2024-02-05 ADR (ellintic completa) + LIFTING

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EXERCISE 1

1.1
$$\begin{cases} -\operatorname{div}(\mu \nabla u) + \operatorname{div}(\vec{b}\, u) + \sigma u = f & \text{in } \Omega \subset \mathbb{R}^2 \\ u = g & \text{on } \Gamma_D \\ \mu \dfrac{\partial u}{\partial \vec{n}} - (\vec{b}\cdot\vec{n})\, u = \phi & \text{on } \Gamma_N \end{cases}$$

on Γ_D → Dirichlet
on Γ_N → Neumann

$\mu > 0$ constant, $\sigma \geq 0$ constant, $\vec{b} \in \mathbb{R}^2$ constant vector, f, g, ϕ given functions

$\boxed{d = 1}$ (dim = 2 codice)

WEAK FORMULATION:

Spazio funzioni test: $V_0 = H^1_{\Gamma_D}(\Omega) = \{v \in H^1(\Omega) \text{ s.t. } v|_{\Gamma_D} = 0\}$

Spazio delle soluzioni: $V = \{v \in H^1(\Omega) \text{ s.t. } \underbrace{v|_{\Gamma_D} = g}_{\text{Dirichlet essential B.C.}}\}$

Let $v \in V_0$:

$$\int_\Omega -\operatorname{div}(\mu \nabla u)\cdot v\, dx + \int_\Omega \operatorname{div}(\vec{b}\cdot u)\cdot v\, dx + \int_\Omega \sigma u \cdot v\, dx = \int_\Omega f \cdot v\, dx$$

$$\int_\Omega \mu \nabla u \nabla v\, dx - \int_{\partial\Omega} \mu \underbrace{\nabla u \cdot \vec{n}}_{= \frac{\partial u}{\partial n}} \cdot v\, d\gamma + \boxed{\int_\Omega \operatorname{div}(\vec{b} u)\, v\, dx} + \int_\Omega \sigma u v\, dx = \int_\Omega f v\, dx$$

$$\int_\Omega \operatorname{div}(u \cdot \vec{b})\cdot v\, dx = -\int_\Omega u \vec{b}\, \nabla v\, dx + \int_{\partial\Omega} u \vec{b}\cdot\vec{n}\, v\, dx$$

$$\nabla v = \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix}$$

$$\mu \int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\, dx - \int_{\partial\Omega} \mu \frac{\partial u}{\partial n}\cdot v\, d\gamma + \left(-\int_\Omega \nabla(v)\, \vec{b}\, u\, dx + \int_{\partial\Omega} \vec{b}\, u\, \vec{n}\, v\, dx \right)$$
$$+ \int_\Omega \sigma u v\, dx = \int_\Omega f \cdot v\, dx$$

$$\mu \int_\Omega \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}\, dx - \underbrace{\left(\int_{\partial\Omega} \left(\mu \frac{\partial u}{\partial n} - \vec{b}\,\vec{n}\cdot u \right) \cdot v\, dx \right)}_{= \int_{\Gamma_N} \underbrace{(\dots)}_{\phi} \cdot v\, dx + \cancel{\int_{\Gamma_D} (\dots)\cdot v\, dx}} - \int_\Omega \frac{\partial v}{\partial x}\cdot \vec{b}\, u\, dx + \int_\Omega \sigma u v\, dx = \int_\Omega f \cdot v\, dx$$

since $v|_{\Gamma_D} = 0$

$$\mu \int_\Omega \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}\, dx \boxed{- \int_{\Gamma_N} \phi \cdot v\, dx} - \vec{b} \int_\Omega \frac{\partial v}{\partial x}\cdot u\, dx + \int_\Omega \sigma u v\, dx = \int_\Omega f \cdot v\, dx$$

$$\underbrace{\mu \int_\Omega \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}\, dx - \vec{b} \int_\Omega \frac{\partial v}{\partial x}\cdot u\, dx + \sigma \int_\Omega u v\, dx}_{a(u,v)} = \underbrace{\int_\Omega f \cdot v\, dx + \int_{\Gamma_N} \phi v\, dx}_{F(v)}$$

$$\boxed{\begin{array}{l} \text{Find } u \in V \text{ s.t.:} \\ a(u,v) = F(v) \quad \forall v \in V_0 \end{array}}$$

(WP)

LIFTING: voglio avere soluzione in V_0; suppose $\exists R_g \in V$ s.t.

