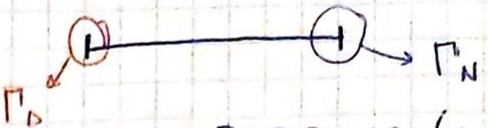
2024-06-10

EXERCISE 1



1.1 Consider the problem:

PARABOLIC (TIME-DEPEND)

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} + Ku = 0 & 0 < x < 1,\ t > 0 \\ u(x=0,t) = \alpha & t > 0 \rightarrow \Gamma_D \\ \dfrac{\partial u}{\partial x}(x=1,t) = \beta & t > 0 \rightarrow \Gamma_N \\ u(x,t=0) = u_0(x) & 0 < x < 1 \rightarrow \text{I.C.} \end{cases}$$

Γ_D, Γ_N } B.C. $\rightarrow$ LIFTING OPERATOR

and $K \geq 0$, α, β are constants.

SPAZIO FUNZIONI TEST — SPAZIO DELLE SOL.

$V_0 = H^1_{\Gamma_D}(\Omega)$; $V = \{ v \in H^1((0,1)) \text{ s.t. } \underbrace{v(x=0,t) = \alpha}_{\text{DIRICHLET}} \}$

DIRICHLET (ESSENTIAL B.C. $\downarrow$ vanno imposte nello spazio)

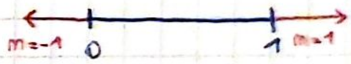
Let $v \in V_0$:

$$\int_\Omega \frac{\partial u}{\partial t} \cdot v\, dx \boxed{- \int_\Omega \frac{\partial^2 u}{\partial x^2} v\, dx} + \int_\Omega K u \cdot v\, dx = 0$$

$\downarrow$ GG

$$\boxed{- \int_\Omega \frac{\partial^2 u}{\partial x^2} \cdot v\, dx} = \boxed{\int_\Omega \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} dx - \int_{\partial\Omega} \frac{\partial u}{\partial x} \vec{n}\, v\, d\gamma}$$

$\frac{\partial u}{\partial n}$; $\vec{n}$ non è un vettore



$$\int_0^1 \frac{\partial u}{\partial t} \cdot v\, dx + \boxed{\int_0^1 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} dx - \int_0^1 \frac{\partial u}{\partial x} \cdot v\, d\gamma} + \int_0^1 K u \cdot v\, dx = 0$$

$\uparrow \partial\Omega \downarrow$

$$\left[\frac{\partial u}{\partial x} \cdot n\, v \right]_0^1 = -\frac{\partial u}{\partial x} \cdot n\, v \Big|_0 + \underbrace{\frac{\partial u}{\partial x}}_{=\beta} n\, v \Big|_1$$

$v \in H^1_{\Gamma_D}((0,1))$

$\beta v(1,t)$

$\beta \cdot v(1,t)$

$$\underbrace{\int_0^1 \frac{\partial u}{\partial t} \cdot v\, dx + \int_0^1 \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} dx}_{a(u,v)} - \overbrace{\beta \cdot v(1,t)}^{F(v)} + \underbrace{\int_0^1 K u \cdot v\, dx}_{} = 0$$

- $a(u,v) = \displaystyle\int_0^1 \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} dx + \int_0^1 K u \cdot v\, dx$

- $F(v) = \beta v(1,t)$

$\forall t > 0$, find $u \in V$ s.t.

$$\int_0^1 \frac{\partial u}{\partial t} \cdot v\, dx + a(u,v) = F(v) \quad \forall v \in V_0 \qquad \text{(WP)}$$

and $u(x,t=0) = u_0(x) \qquad 0 < x < 1$

LIFTING: voglio avere incognita in V_0; suppose $\exists R_g \in H^1(\Omega)$ s.t.

$u = \hat{u}_0 + R_g$ and $\hat{u}_0 \in V_0$

$R_g \in V$ SEMPRE

substitute in (WP):

$$\int_0^1 \frac{\partial(\hat{u}_0 + R_g)}{\partial t} \cdot v\, dx + a(\hat{u}_0 + R_g, v) = F(v)$$

$$\int_0^1 \frac{\partial \hat{u}_0}{\partial t} \cdot v\, dx + \boxed{\int_0^1 \frac{\partial R_g}{\partial t} \cdot v\, dx} + a(\hat{u}_0, v) + \boxed{a(R_g, v)} = F(v)$$

$$\int_0^1 \frac{\partial \hat{u}_0}{\partial t} \cdot v\, dx + a(\hat{u}_0, u) = \underbrace{F(v) - \int_0^1 \frac{\partial R_g}{\partial t} \cdot v\, dx - a(R_g, v)}_{G(v)}$$

