

# DERIVING MOMENTS IN THE AGE OF GOSSIP PROCESS FROM PERCOLATION

Thomas Jacob Maranzatto and Sennur Ulukus

Department of Electrical and Computer Engineering, University of Maryland, College Park, MD

## ABSTRACT

This paper concerns fundamental identities in the study of age of information (AoI) in gossip networks. We recover known recursive identities for arbitrary  $k$ th moments of the age process based on the recent connection between AoI and first passage percolation. Apart from the connection to percolation, our proofs are more concise and can be followed using only elementary facts from probability. Our argument generalizes some techniques known in the statistical physics community, and we remark on connections to the Eden model.

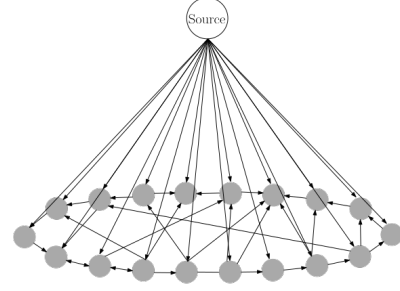
**Index Terms**— Gossip networks, age of information, first passage percolation.

## 1. INTRODUCTION

Given a network of interacting users, a fundamental question is to quantify the freshness of their stored data. This paper considers the *age of information* (AoI) metric over systems of gossiping users, where users send packets to neighbors randomly. The AoI metric was first introduced by Kaul, Yates, and Gruteser [1], and we direct the reader to the survey [2] for background on related metrics over a wide class of communication and queuing models.

Our focus is on a gossiping model introduced by Yates [3]. Here, a directed network of  $N$  users  $G = (\mathcal{N}, E)$  communicate in a decentralized manner by sending packets to their immediate neighbors. Without loss of generality, take  $\mathcal{N} = [0, \dots, N]$ , and distinguish the node 0 (sometimes referred to as node  $n_0$ ) as the source so only directed edges of the type  $(0, u)$  exist, never  $(u, 0)$ . The source acts as the *state of the world*, and does not control information flow or packets sent in the gossip network. If  $(u, v) \in E$ , then  $u$  sends packets to  $v$  via a Poisson process with rate  $\lambda_{uv}$ , independent of all other processes in the network. For every time  $t \geq 0$ , each node  $u$  stores a real number  $N_u(t)$  denoting the timestamp of its current packet. We initialize  $N_u(0) = 0$  for all  $u$ , and for all  $t \geq 0$  the source maintains  $N_0(t) = t$ . When the Poisson clock  $(u, v)$  rings at time  $t$ , the counter of  $v$  updates as  $N_v(t) = \max\{N_v(t), N_u(t)\}$ . The *age of information* (AoI) of  $v$  is defined as  $X_v(t) := t - N_v(t)$ . It is helpful to think of the AoI as the time elapsed since the packet stored at node  $v$  was generated by the source.

This model and variants have been studied extensively [3–



**Fig. 1.** A typical gossip network we consider. In general, the source does not have to update all vertices with the same rate.

[10], and we point the reader to a survey for more context [6]. Majority of the work in the area has used the stochastic hybrid system (SHS) framework, and the most well-known example of this is Yates' recursive formula for the long-term expected age of a subset of vertices [3, 10] which we briefly recall. Define the total rate of  $u$  into a subset of vertices  $S$  as

$$\lambda_u(S) = \begin{cases} \sum_{v \in S} \lambda_{uv}, & u \notin S, \\ 0, & u \in S, \end{cases} \quad (1)$$

and let  $v_S = \lim_{t \rightarrow \infty} \mathbb{E}[\min_{u \in S} N_u(t)]$  be the long-term expected minimum age of any vertex in  $S$ . Then, for any subset  $S$ , Yates showed,

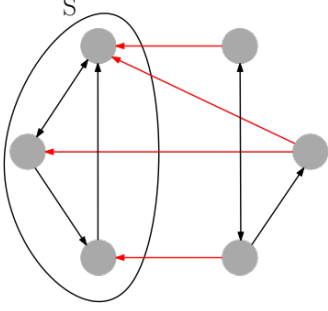
$$v_S = \frac{1 + \sum_{u \notin S} \lambda_u(S) v_{S \cup \{u\}}}{\lambda_0(S) + \sum_{u \notin S} \lambda_u(S)}. \quad (2)$$

Similar recursive identities for the  $k$ th moments were recovered by Abd-Elmagid and Dhillon [11]. Letting  $v_S^k = \lim_{t \rightarrow \infty} \mathbb{E}[(\min_{u \in S} N_u(t))^k]$  be the long-term average  $k$ th moment of the age process, they proved,

$$v_S^k = \frac{k v_S^{k-1} + \sum_{u \notin S} \lambda_u(S) v_{S \cup \{u\}}^k}{\lambda_0(S) + \sum_{u \notin S} \lambda_u(S)}. \quad (3)$$

