Lattices for Space/Belief and Extrusion/Utterance

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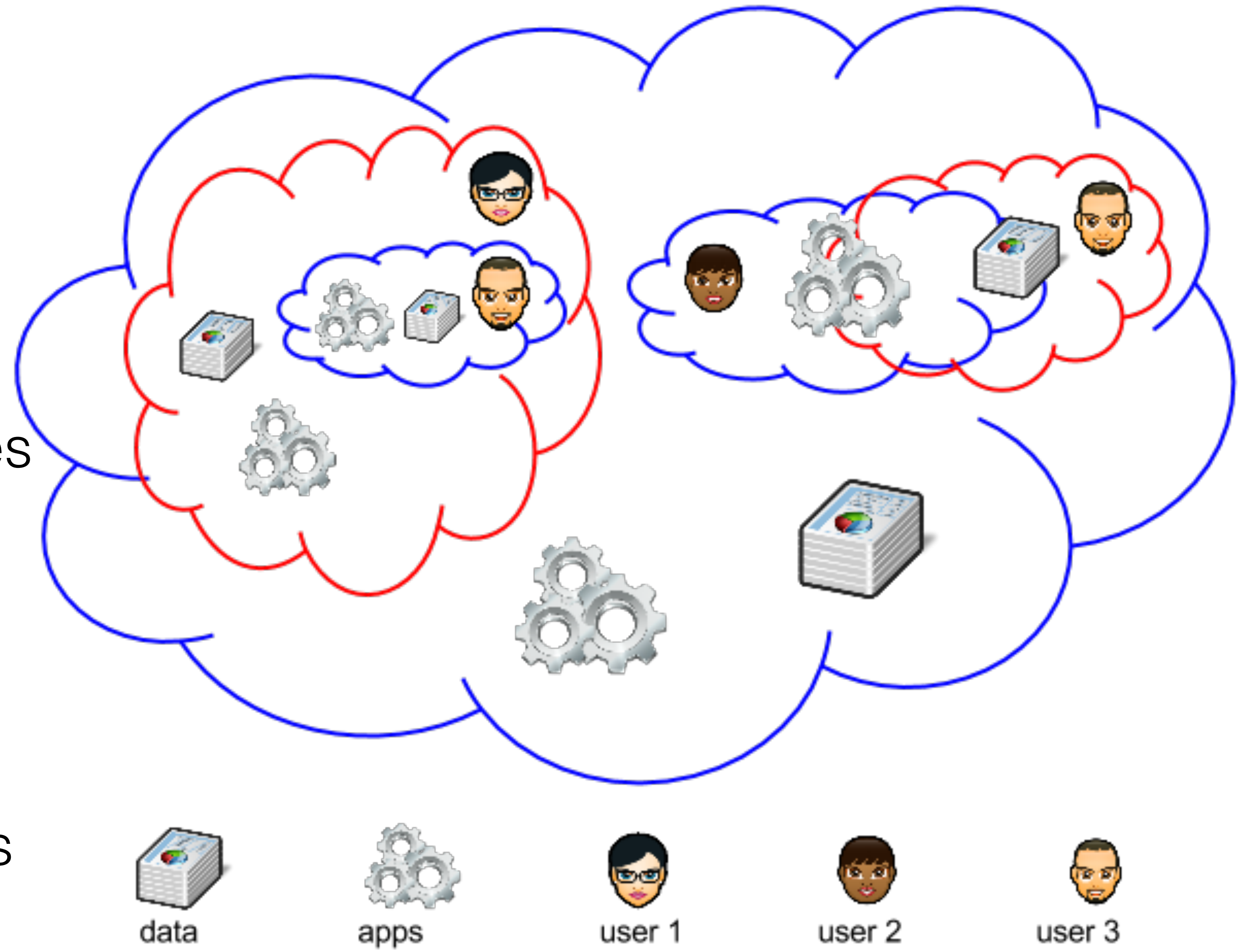
Context: Spatial Multi-Agent Systems

- An agent's <u>space</u> has:
  - Information
  - Processes
  - Other agents' spaces
- Agents are able to:
  - Run apps
  - Move data and apps
  - Share opinions and/or lies