$u = \hat{u} + R_g$, and $\hat{u} \in V_0$

$$a(\hat{u} + R_g, v) = F(v)$$

$$a(\hat{u}, v) + \boxed{a(R_g, v)} = F(v)$$

$$a(\hat{u}, v) = \underbrace{F(v) - a(R_g, v)}_{= G(v)}$$

Final (WP):

$$\boxed{\begin{array}{l}\text{Find } \hat{u} \in V_0 \text{ s.t.:} \\ \quad a(\hat{u}, v) = G(v) \qquad \forall v \in V_0\end{array}}$$

1.2 PROVE (WP) has UNIQUE SOLUTION:

The L-M lemma should be satisfied;

"Assume:

- V is an Hilbert space w/ defined norm $\|\cdot\|$
- G linear functional on V
  - G linear
  - G bounded

- $$|G(v)| = \left|\int_\Omega f \cdot v\, dx + \int_{\Gamma_N} \phi \cdot v\, dx\right| \leq \left|\int_\Omega f v\, dx\right| + \left|\int_{\Gamma_N} \phi \cdot v\, dx\right| \leq$$

$$\leq \|f\|_{L^2(\Omega)} \cdot \|v\|_{L^2(\Omega)} + \|\phi\|_{L^2(\Gamma_N)} \cdot \|v\|_{L^2(\Gamma_D)} \leq$$

$$\leq \underbrace{\left(\|f\|_{L^2(\Omega)} + \|\phi\|_{L^2(\Gamma_N)}\right)}_{C} \cdot \|v\|_{L^2(\Omega)} \longrightarrow G \text{ is bounded}$$

- a is a bilinear form, continuous & coercive:

$$\begin{cases} \text{continuity: } \exists M > 0 : |a(u,v)| \leq M \cdot \|u\| \cdot \|v\| \quad \forall u, v \in V \\ \text{coercivity: } \exists \alpha > 0 : a(v,v) \geq \alpha \|v\|^2 \quad \forall v \in V \end{cases}$$

- $$|a(u,v)| = \left|\mu \int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} dx - \vec{b}\int_\Omega \frac{\partial v}{\partial x} u\, dx + \sigma\int_\Omega uv\, dx\right| \leq$$

$$\leq \left|\int \ldots\right| + \left|\int \ldots\right| + \left|\int \ldots\right| \leq$$

$$\leq \mu \|u'\|_{L^2(\Omega)} \cdot \|v'\|_{L^2(\Omega)} + |\vec{b}| \cdot \|v'\|_{L^2(\Omega)} \cdot \|u\|_{L^2(\Omega)} +$$

$$+ \sigma \|u\|_{L^2(\Omega)} \cdot \|v\|_{L^2(\Omega)} \leq$$

$$\leq \mu \|u\|_{L^2(\Omega)} \cdot \|v\|_{L^2(\Omega)} + |b| \cdot \|v\|_{L^2} \|u\|_{L^2}\ \sigma \|u\|_{L^2} \|v\|_{L^2} =$$

$$= \underbrace{(\mu + |\vec{b}| + \sigma)}_{M>0} \cdot \|u\|_{L^2(\Omega)} \cdot \|v\|_{L^2(\Omega)}$$

↳ no assumptions bc $\mu > 0$; $|\vec{b}|$ always ≥ 0 ($=0$ if $\vec{b} = \vec{0}$); $\sigma \geq 0$

• $a(v,v) = \mu \int_\Omega \left(\frac{\partial v}{\partial x}\right)^2 dx - \vec{b} \int_\Omega \frac{\partial v}{\partial x} \cdot v \, dx + \sigma \int_\Omega v^2 dx \geq$

$$\geq \mu \left\| \frac{\partial v}{\partial x} \right\|^2_{L^2(\Omega)} + \frac{1}{2} \int_\Omega \vec{b} \, (v^2)' dx + \sigma \int_\Omega v^2 dx = (*)$$

$$= -\frac{1}{2} \int_\Omega (\operatorname{div} \vec{b}) \, v^2 + \underbrace{\int_{\partial\Omega} \frac{1}{2} \vec{b} \vec{n} \, v^2}_{\geq 0}$$

$$= \frac{1}{2} \cancel{\int_{\Gamma_D} \vec{b} \cdot \vec{n} \cdot v^2}^{\to 0} + \frac{1}{2} \int_{\Gamma_N} \underbrace{\boxed{\vec{b} \cdot \vec{n}}}_{\geq 0} \cdot v^2$$

$v|_{\Gamma_D} = 0$ therefore

I assume $\vec{b} \cdot \vec{n} \geq 0$

$(*)$ $\geq \mu \|\nabla v\|^2_{L^2(\Omega)} + \int_\Omega \underbrace{(\sigma - \frac{1}{2} \operatorname{div} \vec{b})}_{= \gamma} v^2 \geq$

