^n[\gamma_i] = \frac{\mu[\gamma_i]}{q_n}$ for $i \in \{1, \dots, n\}$
3. $\mu^n[\gamma_i] = 0$ for $i > n$
4. The induced measure $\mu^n|_{[a_1 \cdots a_m]}$ is a Bernoulli measure on choices of $[\gamma_i]$.

The measure $\mu^n|_{[a_1 \cdots a_m]}$, is a standard Bernoulli measure on a full shift with n -symbols; the existence of such measures is classical. From this we can use additivity and shift invariance to uniquely define the measure μ^n on Σ .

We note that

$$\mu^n(\Sigma) = \sum_{i=1}^n \sum_{k=0}^{l(\gamma_i)-m} \mu^n(\sigma^k[\gamma_i]) < \sum_{i=1}^n \sum_{k=0}^{l(\gamma_i)-m} \mu^n[a_1 \cdots a_m] < \infty$$

since each γ_i contains some occurrence of $a_1 \cdots a_m$, and $\mu^n[a_1 \cdots a_m] = 1$. The reason that we divide by q_n is to make $\mu^n[a_1 \cdots a_m] = 1$, allowing us to make $\mu^n|_{[a_1 \cdots a_m]}$ a Bernoulli measure.

We now investigate $\int f d\mu^n$. For \underline{x} and \underline{y} in $\sigma^k[\gamma_i]$, $|f(\underline{x}) - f(\underline{y})| < \text{var}_{l(\gamma_i)-k}(f)$, because f has summable variation and depends only on future coordinates. We choose \underline{x} in $[\gamma_i]$. Then writing $a = b \pm c$ as shorthand for the statement $a - c \leq b \leq a + c$, we have that

$$\begin{aligned} \int_{\sigma^k[\gamma_i]} f d\mu^n &= (f(\sigma^k(\underline{x})) \pm \text{var}_{l(\gamma_i)-k}(f)) \mu^n(\sigma^k[\gamma_i]) \\ &= f(\sigma^k(\underline{x})) \mu^n[\gamma_i] \pm \text{var}_{l(\gamma_i)-k}(f) \mu^n[\gamma_i], \end{aligned}$$

since $\mu^n(\sigma^k[\gamma_i]) = \mu^n[\gamma_i]$. The same argument works replacing μ^n with μ . This gives us

$$\sum_{k=0}^{l(\gamma_i)-m} \int_{\sigma^k[\gamma_i]} f d\mu^n = \left(\sum_{k=0}^{l(\gamma_i)-m} f(\sigma^k(\underline{x})) \mu^n[\gamma_i] \right) \pm \mu^n[\gamma_i] \sum_{k=m}^{\infty} \text{var}_k(f).$$

Then

$$\begin{aligned}
\int_{\Sigma} f d\mu^n &= \sum_{i=1}^n \sum_{k=0}^{l(\gamma_i)-m} \left(\int_{\sigma^k[\gamma_i]} f d\mu^n \right) \\
&= \sum_{i=1}^n \left(\left(\sum_{k=0}^{l(\gamma_i)-m} f(\sigma^k(\underline{x})) \mu^n[\gamma_i] \right) \pm \mu^n[\gamma_i] \cdot \sum_{k=m}^{\infty} \text{var}_k(f) \right) \\
&= \frac{1}{q_n} \left(\sum_{i=1}^n \sum_{k=0}^{l(\gamma_i)-m} \int_{\sigma^k[\gamma_i]} f d\mu \right) \pm 2\mu^n[a_1 \cdots a_m] \sum_{k=m}^{\infty} \text{var}_k(f).
\end{aligned}$$

Then, since $\sum_{k=m}^{\infty} \text{var}_k(f) < \delta$ and $\mu^n[a] = 1$, we can take limits as n tends to infinity in the above equation to get

$$\lim_{n \rightarrow \infty} \int_{\Sigma} f d\mu^n = \int_{\Sigma} f d\mu \pm 2\delta.$$