Goal: A sound mathematical model for mobility

$$[c \sqcup \uparrow_i([d]_j)]_i \sqcup [e]_j = [c]_i \sqcup [d \sqcup e]_j$$

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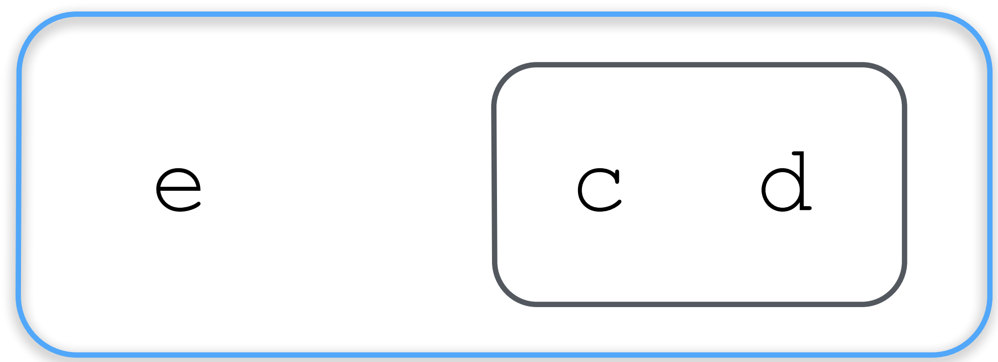
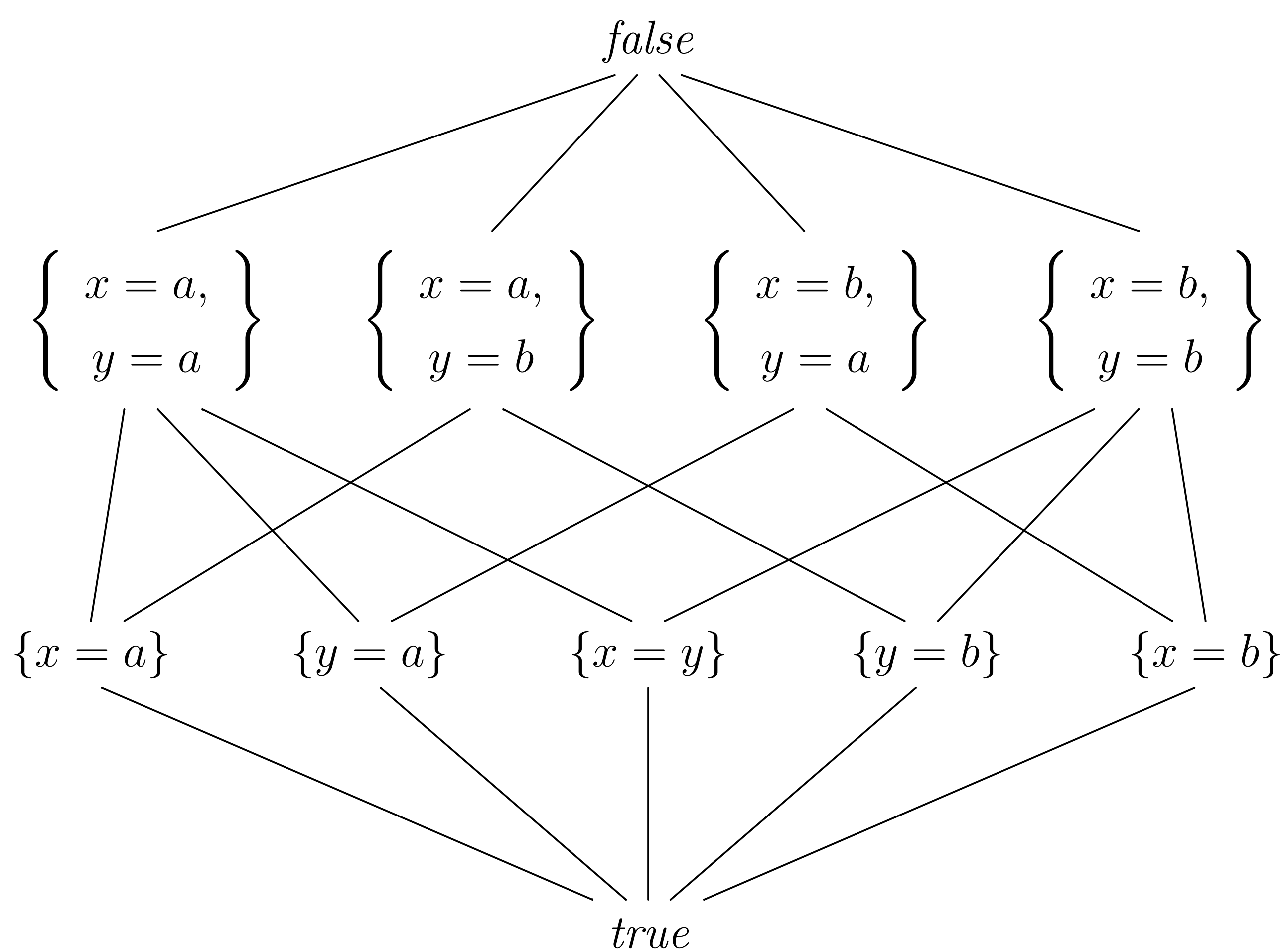