Our contribution is to reprove these fundamental identities without the use of the SHS framework. The authors of [11] proved many other identities for the moment generating function of the age process, and generalized to joint moment identities. We comment on how our approach applies to the joint moment recursion in the conclusion. We also remark that (2)





**Fig. 3.** In the proofs of Theorems 2 and 3, the random variable  $Y$  is one of those edges in  $E(S)$ , highlighted in red above. The black edges are not included in  $E(S)$ . We have not included the source vertex  $n_0$ , but implicitly, the edges from  $n_0$  to the network are still present, though not in  $E(S)$ .

invariance to hold. By the discussion above, one can equivalently view  $X_S$  as the first passage time from  $n_0$  to  $S$  in the dual graph  $G'$ , so that  $X_S \stackrel{D}{=} T_{G'}(n_0, S)$ . In what follows, for any set of vertices  $S$ , we use  $E(S)$  to denote the in-edge boundary of  $S$ , minus those edges adjacent to  $n_0$ ; since  $G$  is directed, these directed edges originate outside  $S$  and terminate in  $S$ ; see Fig. 3 for an example. We implicitly assume  $n_0 \notin S$ . For any edge  $e = (u, v)$ , we will use the notation  $S \cup e$  to mean  $S \cup \{u, v\}$ . Using this notation, the presentation of the main theorems are slightly different from (2), (3), but will make our algebraic manipulations more convenient. To see the equivalence, notice that for any function  $f : 2^N \rightarrow \mathbb{R}$ , we can re-index the sum and combine like terms to obtain,

$$\sum_{e \in E(S)} \lambda_e f(S \cup e) = \sum_{i \in N(S)} \lambda_i(S) f(S \cup i). \quad (5)$$

Applying this change of variable recovers the original form found in [3, 10, 11].

We choose to present our results in two parts: Theorem 2 gives a new proof for Yates' recursion of the first moment, and Theorem 3 gives the higher moments in [11]. Our reasoning is that, Yates' recursion is much more prominent in the literature, and we feel the standalone proof is pedagogically useful, though Theorem 3 encompasses all cases and is our main contribution in this paper. We also believe this first moment proof is useful as it avoids much of the tedious algebra in the sequel. The following argument is inspired by the classical works of Dhar [13, 14] on the Eden model.

**Theorem 2 ([3], Theorem 1)** *For any subset  $S \subset G$  and large enough time  $t$ , the AoI of  $S$  satisfies,*

$$\mathbb{E}[X_S] = \frac{1 + \sum_{e \in E(S)} \lambda_e \mathbb{E}[X_{S \cup e}]}{\lambda_0(S) + \sum_{e \in E(S)} \lambda_e}. \quad (6)$$

**Proof:** Choose  $t > t_0$  as in Theorem 1. For any subset  $S$ , let  $Y$  be the random variable indexing the next edge that rings in  $E(S)$ ; the notation  $\{Y = e\}$  then denotes the event that edge  $e$  is the first edge to ring among all boundary edges of  $S$ . Let  $D = \lambda_0(S) + \sum_{e \in E(S)} \lambda_e$  denote the total update rate into

the set  $S$ . We need the following fact, which is a restatement of the well-known formula for the index of the minimum of exponential random variables,

$$\mathbb{P}[Y = e] = \frac{\lambda_e}{D}. \quad (7)$$

Furthermore, by the Markov property and the equivalence  $X_S = T_{G'}(n_0, S)$  as distributions, we have,

$$\mathbb{E}[X_S | Y = e] = \begin{cases} \mathbb{E}[\text{Exp}(D)], & n_0 \in e, \\ \mathbb{E}[X_{S \cup e} + \text{Exp}(D)], & n_0 \notin e. \end{cases} \quad (8)$$

