2025-02-10

→ EXERCISE 1

Consider the ~~parabolic~~ elliptic:

Questo in realtà non è una parabolic ma è una elliptic perché NON c'è la condizione iniziale sul tempo

$$\boxed{\begin{cases} -u'' - xu' + u = f & 0<x<1 \\ u(0) = u(1) = 0 \end{cases} \quad \text{where } f(x) = 0.}$$

1.1 We have $\Omega = (0,1)$, and we have 0 boundary conditions, therefore: $V = H_0^1(\Omega)$;

Let's take the generic function $v \in V$:

$$-\int_0^1 \frac{\partial^2 u}{\partial x^2} v\,dx \underbrace{- \int_0^1 x \frac{\partial u}{\partial x} v\,dx + \int_0^1 u\,v\,dx = \int_0^1 f \cdot v\,dx}_{\text{tutto questo lo lascio stare}}$$

Gauss-green

$$\int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}dx - \cancel{\left[\nabla u \cdot n \cdot v\right]_{x=0}^{x=1}}^{\,0} - \int_0^1 x\frac{\partial u}{\partial x} v\,dx + \int_0^1 u \cdot v\,dx = \int_0^1 f \cdot v\,dx$$

perché abbiamo preso uno spazio in cui la u è 0 al bordo

So, in the end, our weak formulation reads as:

> "Find, $\forall t>0$, $u \in V = H_0^1(\Omega)$ s.t. $a(u,v) = F(v)$ $\forall v \in V$,
>
> where:
>
> $$a(u,v) = \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}dx - \int_0^1 x\frac{\partial u}{\partial x} v\,dx + \int_0^1 u \cdot v\,dx$$
>
> $$F(v) = \int_0^1 f \cdot v\,dx \text{"}$$

1.2 CONTINUITY: by definition we have that $a(\cdot,\cdot)$ bilinear form is continuous if:

$$\exists M > 0 : |a(u,v)| \leq M \cdot \|u\| \cdot \|v\| \qquad \forall u,v \in V$$

> Generally speaking we have 3 tools for proving continuity:
>
> ① TRIANGLE INEQUALITY: $\left|\int \ldots \pm \int \ldots\right| \leq \left|\int \ldots\right| + \left|\int \ldots\right|$
>
> ② $\left|\int \ldots\right| \leq \int |\ldots|$
>
> ③ Cauchy-Schwarz: $\int |uv| \leq \|u\|_{L^2} \cdot \|v\|_{L^2}$

$$|a(u,v)| = \left| \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx - \int_0^1 x\frac{\partial u}{\partial x} v\,dx + \int_0^1 u\cdot v\,dx \right| \leq \leftarrow ① + ②$$

$$\leq \int_0^1 \left|\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right| dx + \int_0^1 \left| x\frac{\partial u}{\partial x} v\right| dx + \int_0^1 |u\cdot v|\,dx \leq \leftarrow ③$$

$$\leq \left\|\frac{\partial u}{\partial x}\right\|_{L^2}\cdot\left\|\frac{\partial v}{\partial x}\right\|_{L^2} + \left\|\frac{\partial u}{\partial x}\right\|_{L^2}\cdot\|v\|_{L^2} + \|u\|_{L^2}\cdot\|v\|_{L^2} \leq$$

$$\leq \|u\|_{H^1}\cdot\|v\|_{H^1} + \|u\|_{H^1}\cdot\|v\|_{H^1} + \|u\|_{H^1}\cdot\|v\|_{H^1} =$$

$$= 3\|u\|_{H^1}\cdot\|v\|_{H^1} \quad \longrightarrow M = 3$$

in H_0^1 we have two equivalent norms

$$\underbrace{\left\|\frac{\partial u}{\partial x}\right\|_{L^2}}_{\|u\|_{H_0^1}} \quad \text{and} \quad \underbrace{\left(\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2} + \|u\|^2_{L^2}\right)^{1/2}}_{\|u\|_{H^1}}$$

Also note that:

$$\left\|\frac{\partial u}{\partial x}\right\|_{L^2} \leq \|u\|_{H^1} \quad \text{and} \quad \|u\|_{L^2} \leq \|u\|_{H^1}$$

COERCIVITY: $\boxed{a(\cdot,\cdot) \text{ is coercive if } \exists \alpha>0 : a(v,v) \geq \alpha\|v\|^2 \ \forall v\in V}$

$$a(v,v) = \int_0^1 \left(\frac{\partial v}{\partial x}\right)^2 dx - \int_0^1 x\frac{\partial v}{\partial x} v\,dx + \int_0^1 v^2\,dx$$

$$-\int_0^1 x\frac{\partial v}{\partial x} v\,dx = -\frac{1}{2}\boxed{\int_0^1 x(v^2)'\,dx}$$

