Diameter of polytopes and Hirsch conjecture

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January 14^{th}, 2016

Polytopes

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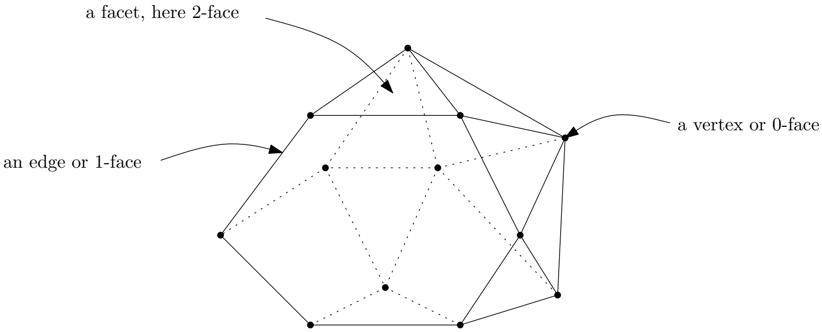

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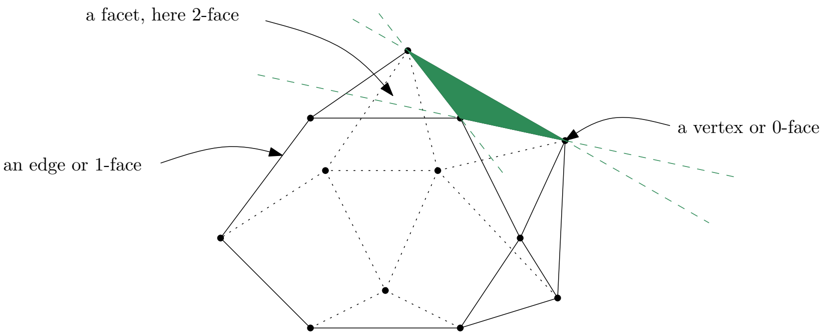

$P \subseteq \mathbb{R}^d$ is a **polytope** if $P = \bigcap_{\mathcal{H} \in \mathbb{H}} \mathcal{H}^+$ and P is bounded.

Polytopes

mathematically: $\mathrm{conv}(S) \iff \bigcap_{\mathcal{H}\in\mathbb{H}} \mathcal{H}^+ +$ bounded.

BUT

algorithmically: $\mathrm{conv}(S) \longleftrightarrow \bigcap_{\mathcal{H}\in\mathbb{H}} \mathcal{H}^+$ is *NP*-hard.

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Prop. The intersection of two polytopes is a polytope.

Prop. The projection of a polytope is a polytope.

Linear Programming

$$\max_{A.x \leq b} \quad c.x \; = \; ??$$

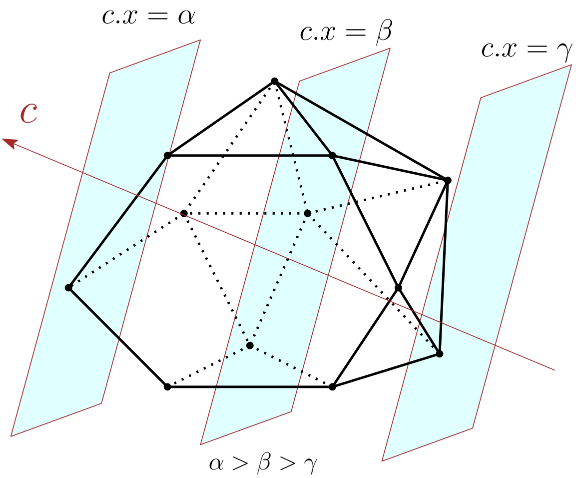
economics, physics, flow problems...

Linear Programming

$$\max_{A.x \leq b} c.x = \max_{x \in P} c.x$$

where $\mathcal{H}_i^+ : \sum_{1 \leq j \leq n} a_{i,j} x_j \leq b_i$ and $P = \bigcap_{i \in [m]} \mathcal{H}_i^+$

Linear Programming



$\Rightarrow$ test all vertices!

Solving linear Programming

- $\diamondsuit$ ellipsoid method: polynomial but inefficient.
- $\diamondsuit$ simplex algorithm: not polynomial but efficient in practice.

The simplex algorithm

Pick a vertex and move to a better neighbor with a simple **pivot rule**.

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A problem of the century: find a polynomial version of the simplex algorithm.

Strategy:

- keep pivot rules simple;
- achieve paths of polynomial length.