An identical argument shows that

$$\lim_{n \rightarrow \infty} \int_{\Sigma} \Delta_g d\mu^n = \int_{\Sigma} \Delta_g d\mu \pm 2\delta.$$

Now we consider the entropy.

$$\begin{aligned}
h_{\mu^n}(\sigma) &= h_{\mu^n|_{[a_1 \cdots a_m]}}(\sigma|_{[a_1 \cdots a_m]}) \text{ (since } [a_1 \cdots a_m] \text{ is a sweep out set)} \\
&= - \sum_{i=1}^n \mu^n[\gamma_i] \log(\mu^n[\gamma_i]) \\
&\quad \text{(since } \mu^n|_{[a_1 \cdots a_m]} \text{ is a Bernoulli measure on choices of } \gamma_i) \\
&= - \frac{1}{q_n} \sum_{i=1}^n \mu[\gamma_i] \log(\mu[\gamma_i]) + \log(q_n).
\end{aligned}$$

Hence, since $\lim_{n \rightarrow \infty} q_n = 1$,

$$\lim_{n \rightarrow \infty} h_{\mu^n}(\sigma) = - \sum_{i=1}^n \mu[\gamma_i] \log(\mu[\gamma_i]).$$

Now $-\sum_{i=1}^n \mu[\gamma_i] \log(\mu[\gamma_i]) < \infty$, since otherwise $\lim_{n \rightarrow \infty} h_{\mu^n}(\sigma)$ would be infinite, giving $\lim_{n \rightarrow \infty} h_{\mu_f^n}(\phi) = \infty$ and contradicting the finiteness of $h(\phi)$. We choose n and δ (and hence m) such that

$$\left| \frac{-\sum_{i=1}^n \mu[\gamma_i] \log(\mu[\gamma_i])}{\int f d\mu} + \frac{\int \Delta_g d\mu}{\int f d\mu} - \frac{h_{\mu^n}(\sigma)}{\int f d\mu^n} - \frac{\int \Delta_g d\mu^n}{\int f d\mu^n} \right| < \epsilon.$$

Now

$$- \sum_{i=1}^n \mu[\gamma_i] \log(\mu[\gamma_i]) \geq h_{\mu|_{[a_1 \cdots a_m]}}(\sigma|_{[a_1 \cdots a_m]}) = h_{\mu}(\sigma).$$

with the inequality coming since $H(\sigma, \vee_{i=0}^n \sigma^{-i} \zeta)$ decreases to $H(\sigma, \zeta)$. So we have

$$\frac{h_\mu(\sigma)}{\int f d\mu} + \frac{\int \Delta_g d\mu}{\int f d\mu} - \frac{h_{\mu^n}(\sigma)}{\int f d\mu^n} - \frac{\int \Delta_g d\mu^n}{\int f d\mu^n} < \epsilon$$

giving

$$\left(h(\phi, \mu_f) + \int g d\mu_f \right) - \left(h_{\mu_f^n}(\phi) + \int g d\mu_f^n \right) < \epsilon$$

and hence

$$V(g) - \left(h_{\mu_f^n}(\phi) + \int g d\mu_f^n \right) < 2\epsilon$$

as required. Each of our μ^n are finite measures, so we scale them to be probability measures without affecting μ_f^n . This makes each μ_f^n an element of $\mathcal{E}_{\phi, g}^p$, and completes the proof of proposition 6.1. \square

Combining proposition 6.1 with lemma 6.1 we have that

$$\begin{aligned} P_\phi(g) &= \sup\{h_{\mu_f}(\phi) + \int g d\mu_f : \mu \in \mathcal{E}_{\phi, g}^p\} \\ &= \sup\{h_\mu(\phi) + \int g d\mu : \mu \in \mathcal{E}_{\phi, g}\}, \end{aligned}$$

completing our proof of theorem 6.1

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