Final formulation:

$\forall t > 0$, find $\hat{u}_0 \in V_0$ s.t.

$$\int_0^1 \frac{\partial \hat{u}_0}{\partial t} \cdot v\, dx + a(\hat{u}_0, v) = G(v) \qquad \forall v \in V_0$$

$u(x, 0) = u_0(x)$ → lascio così anche se ho lifting

1.2 COERCIVITY of $a(u,v)$:

$$a(v,v) = \int_0^1 \nabla v^2\, dx + \int_0^1 K v^2\, dx = \|\nabla v\|^2_{L^2(\Omega)} + K \|v\|^2_{L^2(\Omega)} \geq$$

$$\geq \underbrace{\min(1, K)}_{\alpha} \cdot \|v\|^2_{V} \quad (V = H^1_1)$$

1.3 PIECEWISE LINEAR FINITE ELEMENTS IN SPACE

① We don't take $\cancel{V_h}$ $\Rightarrow$ $\boxed{V_{0,h}} = V_0 \cap X_h^r = \boxed{V_0 \cap X_h^1(\Omega)}$ $\dim(V_{0,h}) < +\infty$

② Let's define: a partition of Ω K_c, $c = 1, \dots, N_{el}$ s.t.

$\bigcup_{i=1}^{N_{el}} K_i = \Omega$ and $K_i \cap K_j = \emptyset$ $\forall i, j$

Approximate formulation: (SEMI-DISCR, for space):

$\forall t > 0$, find $\hat{u}_{0,h} \in V_{0,h}$ s.t. $\qquad \forall v_h \in V_{0,h}$

$$\int_0^1 \frac{\partial \hat{u}_{0,h}}{\partial t} \cdot v_h\, dx + a(\hat{u}_{0,h}, v_h) = G(v_h)$$

$u_h(x, 0) = u_{0,h}(x)$

basis for $V_{0,h}$: $\{\varphi_j(x)\}_{j=1}^{N_{V_h}}$ $\Rightarrow$ $\hat{u}_h(x,t) = \sum_{j=1}^{N_{V_h}} \hat{u}_j(t) \cdot \varphi_j(x)$

$v_h(x,t) = \varphi_i(x)$

$\forall i$

$$\int_0^1 \frac{\partial \sum_{j=1}^{N_{V_h}} \hat{u}_j(t)\cdot\varphi_j(x)}{\partial t}\cdot \varphi_i(x)\,dx + a\left(\sum_{j=1}^{N_{V_h}} \hat{u}_j(t)\cdot\varphi_j(x),\ \varphi_i(x)\right) = G(\varphi_i(x))$$

$$\sum_{j=1}^{N_{V_h}} \dot{\hat{u}}_j(t) \underbrace{\int_0^1 \varphi_j(x)\cdot\varphi_i(x)\,dx}_{M_{ij}} + \sum_{j=1}^{N_{V_h}} \hat{u}_j(t)\cdot \underbrace{a(\varphi_j(x), \varphi_i(x))}_{A_{ij}} = \underbrace{G(\varphi_i(x))}_{G_i} \quad \forall i$$

↳ separated time & space

↓ I'm integrating in space ⇒ time goes out

- $\vec{\dot{\hat{u}}}(t) = [\dot{\hat{u}}_1(t), \dots, \dot{\hat{u}}_{N_{V_h}}(t)]$
- $\vec{\hat{u}}(t) = [\hat{u}_1(t), \dots, \hat{u}_{N_{V_h}}(t)]$

$$\Rightarrow \boxed{M\vec{\dot{\hat{u}}} + A\vec{\hat{u}} = G}$$

semidiscrete formulation in space

SEMI-DISC IN TIME:

$$\begin{cases} M\dfrac{\vec{u}^{n+1} - \vec{u}^{n}}{\Delta t} + A\vec{u}^{n+1} = \vec{F}^{n+1} & n = 0, \dots, N-1 \\ \vec{u}^0 = \vec{u}_0 \end{cases}$$

#timesteps

$0 \quad \Delta t \quad \dots \quad t^n \quad t^{n+1} \quad T \quad t$

1.4 • With $\theta \geq \frac{1}{2}$, we have UNCONDITIONAL STABILITY;

• $\| \quad \| < C[h^{r+1} + \Delta t^{q}$

STABILITY IN SPACE?