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Conjecture (Hirsch)

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WRONG

Santos

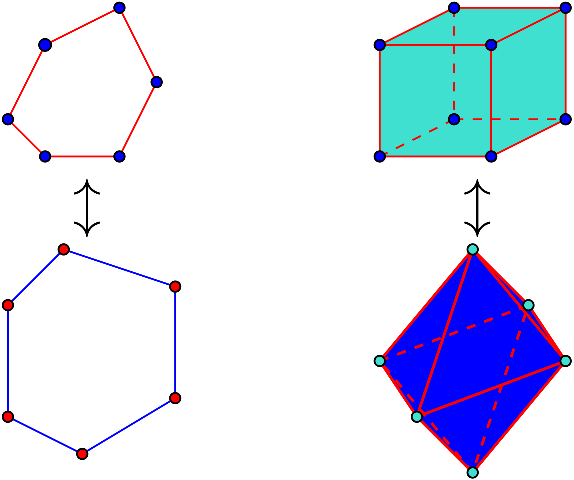
A counterexample in dimension 43.

Conjecture (Polynomial Hirsch)

The diameter of $\mathrm{conv}(v_1, \ldots, v_n) \in \mathbb{R}^d$ *is smaller than* $f(n, d)$.

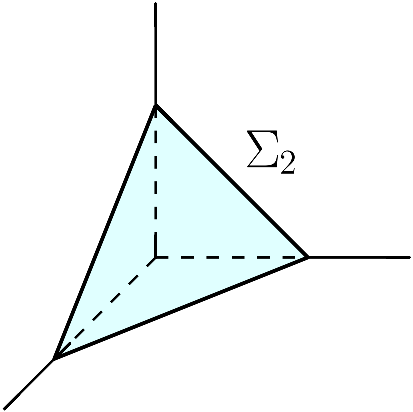
Polytopes

duality on polytopes: reversing the inclusion order on faces.



Geometry

Geometrical **simplex** of dimension n: $\Sigma_n = \mathrm{conv}\{e_i\}_{i \in [n+1]}$



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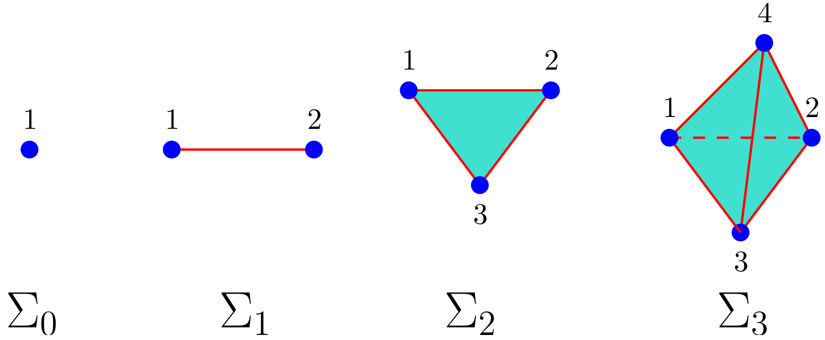

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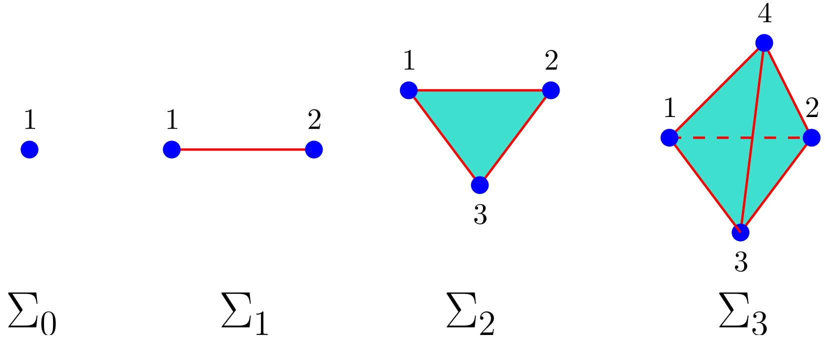

$\forall I \subseteq [n+1], \mathrm{conv}\{e_i\}_{i \in I}$ is a face of Σ_n.
Simplexes: only polytopes with this property.

Simplicial complexes

Definition

$\Delta \subseteq \mathcal{P}(S)$ is a **simplicial complex** if $\sigma' \subseteq \sigma \in \Delta \Longrightarrow \sigma' \in \Delta$.
$\sigma \in \Delta$: **simplex** or **face** of Δ.

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Example $S = \{1, 2, 3, 4\}$

$$\Delta = \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{3,4\},\{1,3,4\}\}$$

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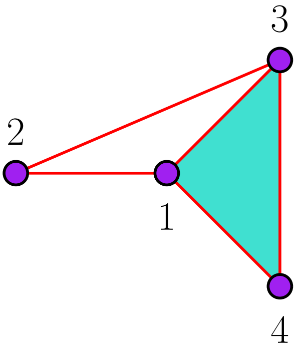
$$\Delta = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{3,4\}, \{1,3,4\}\}$$

geometrical representation:



THANK YOU FOR YOUR CAREFUL ATTENTION!