`c?`

$$[e \sqcup [c \ \sqcup (c \rightarrow \uparrow_j d)]_j]_i = [e \sqcup d \sqcup [c]_j]_i$$

A bit of background: Lattices



Complete Algebraic Lattice

$$\mathbf{C} = (Con, \sqsubseteq, \sqcup, \text{false}, \text{true})$$

A bit of background: Modal Logics

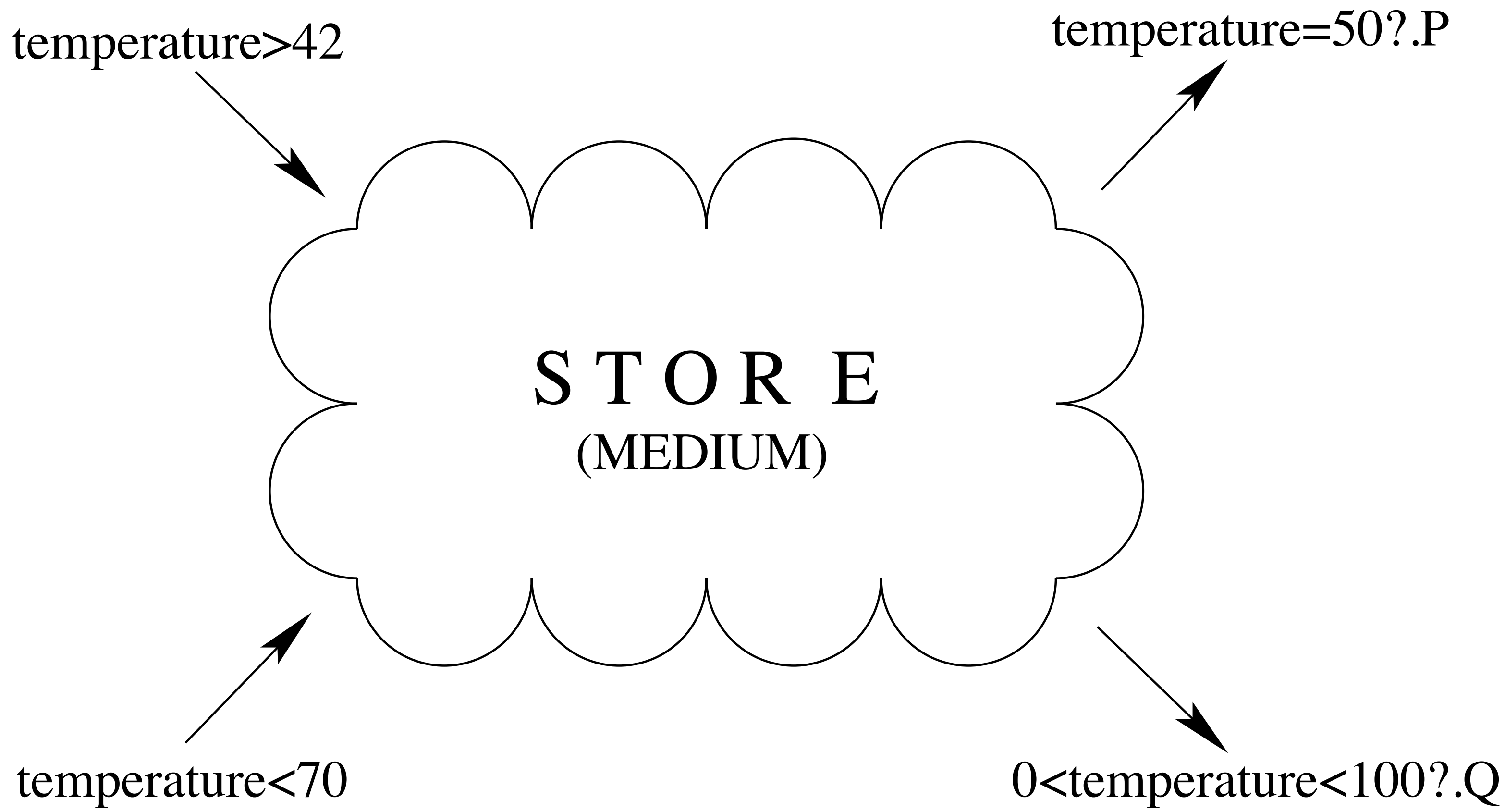
$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_i \varphi$$

$B_i(\varphi)$: Agent i believes φ $\qquad \neg B_i(\text{false})$

$K_i(\varphi)$: Agent i knows φ $\qquad K_i(\varphi) \Rightarrow \varphi$

A bit of background: CCP

$$P ::= \mathbf{tell}(c) \mid \mathbf{ask}(c)\ \mathbf{do}\ P \mid P \parallel Q \mid \mu X.P$$



Processes, constraints and formulas

Process Calculus	Lattice of Information	Modal Logic
tell(c)	c	p
P || Q	$c \sqcup d$	$\varphi \wedge \varphi$
skip	$\bot$	true
abort	$\top$	false
⋮	⋮	⋮
[	[c]	B
[	?	?