Gauss-Green:

$$\boxed{\int_0^1 \underbrace{x}_{F}\underbrace{(v^2)'}_{g}\,dx} = \cancel{\left[x v^2\right]_{x=0}^{x=1}}^{\,0} - \int_0^1 1\cdot v^2\,dx$$

perché $v\in H_0^1(0,1)$
quindi $v(0) = v(1) = 0$

$$\Longrightarrow -\int_0^1 x\frac{\partial v}{\partial x} v\,dx = +\frac{1}{2}\boxed{\int_0^1 v^2\,dx}$$

$$\Longrightarrow a(v,v) = \int_0^1\left(\frac{\partial v}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^1 v^2\,dx + \int_0^1 v^2\,dx = \int_0^1\left(\frac{\partial v}{\partial x}\right)^2 dx + \frac{3}{2}\int_0^1 v^2\,dx =$$

$$= \left\|\frac{\partial v}{\partial x}\right\|^2_{L^2(0,1)} + \frac{3}{2}\|v\|^2_{L^2(0,1)} = \left\|\frac{\partial v}{\partial x}\right\|^2_{L^2(0,1)} + \|v\|^2_{L^2(0,1)} + \frac{1}{2}\|v\|^2_{L^2(0,1)}$$

definizione $\|\cdot\|_{H^1}$ $\rightarrow$ $= \|v\|^2_{H^1(0,1)} + \frac{1}{2}\|v\|^2_{L^2(0,1)} \geq \|v\|^2_{H^1(0,1)}$

1.3 Dra $L^2(0,1)$

$$\Longrightarrow a(v,v) \geq 1\cdot\|v\|^2_{H^1(0,1)} \longrightarrow \alpha = 1$$

1.3 Expression of the exact solution: $u(x) = \dots$

- V spazio Hilbert
- a continuous & coercive

$\longrightarrow$ Lax-Milgram is satisfied $\Rightarrow$ $\exists!\ u$ solution

$\leadsto$ Let's try with $u=0$:

$$u \equiv 0 \quad \in H^1_0((0,1)) \quad \begin{cases} u = 0 \in L^2 \\ u' \in L^2(0,1) \\ u(0) = u(1) = 0 \end{cases}$$

$u \equiv 0$ is the only exact solution

1.4 "Find $u_h \in V_h$: $a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$"; we take:

Galerkin formulation

$$X_h^2((0,1)) = \{ v_h \in \mathcal{C}^0(\overline{\Omega}) : v_h(x) \in \mathbb{P}^2 \quad \forall x \in K_c, \ \forall c = 1, \dots, N_{el} \}$$

$\overline{\Omega} = (0,1)$

$$\Rightarrow V_h = X_h^2((0,1)) \cap H^1_0((0,1)), \qquad N_h = \dim(V_h)$$

Moreover, let $\varphi_i(x)$, $i = 1, 2, \dots, N_h$ be the Lagrangian basis functions of V_h; therefore we want to look for $u_h \in V_h$ s.t.

$$u_h(x) = \sum_{j=1}^{N_h} U_j \varphi_j(x) \qquad x \in \overline{(0,1)}$$

The discrete weak formulation rewrites as:

"Find U_j, for $j = 1, 2, \dots, N_h$ such that:

$$\sum_{j=1}^{N_h} U_j \, a(\varphi_j, \varphi_i) = F(\varphi_i) \qquad \text{for } i = 1, 2, \dots, N_h\text{"}$$

$\downarrow$ approximation of the weak problem by piecewise quadratic FEMs

$\downarrow$ error estimates for both $H^1(0,1)$ norm and for $L^2(0,1)$ norm

THEOREM (INTERPOLATION BOUNDS)

Let $v \in H^{r+1}$

$$\| v - \Pi_h^r v \|_{L^2} \leq C(r) \Big(\sum_{K \in \mathcal{T}_h} h_K^{2(r+1)} \Big) \sim$$

$$\| v - \Pi_h^r v \|_{H^1} \leq C_2(r) \Big(\sum_{K \in \mathcal{T}_h}$$

$\rightarrow$ su slides

So from (A) we get

$\Rightarrow$ In our case:
$$\| u - u_h \|_{H^1} \leq C_1 \cdot \frac{M}{\alpha} h^r |u|_{H^{r+1}} \leq 3 C_1 h^2 |u|_{H^3}$$

$$\| u - u_h \|_{L^2} \leq \tilde{C} h^{r+1} |u|_{H^{r+1}} \leq \tilde{C} \cdot h^3 |u|_{H^3}$$

1.5 $u_{ex} = x(1-x)$

Per essere completi, dovremmo verificare che la soluzione f è in H_0^1, ma essendo super continua, dovrebbe essere tutto ok