ASSUMPTION: $\gamma > 0$ → $\sigma > \frac{1}{2} \operatorname{div}(\vec{b})$

$$\geq \mu \|\nabla v\|^2_{L^2(\Omega)} + \gamma \|v\|^2_{L^2(\Omega)} \geq$$

$$\geq \underbrace{\min(\mu, \gamma)}_{\alpha} \underbrace{(\|\nabla v\|^2_{L^2} + \|v\|^2_{L^2})}_{\|v\|^2} = \alpha \|v\|^2$$

Then: $\exists!$ solution u to (P(x));

Moreover: $\|u\| \leq \frac{1}{\alpha} \|f\|_{V'}$ (bounded) "

1.3 APPROXIMATION OF WEAK FORMULATION → PIECEWISE FINITE elem

① MESH: let's define a partition of Ω K_c, $c = 1, \dots, N_{el}$ s.t. $\bigcup_{i=1}^{N_{el}} K_i = \Omega$ and $K_i \cap K_j = \emptyset$ $\forall i \neq j$

② $X_h^2(\Omega) = \{ v \in C^0(\Omega), \, v_h|_{K_i} \in \mathbb{P}^2(K_j) \quad \forall j \}$

$\Rightarrow V_h = V \cap X_h^2(\Omega)$ → fem space for velocity ($! N_{V_h} < +\infty$)

$$\boxed{\begin{array}{l} \text{Find } \hat{u}_h \in V_{0,h} \text{ s.t.} \\ a(\hat{u}_h, v_h) = G(v_h) \quad \forall v_h \in V_{0,h} \end{array}} \quad (G)$$

$$\left\{\hat{\varphi}_j(x)\right\}_{j=1}^{N_{V_{h,0}}} \rightarrow \text{basis for } V_{h,0}$$

$$\hat{u}_h(x) = \sum_{j=1}^{N_{V_h}} \hat{u}_j \varphi_j(x) \qquad v_h(x) = \varphi_i(x)$$

$$\Rightarrow a\left(\sum_{j=1}^{N_{V_{h,0}}} \hat{u}_j \varphi_j(x), \varphi_i(x)\right) = G(\varphi_i(x)) \qquad \forall i = 1, \dots, N_{V_h}$$

$$\sum_{j=1}^{N_{V_{h,0}}} \hat{u}_j \underbrace{a(\varphi_j(x), \varphi_i(x))}_{A_{ij}} = \underbrace{G(\varphi_i(x))}_{G_i}$$

- $\vec{\hat{u}} = [\hat{u}_1, \dots, \hat{u}_{N_{V_{h,0}}}]$
- $\vec{G} = [G_1(\varphi_i(x)), \dots, G_{N_{V_{h,0}}}(\varphi_i(x))]$

$$\Rightarrow \boxed{A\vec{\hat{u}} = G}$$

STABILITY:

Since (L-M) holds, (G) has unique solution, and $\|u_h\|_V \leq \frac{1}{\alpha}\|F\|_V$

Galerkin

this is a measure of stability

CONVERGENCE:

If ~~....~~ $\inf_{v_h \in V_h} \|v - v_h\| \xrightarrow[h \to 0]{} 0 \quad \forall v \in V \quad \Rightarrow \|u - u_h\| \xrightarrow[h \to 0]{} 0$

Moreover: if $\hat{u} \in H^3(\Omega) \cap V$, $\inf_{v_h \in V_h} \|\hat{u} - \hat{u}_h\| \leq C h^2 |\hat{u}|_{H^3(\Omega)}$

$C = \frac{M}{\alpha}$

1.4 $\left(\frac{h}{5}\right)^2 = \frac{h^2}{25} \rightarrow$ the error will be reduced by a factor 25

EXERCISE 2

1 Consider the problem:

$$\begin{cases} Lu = f & 0 < x < 1 \\ u(0) = u(1) = 0 & \rightarrow \Gamma_D \end{cases}$$

where $Lu = -u'' + \alpha u$ and $\alpha > 0$

2.1

$\Omega = (0,1) \Rightarrow \Omega_1 = (0,\gamma),\ \Omega_2 = (\gamma, 1)$ for suitable $\gamma \in (0,1)$

↳ partition of Ω into two disjoint subdomains

(D-N):

$\forall K \geq 1$:

$$\begin{cases} L u_1^{(K)} = f & \text{in } \Omega_1 \\ u_1^{(K)} = u_2^{(K-1)} & \text{on } \Gamma \\ u(0) = u(1) = 0 \end{cases}$$

$$\begin{cases} L u_2^{(K)} = f & \text{in } \Omega_1 \\ \mu u_2'^{(K)} = \mu u_1'^{(K)} & \text{on } \Gamma \\ u(0) = u(1) = 0 \end{cases}$$