This Talk

- **Contribution**: Algebraic structure to formalize space and extrusion operations for multi-agent systems
- **Roadmap**:
  1. ~~Motivation & Background~~
  2. Spatial Constraint Systems
  3. The Extrusion Problem
  4. Properties of Extrusion
  5. An application as a bimodal logic

Spatial CS

A **space function** of an agent $[\cdot]_i$ is a self-map on the elements of the lattice $(\mathrm{Con}, \sqsubseteq)$.

- $[c]_i$ is interpreted as **locality** (information c is present in agent **i**'s space)
- $[[c]_i]_j$ is viewed as **nesting** of spaces (Agent **i**'s space inside agent **j**'s space)
- **Joining** space functions is possible (i.e $[c]_i \sqcup [d]_j$)

Spatial CS

A **spatial constraint system (scs)** is a cs equipped with n spatial functions.

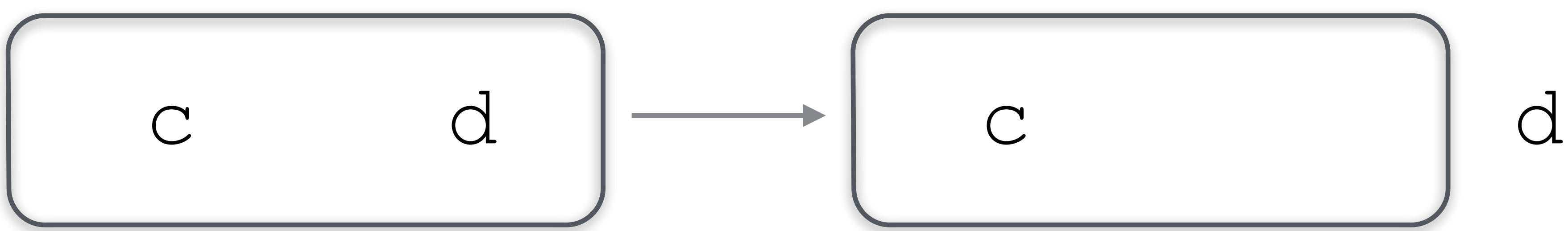
- The spatial functions must fulfill:
  - **S.1** $[\mathrm{true}]_i = \mathrm{true}$ (emptiness)
  - **S.2** $[c]_i \sqcup [d]_i = [c \sqcup d]_i$ (distribution)
- Distribution yields the next property:
  - **S.3** if $c \sqsubseteq d$ then $[c]_i \sqsubseteq [d]_i$ (monotonicity)

Spatial CS

- SCS's allow the following:
  - $[\text{false}]_i \neq \text{false}$
    (Inconsistency Confinement)
  - $[c]_i \sqcup [d]_j \neq \text{false}$ even if $c \sqcup d = \text{false}$
    (Freedom of opinion)
  - $[\text{red}]_i = [\text{green}]_i$ still when $\text{red} \neq \text{green}$
    (Information Blindness)

Spatial CS with Extrusion

- **Extrusion**: Movement of information from one space to the outside with no side effect.



- We call the epistemic interpretation of this action **Utterance**

Spatial CS with Extrusion

An **extrusion function** $\uparrow_i$ is a self-map on the elements of (Con, $\sqsubseteq$), where:
E.1 $[\uparrow_i c]_i = c$

- We then define the extrusion as the right inverse of a space. From **S.2** and **E.1**: $[d \sqcup \uparrow_i c]_i = [d]_i \sqcup c$.

