Faster Algorithms for Computing All Prime Implicants of a Boolean Expression

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Context

The initial goal of this project was to find faster algorithms for computing Grobner basis of a system of polynomial equations. However, as I studied more about this problem, I realized it was out of scope for this thesis. So, I found a special case of the problem to work on. That is the problem of computing all prime implicants of a boolean expression[1].

[1] Please see my final report for the precise mathematical connection

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- Computing all prime implicants of a boolean expression is an np-hard problem
- We are interested in parameterized algortihms for this problem

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- We will focus on tree-decomposition in this thesis

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- Find relationship between the two problems

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- So, parameterization can happen in two ways: input or output. We can also parameterize by multiple parameters.

Runtime

- Given an instance $(x, k) \in L$, L is FPT if there is an algorithm $\mathcal{A}$, a computable function $f : \mathbb{N} \to \mathbb{N}$, and a constant c such that given any $(x, k) \in \Sigma^* \times \mathbb{N}$, $\mathcal{A}$ can decide if $(x, k) \in L$ in time at most $f(k) \cdot |(x, k)|^c$

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- It is XP, if there is an algorithm $\mathcal{A}$ and two computable functions $f, g : \mathbb{N} \to \mathbb{N}$ such that given any $(x, k) \in \Sigma^* \times \mathbb{N}$, $\mathcal{A}$ can decide if $(x, k) \in L$ in time at most $f(k) \cdot |(x, k)|^{g(k)}$

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- checking if a subset is a vertex cover can by done in polynomial time, say $poly(|\text{input}|)$
- Then total runtime is $n^k \cdot poly(|\text{input}|)$
- If k is constant (say ≤ 10), then this is essentially a polynomial time algorithm

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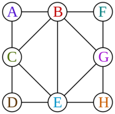
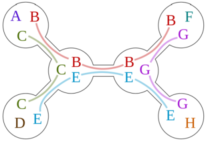
- If at any point we have more than k vertices in our output set, we back-track
- Note that the branching tree has depth at most k and each node has 2 children (whether to take that node or no). So, runtime is $2^k \cdot poly(|\text{input}|)$ for some polynomial $poly$

Tree-decomposition

- Given a graph $G = (V, E)$, a pair $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ (where T is a tree and $X_t \subseteq V(G)$ for each t) is a tree decomposition of G if
  1. $\bigcup_{t \in V(T)} X_t = V(G)$
  2. $\forall uv \in E(G)$, there is $t \in V(T)$ such that $u, v \in X_t$.
  3. For every $u \in V(G)$, the set $T_u = \{t \in V(T) \ : \ u \in X_t\}$ is a connected sub-tree of T.





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- This is capturing how closely G resembles a tree

Application of Tree-width

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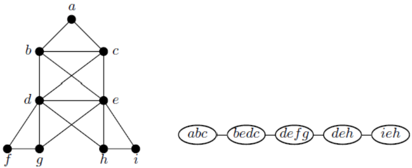
- Many np-hard problems become FPT when parameterized by tree-width
- For solving system of linear equations, the runtime improves from cubic to linear when parameterized by the tree-width of the primal graph

Path-width

- one can define path-width $pw(G)$, where instead of having a tree-decomposition, we have path-decomposition (i.e., we require T to be a path instead of a tree)



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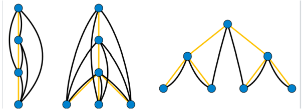
$$td(G) = \begin{cases} 1, & \text{if } |G| = 1 \\ 1 + \min_{v \in V} td(G - v), & \text{if } G \text{ is connected and } |G| > 1 \\ \max_i td(G_i), & \text{Otherwise} \end{cases}$$



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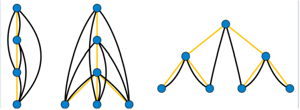
- It is well-known that $tw(G) \leq pw(G) \leq td(G) - 1 \leq tw(G) \log n - 1$ for any graph G, where n is number of vertices.



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- A conjunct c is a prime implicant of a boolean expression b, if c is an implicant of b and for every conjunct c' whose literals are also literals of c (i.e., c' is a "subset" of c), c' is not an implicant of b (i.e., c is a "minimal" implicant of b)

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- For example $a \wedge b$ is an implicant of a but not prime implicant.
- we can similarly define disjunct (disjunction of several literals) and conjunctive normal form c.n.f. (conjunction of several disjuncts)

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- This d.n.f expression is minimal in the sense that if we remove one conjunct from it or if we remove some literals from a conjunct, the expression won't be equivalent anymore
- We will parameterize the primal graph: vertices are variables and two vertices are connected if they appear in the same disjunct (for c.n.f.) or conjunct (for d.n.f.).

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- So, it can be simplified to a.

Initial Results

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- **Lemma 2:** Given a conjunct c and a c.n.f. b, it can be checked in polynomial time if c is a prime implicant of b.

Algorithm for Lemma-2

The following is an algorithm to check if conjunct c is an implicant of c.n.f. b.

Algorithm Sketch.

- Substitute truth values according to b in c

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- Use the algorithm from Lemma-1 to check if the remaining expression is true.

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Algorithm for Lemma-2 (Cont'd)

The following is an algorithm to check if conjunct c is a prime implicant of c.n.f. b.