By the law of total expectation, we have,

$$\mathbb{E}[X_S] = \mathbb{E}[\mathbb{E}[X_S | Y]] \quad (9)$$

$$= \sum_{e \in E(S)} \mathbb{E}[X_S | Y = e] \mathbb{P}[Y = e] \quad (10)$$

$$= \frac{1}{D} \left( \sum_{\substack{e \in E(S) \\ n_0 \in e}} \lambda_e \mathbb{E}[\text{Exp}(D)] + \sum_{\substack{e \in E(S) \\ n_0 \notin e}} \lambda_e \mathbb{E}[X_{S \cup e} + \text{Exp}(D)] \right) \quad (11)$$

$$= \frac{1}{D} \left( \frac{\lambda_0(S)}{D} + \sum_{\substack{e \in E(S) \\ n_0 \notin e}} \lambda_e \mathbb{E}[X_{S \cup e}] + \frac{\lambda_e}{D} \right) \quad (12)$$

$$= \frac{1}{D} \left( 1 + \sum_{\substack{e \in E(S) \\ n_0 \notin e}} \lambda_e \mathbb{E}[X_{S \cup e}] \right), \quad (13)$$

where (11) is obtained by applying (8), and (12) is obtained by expanding the expectations. Then, (13) follows by combining terms as  $\lambda_0/D + \sum \lambda_e/D = 1$ . ■

The proof technique for an arbitrary  $k$ th moment is essentially the same. We will again apply the law of total expectation conditioned with respect to  $Y$ . The difference is there are now terms of the form  $\mathbb{E}[(X + \text{Exp}(D))^k]$  that we have to deal with. By independence, we can expand this expectation using the binomial theorem and the rest of the proof is rearranging and combining the resulting terms.

**Theorem 3 ([11] Theorem 1)** *For any subset  $S \subset G$  and large enough time  $t$ , the  $k$ th AoI moment of  $S$  satisfies,*

$$\mathbb{E}[X_S^k] = \frac{k \mathbb{E}[X_S^{k-1}] + \sum_{e \in E(S)} \lambda_e \mathbb{E}[X_{S \cup e}^k]}{\lambda_0(S) + \sum_{e \in E(S)} \lambda_e}. \quad (14)$$

**Proof:** Define  $t, Y, D$  as above. Before starting, we need the following generalization of (8),

$$\mathbb{E}[X_S^k | Y = e] = \begin{cases} \mathbb{E}[\text{Exp}(D)^k], & n_0 \in e, \\ \mathbb{E}[(X_{S \cup e} + \text{Exp}(D))^k], & n_0 \notin e, \end{cases} \quad (15)$$

which follows by considering the conditional distribution of  $X_S^k$  given  $\{Y = e\}$  for the two classes of  $e$ . Furthermore, the exponential is independent from  $X_{S \cup e}$  by the Markov property. Again, by the law of total expectation,

$$\mathbb{E}[X_S^k] = \mathbb{E}[\mathbb{E}[X_S^k | Y]] \quad (16)$$

$$= \sum_{e \in E(S)} \mathbb{E}[X_S^k | Y = e] \mathbb{P}[Y = e] \quad (17)$$

$$= \frac{1}{D} \left( \sum_{\substack{e \in E(S) \\ n_0 \notin e}} \lambda_e \mathbb{E}[\text{Exp}(D)^k] + \sum_{\substack{e \in E(S) \\ n_0 \in e}} \lambda_e \mathbb{E}[(X_{S \cup e} + \text{Exp}(D))^k] \right). \quad (18)$$

Since  $X_{S \cup e}$  and the exponential are independent, we can expand the right-hand term, as

$$\begin{aligned} & \frac{1}{D} \left( \sum_{\substack{e \in E(S) \\ n_0 \notin e}} \lambda_e \mathbb{E}[\text{Exp}(D)^k] + \sum_{\substack{e \in E(S) \\ n_0 \in e}} \lambda_e \sum_{j=0}^k \binom{k}{j} \mathbb{E}[\text{Exp}(D)^j] \mathbb{E}[X_{S \cup e}^{k-j}] \right) \\ &= \frac{1}{D} \left( \lambda_0(S) \mathbb{E}[\text{Exp}(D)^k] + \sum_{\substack{e \in E(S) \\ n_0 \notin e}} \lambda_e \sum_{j=1}^k \binom{k}{j} \mathbb{E}[\text{Exp}(D)^j] \mathbb{E}[X_{S \cup e}^{k-j}] \right) \\ & \quad + \frac{1}{D} \left( \sum_{\substack{e \in E(S) \\ n_0 \in e}} \lambda_e \mathbb{E}[X_{S \cup e}^k] \right), \end{aligned} \quad (19)$$

where in the second step we combined the  $n_0 \in e$  terms and split up the inner sum according to the  $j = 0$  term. We now demonstrate the first term in (19) is actually  $\frac{k}{D} \mathbb{E}[X_S^{k-1}]$ , which will complete the proof. To that end, we write,

$$\begin{aligned} & \lambda_0(S) \mathbb{E}[\text{Exp}(D)^k] \\ & + \sum_{\substack{e \in E(S) \\ n_0 \notin e}} \lambda_e \sum_{j=1}^k \binom{k}{j} \mathbb{E}[\text{Exp}(D)^j] \mathbb{E}[X_{S \cup e}^{k-j}] \\ &= \frac{k}{D} \lambda_0(S) \mathbb{E}[\text{Exp}(D)^{k-1}] \end{aligned}$$