A **scs with extrusion (scs-e)** is an scs equipped with n extrusion functions satisfying **E.1**

Derived Constructions in scs-e

- $c \rightarrow d$ is defined as $\sqcap\{ e \mid e \sqcup c \sqsupseteq d \}$
(Heyting **implication**)
  - This implication satisfies **modus ponens**:
  $c \sqcup (c \rightarrow d) \sqsupseteq d$
- $\neg c$ is defined as $c \rightarrow$ false
(**complement**)

Communication Example

- Guarded extrusion of inconsistency (w.r.t agent **i**'s information):

  - Suppose: $[c \sqcup c \rightarrow (\uparrow_{\mathbf{i}} [d]_{\mathbf{j}})]_{\mathbf{i}} \sqcup [e]_{\mathbf{j}}$

  - Equivalent to: $[c]_{\mathbf{i}} \sqcup [d \sqcup e]_{\mathbf{j}}$

Information d is now in the space of agent **j**

This Talk

- Contribution: Algebraic structure to formalize space and extrusion operations for multi-agent systems

- **Roadmap**:

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  2. ~~Spatial Constraint Systems~~

  3. The Extrusion Problem

  4. Properties of Extrusion

  5. An application as a bimodal logic

Finding an extrusion operator

Problem: Given a scs (Con, $\sqsubseteq$, $[\cdot]_i$, …, $[\cdot]_n$) find maps $\uparrow_i$, …, $\uparrow_n$ that satisfy **E.1** $[\uparrow_i c]_i = c$

$[\cdot]_i$ is surjective iff $\uparrow_i$ exists (Axiom of Choice)

Theorem: If $[\cdot]_i$ is a **surjective** and **continuous** function then the map $\uparrow_i c = \sqcup\{d \mid [c]_i^{-1}\}$ satisfies **E.1.**

Max. extrusion

Distributed Extrusion

What about S.1 and S.2 counterparts? We say that the extrusion distributes globally/objectively if:

E.2 $\uparrow_i(\text{true}) = \text{true}$

E.3 $\uparrow_i(c \sqcup d) = \uparrow_i(c) \sqcup \uparrow_i(d)$

Theorem: If $[\![\cdot]\!]_i$ is **surjective** and **meet-complete** then $\uparrow_i c = \sqcap\{d \mid [\![d]\!]_i \sqsupseteq c\}$ satisfies **E.1 - E.3**

Result: Not always the right inverse of a surjective function (even for continuous maps)

Min. extrusion

Sp SCS-E CS

- ~~SCS's~~ SCS-E's allow the following:
  - ~~$[\text{false}]_i \neq \text{false}$~~
    ~~(Inconsistency Confinement)~~

    $[\text{false}]_i = \text{false}$
    Inconsistencies propagate

  - $[c]_i \sqcup [d]_j \neq \text{false}$ even if $c \sqcup d = \text{false}$ ✓
    (Freedom of opinion)

  - $[\text{red}]_i = [\text{green}]_i$ still when $\text{red} \neq \text{green}$ ✓
    (Information Blindness)

    $[\cdot]_i$ is <u>not necessarily</u> injective

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Properties of Extrusion

- $\uparrow_i$ is distributed then $\boxed{\uparrow_i \text{ is an order-embedding}}$
- $[\cdot]_i$ is injective then $\boxed{[\cdot]_i \text{ is an order-embedding}}$
- $\uparrow_i$ is defined as min. extrusion then $\boxed{\text{The pair } (\uparrow_i, [\cdot]_i) \text{ is a Galois connection}}$

Application: Bimodal Logic with Utterance/Belief

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid B_i\varphi \mid U_i\varphi$$

right inverse of belief

$$\mathcal{O}_i(\varphi) \stackrel{\mathtt{def}}{=} B_i(\varphi \wedge U_i(\varphi))$$

$$\mathcal{H}_i(\varphi) \stackrel{\mathtt{def}}{=} B_i(\neg\varphi \wedge U_i(\varphi))$$

These equivalences hold:

$$\mathcal{O}_i(\varphi) \Leftrightarrow (B_i\varphi) \wedge \varphi$$

$$\mathcal{H}_i(\varphi) \Leftrightarrow (B_i\neg\varphi) \wedge \varphi$$

Application: Bimodal Logic with Utterance/Belief

$$(\mathcal{P}(\Delta), \supseteq, [\cdot]_1, \ldots, [\cdot]_n, \uparrow_1, \ldots, \uparrow_n) \text{ where :}$$

Δ : all pairs (M, s) where s is a state of $M \in \mathcal{M}_{ltu}$

$\sqcup : \cap$

$[c]_i = \{(M, s) \in \Delta \mid \forall_t \text{ if } s \rightarrow_i t \text{ then } (M, t) \in c\}$

$\uparrow_i$: max. extrusion

- Left Unique
- Left Total

Summarizing…

- We proposed a **lattice structure** to represent information **locality** and **extrusion** in a distributed multi-agent system.

- The extrusion is seen as the **right-inverse** operation of locality. We studied different constructions and their properties.

- The structure can be used to derive a **bimodal logic** with **belief** and **utterance** behaviors.