(NN): λ is the unknown value of the solution u on the interface Γ:

$$\boxed{\lambda = u_i \quad \text{on } \Gamma \quad (i = 1, 2)}$$

$\forall \lambda^{(0)}$ on Γ, for $K \geq 0$ and $i = 1,2$, solve the following problems:

$$\begin{cases} L u_i^{(K+1)} = f & \text{in } \Omega_i \\ u_i^{(K+1)} = \lambda^{(K)} & \text{on } \Gamma \\ u_i^{(K+1)} = 0 & \text{on } \partial\Omega_i \setminus \Gamma \end{cases}$$

$$\begin{cases} L \psi_i^{(K+1)} = 0 & \text{in } \Omega_i \\ \dfrac{\partial \psi_i^{(K+1)}}{\partial n} = \dfrac{\partial u_1^{(K+1)}}{\partial n} - \dfrac{\partial u_2^{(K+1)}}{\partial n} & \text{on } \Gamma \\ \psi_i^{(K+1)} = 0 & \text{on } \partial\Omega_i \setminus \Gamma \end{cases}$$

with $\lambda^{(K+1)} = \lambda^{K} - \theta \left(\sigma_1 \psi_{1|\Gamma}^{(K+1)} - \sigma_2 \psi_{2|\Gamma}^{(K+1)} \right)$

positive acceleration param

positive coefficients

2.2 $r = 1$

- $V = H_0^1(\Omega)$
- partition of Ω: K_c, $c = 1, \dots, N_{el}$ s.t. $\bigcup_{i=1}^{N_{el}} K_i = \Omega$, $K_i \cap K_j = \emptyset \quad \forall i \neq j$

$V_h = V \cap \underbrace{\{ v \in C^0(\Omega),\ v_h|_{K_i} \in \mathbb{P}^1(K_j) \ \forall j \}}_{X_h^{1,1}(\Omega)}$ $\quad (N_{V_h} < +\infty)$

Suppose: N_1 = interior FE nodes in Ω_1

N_Γ = interface FE nodes on Γ

N_2 = interior FE nodes ~~ ~~ in Ω_2

$$\vec{u} = \begin{bmatrix} \vec{u}_1 \\ \vdots \\ \vec{u}_{N_h} \end{bmatrix} \Longrightarrow \vec{u} = \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vec{\lambda} \end{bmatrix} \qquad \vec{\lambda} = \vec{u}|_\Gamma$$

nodal values on Γ

A = stiffness matrix:

$$A = \begin{bmatrix} \boxed{A_{11}} & \cancel{A_{12}}^{\,0} & A_{1\Gamma} \\ \cancel{A_{21}}^{\,0} & \boxed{A_{22}} & A_{2\Gamma} \\ A_{\Gamma 1} & A_{\Gamma 2} & \boxed{A_{\Gamma\Gamma}} \end{bmatrix} \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vec{u}_\Gamma \end{bmatrix} = \begin{bmatrix} \vec{f}_1 \\ \vec{f}_2 \\ \vec{f}_\Gamma \end{bmatrix} \quad (*)$$

$A_{21} = 0$, $A_{12} = 0$ → assumptions: non-overlapping

~~equations~~ $\vec{u}_1 = A_{11}^{-1}(\vec{f}_1 - A_{1\Gamma}\vec{\lambda})$

$\vec{u}_2 = A_{22}^{-1}(\vec{f}_2 - A_{2\Gamma}\vec{\lambda})$

$$\Rightarrow \Big[\underbrace{(-A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma} - A_{\Gamma 2}A_{22}^{-1}A_{2\Gamma}) + A_{\Gamma\Gamma}}_{N_\Gamma \times N_\Gamma \text{ matrix } \Sigma}\Big]\vec{\lambda} = \vec{f}_\Gamma - A_{\Gamma 1}A_{11}^{-1}\vec{f}_1 - A_{\Gamma 2}A_{22}^{-1}\vec{f}_2$$

$\Longleftrightarrow$ $\boxed{\Sigma \cdot \vec{\lambda} = \vec{X}_\Gamma}$ Schur complement system of (*)

$\Sigma^{(1)} + \Sigma^{(2)} = \Sigma$

local " " $\Longleftrightarrow$ global Schur complem

2.3 CONVERGENCE of DN & NN:

- $P_{DN} = P = \Sigma_2$
- $P_{NN} = P = (\delta_1 \Sigma_1^{-1} + \delta_2 \Sigma_2^{-1})^{-1}$

if $B\vec{x} = \vec{b}$ linear syst, if we use preconditioned Rich:

$\vec{x}^{(0)}$ given, $\forall k \geq 1$:

$$P\underbrace{(\vec{x}^{(k)} - \vec{x}^{(k-1)})}_{\text{increment}} = \underset{\substack{\uparrow \\ \text{Richardson parameter}}}{\theta}\underbrace{(\vec{b} - B\vec{x}^{(k-1)})}_{\text{Residual}}$$

Risultati di convergenza