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- for each conjunct, which is obtained by removing one literal from b, check if it is an implicant (using previous slide's algorithm)

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- check if b is an implicant. If so, output *TRUE*, otherwise *FALSE*

$\square$

Parameterized by Vertex Cover

- We present an FPT algorithm in the following slide

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- The runtime is $3^k \times poly(\text{input size})$ for some polynomial *poly*, where k is the size of the smallest vertex cover.

Parameterized by Vertex Cover (Example)

Let $b = (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_4) \wedge (\neg x_1 \vee x_2 \vee x_5)$ Note that

- $VC = \{x_1, x_2\}$ is a vertex cover
- other vertices form an independent set (follows from definition of vertex cover)
- means that each of x_3, x_4, x_5 appear alone in disjuncts.

Now suppose $VC' = \{x_1\}$, and $c = \neg x_1$ Then $b' = (x_2 \vee x_3) \wedge (\neg x_2 \vee x_4)$. Then $\neg x_1 \wedge x_3 \wedge x_4$ is an implicant as if x_3 and x_4 are true, then $b' = \mathit{true}$. Thus it is added to the set PI.

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Sketch Of Algorithm.

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- One can show that all variables of $V \setminus VC$ will appear alone in disjuncts and there should be no disjunct with variables only from $VC \setminus VC'$

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- Remove those variables from b', so $c \wedge b'$ will be an implicant

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- One can show that all variables of $V \setminus VC$ will appear alone in disjuncts and there should be no disjunct with variables only from $VC \setminus VC'$
- Remove those variables from b', so $c \wedge b'$ will be an implicant
- This gives a set of 3^k implicants. We use previous algorithm to check which of those is a prime implicant

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Parameterized by Maximum Size of a Prime Implicant

Sketch of an XP algorithm.

- For any subset of the variables of size at most k, assign all possible to truth assignment to these variables

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Parameterized by Maximum Size of a Prime Implicant

Sketch of an XP algorithm.

- For any subset of the variables of size at most k, assign all possible to truth assignment to these variables
- Use the algorithm from lemma 2 to check if this conjunct is a prime implicant

$\square$

Parameterized by Maximum Size of a Prime Implicant

Sketch of an XP algorithm.

- For any subset of the variables of size at most k, assign all possible to truth assignment to these variables
- Use the algorithm from lemma 2 to check if this conjunct is a prime implicant
- Overall runtime is $\mathcal{O}(n^k \cdot 2^k \cdot poly(n, k))$ where $poly(n, k)$ is some polynomial over n and k, where k is the Maximum Size of a Prime Implicant

$\square$

Parameterized by Number of Disjuncts

- XP algorithm (Lemma 4)

Parameterized by Number of Disjuncts

- XP algorithm (Lemma 4)
- We can show that this is a special case of the previous parameterization

Proof Sketch.

Suppose k is the number of disjuncts. We can show that number of literals in a prime implicant is bounded by k. Any prime implicant intersects with each disjunct in at least one literal and it cannot contain anything extra. $\square$

Hardness Result: Tree-depth and Max Degree

Lemma 5: Number of prime implicants can be exponential even when tree-depth or max degree is bounded.

Proof Sketch.

Consider the example: $b = (x_1 \vee x_2) \wedge (x_3 \vee x_4) \wedge \cdots \wedge (x_{n-1} \vee x_n)$ Then b has $2^{\frac{n}{2}}$ prime implicants. Note that tree-depth is 2 and max degree, tree-width, path-width are 1. $\square$

Hardness Result: Parameterized by number of disjuncts

Lemma 6: The problem is XP or worse

Proof Sketch.

Let $b = (x_1 \vee x_2 \vee \cdots \vee x_{\frac{n}{2}}) \wedge (x_{\frac{n}{2}+1} \vee \cdots \vee x_n)$ It has $\left(\frac{n}{2}\right)^2$ prime implicants. Replace 2 with K to get $\Omega(n^k)$ prime implicants. $\square$

Hardness Result: Parameterized by Vertex Cover for d.n.f. input

Lemma 7: The problem is XP or worse

Proof Sketch.

We can construct an example[a] where number of prime implicants is $\left(\frac{n}{k}\right)^k$, where k is the size of a minimum vertex cover. $\square$

[a] please see my final report

Table of Contents

Summary

the results obtained in this thesis can be grouped into three main categories

- novel parameterized algorithms for computing all prime implicants of a given c.n.f. boolean expression.
    - we developed an FPT algorithm parameterized by the vertex cover of the primal graph
    - an XP algorithm parameterized by the maximum size of a prime implicant
    - an XP algorithm parameterized by the number of disjuncts

Summary

the results obtained in this thesis can be grouped into three main categories

- novel parameterized algorithms for computing all prime implicants of a given c.n.f. boolean expression.
  - we developed an FPT algorithm parameterized by the vertex cover of the primal graph
  - an XP algorithm parameterized by the maximum size of a prime implicant
  - an XP algorithm parameterized by the number of disjuncts
- we established hardness results for several other parameters for the same problem.
  - NP-hard even when one of the parameters tree-width, tree-depth, path-width, and degree is bounded
  - XP or worse when parameterized by number of disjuncts
  - XP or worse when the smallest vertex cover Size is bounded

Future Ideas

- Computing all prime implicants for a given c.n.f. in time that is a linear function of input + output size.

Q&A