$$+ \sum_{\substack{e \in E(S) \\ n_0 \notin e}} \lambda_e \sum_{j=0}^{k-1} \binom{k}{j+1} \mathbb{E}[\text{Exp}(D)^{j+1}] \mathbb{E}[X_{S \cup e}^{k-j-1}] \quad (20)$$

$$= \frac{k}{D} \lambda_0(S) \mathbb{E}[\text{Exp}(D)^{k-1}] + \frac{k}{D} \sum_{\substack{e \in E(S) \\ n_0 \notin e}} \lambda_e \sum_{j=0}^{k-1} \binom{k}{j} \mathbb{E}[\text{Exp}(D)^j] \mathbb{E}[X_{S \cup e}^{k-j-1}] \quad (21)$$

$$= \frac{k}{D} \left( \sum_{\substack{e \in E(S) \\ n_0 \in e}} \lambda_e \mathbb{E}[\text{Exp}(D)^{k-1}] + \sum_{\substack{e \in E(S) \\ n_0 \notin e}} \lambda_e \mathbb{E}[(X_{S \cup e} + \text{Exp}(D))^{k-1}] \right) \quad (22)$$

$$= k \left( \sum_{e \in E(S)} \mathbb{E}[X_S^{k-1} | Y = e] \mathbb{P}[Y = e] \right) \quad (23)$$

$$= k \mathbb{E}[X_S^{k-1}], \quad (24)$$

where in (20) we expand  $\mathbb{E}[\text{Exp}(D)^k] = \frac{k!}{D^k}$  in the first term, and re-index the sum to start from  $j = 0$  instead of  $j = 1$ , in (21) we expand  $\mathbb{E}[\text{Exp}(D)^j]$  inside the sum, pull the  $k$  out of the sum, and absorb the  $j + 1$  in the binomial term, in (22) we apply the binomial theorem since the random variables are independent, in (23) we apply (15), and in (24) we apply the law of total expectation. ■

### 3. CONCLUSION AND REMARKS

We have focused on reproving some identities that have a large influence on the research directions for age processes in networks, we have not made an effort to re-prove all AoI identities that exist in the literature. Notable identities we have excluded appear in Abd-Elmagid and Dhillon [11]. Those authors proved joint moment recursions of the form  $\mathbb{E}[v_S^k v_U^j]$ . We believe that we should be able to recover these identities using similar techniques presented here, but the algebra involved is tedious and the pedagogical value of doing so is limited—we leave this as an open problem.

Our proofs crucially rely on the first passage percolation interpretation of the AoI in gossip networks. As discussed in [12], Poissonian gossip networks with the standard AoI metric are very special, in that the first passage percolation formulation *exactly* recovers the distribution of the age process. Auffinger and Tang [16] managed to push the second moment analysis (a direct inspiration for our proof of the  $k$ th moments) to random variables which look sufficiently close to an exponential distribution near 0. We leave as an open problem the question of whether a similar technique can be applied to Theorem 3 in gossip networks with renewal processes on edges that are sufficiently close to Poissonian.

Finally, we note that while inequality variants of (2) have appeared in the statistical physics literature, we were unable to find reference to the full recursive equality. This is only an extra step of algebra, and the statement makes intuitive sense since the Markov property holds for  $\text{Exp}(1)$  random variables. Letting  $\mathbb{Z}_\ell^d = \{x \in \mathbb{Z}^d : \|x\|_1 \leq \ell\}$  be the  $d$ -dimensional box with side length  $\ell$ , it is not hard to show that the *time constant* for  $\mathbb{Z}^d$  with *i.i.d.*  $\text{Exp}(1)$  edge weights is,

$$\mu(e_1) = \liminf_{\ell \rightarrow \infty} v_{\ell, \{0\}}, \quad (25)$$

with  $v_{\ell, S}$  the recursion over the box  $\mathbb{Z}_\ell^d$  defined as,

$$v_{\ell, S} = \frac{1 + \sum_{e \in E(S)} v_{\ell, S \cup e}}{|E(S)|}. \quad (26)$$

Then, each term in the  $\liminf$  is in principle computable in finite time, and  $\mu(e_1)$  can be approximated to arbitrary precision by taking a large enough value of  $\ell$ . We attempted to use this simple observation to improve results on the time constant and Eden conjecture, but found the resulting computations intractable. One can view the recursive equality as encoding the combinatorics that makes this classical problem notoriously difficult.

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