- $u'_{ex} = 1-2x$
- $u''_{ex} = -2$

} find f

$$\Rightarrow \quad 2 - x(1-2x) + x - x^2 = f$$

$$2 - \cancel{x} + 2x^2 + \cancel{x} - x^2 = f \quad \Rightarrow \quad \boxed{f = x^2+2}$$

EXERCISE 2

Consider the problem:

$\vec{u} \in \mathbb{R}^2$

TIME-DEPENDENT STOKES

PROBLEM
$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \nabla p = 0 & \text{in } \Omega \subset \mathbb{R}^2,\ t>0 \quad \text{(a)} \\ \operatorname{div} u = 0 & \text{in } \Omega,\ t>0 \quad \text{(b)} \end{cases}$$

$d=2$

I.C. $u(x, t=0) = u_0$ in Ω

B.C. (only Neumann) $\left(\dfrac{\partial u}{\partial n} - pn\right)(x,t) = \varphi(x,t)$ on $\partial\Omega$, $t>0$

where u_0 and φ are two given functions, sufficiently regular.

2.1 Weak formulation of the problem:

we have to declare 2 spaces → ① velocity, ② pressure

$V = [H_1(\Omega)]^d = [H_1(\Omega)]^2$ → vector space for VELOCITY

$$Q = \begin{cases} L_0^2(\Omega) & \text{if } \Gamma_D = \partial\Omega \\ L^2(\Omega) & \text{otherwise} \end{cases}$$

nel nostro caso abbiamo tutto bordo di Neumann

$Q = L^2(\Omega)$

EQUATIONS:

(a): $\displaystyle\int_\Omega \frac{\partial \vec{u}}{\partial t} \cdot \vec{v}\, dx - \int_\Omega \Delta\vec{u} \cdot \vec{v}\, dx + \int_\Omega \nabla p \cdot \vec{v}\, dx = 0 \qquad \forall v \in V$

(b): $\displaystyle\int_\Omega (\operatorname{div} \vec{u}) \cdot q\, dx = 0 \qquad \forall q \in Q$ test function

$$-\int_\Omega \operatorname{div}(\nabla\vec{u}) \cdot \vec{v}\, dx = \int_\Omega \nabla\vec{u} : \nabla\vec{v}\, dx - \int_{\partial\Omega} \nabla\vec{u} \cdot \vec{n}\, \vec{v}\, d\gamma$$

! $\nabla\vec{u} : \nabla\vec{v} =$

$$= \frac{\partial\vec{u}}{\partial x}\cdot\frac{\partial\vec{v}}{\partial x} + \frac{\partial\vec{u}}{\partial x}\cdot\frac{\partial\vec{v}}{\partial y} + \frac{\partial\vec{u}}{\partial y}\cdot\frac{\partial\vec{v}}{\partial x} + \frac{\partial\vec{u}}{\partial y}\cdot\frac{\partial\vec{v}}{\partial y}$$

$$\nabla\vec{u}\cdot\vec{n} = \frac{\partial u}{\partial n}$$

$$\begin{cases} \nabla\vec{u} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} \\ \vec{n} = \begin{pmatrix} n_x \\ n_y \end{pmatrix} \end{cases}$$

Termine di pressione:

$$\int_\Omega (\nabla p)\cdot \vec{v}\, dx = -\int_\Omega p\, \nabla\cdot\vec{v}\, dx + \int_{\partial\Omega} p\,(\vec{v}\cdot\vec{n})\, ds$$

Therefore:

$$\text{(a)} \quad \int_\Omega \frac{\partial \vec{u}}{\partial t}\cdot\vec{v}\, dx + \int_\Omega \nabla\vec{u} : \nabla\vec{v}\, dx - \int_{\partial\Omega} \frac{\partial \vec{u}}{\partial n}\cdot\vec{v}\, ds - \int_\Omega p\, \nabla\cdot\vec{v}\, dx + \int_{\partial\Omega} (p\vec{n})\cdot\vec{v}\, ds = 0$$

$$\int_\Omega \frac{\partial \vec{u}}{\partial t}\cdot\vec{v}\, dx + \int_\Omega \nabla\vec{u} : \nabla\vec{v}\, dx - \int_\Omega p\, \nabla\cdot\vec{v}\, dx + \int_{\partial\Omega} \underbrace{\left(p\vec{n} - \frac{\partial \vec{u}}{\partial n}\right)}_{= -\vec{\varphi}}\cdot\vec{v}\, ds = 0$$

! As boundary condition, we have $\frac{\partial \vec{u}}{\partial n} - p\vec{n} = \vec{\varphi}$

$$\Longrightarrow \quad \boxed{\text{(a)} \quad \int_\Omega \frac{\partial \vec{u}}{\partial t}\cdot\vec{v}\, dx + \underbrace{\int_\Omega \nabla\vec{u} : \nabla\vec{v}\, dx}_{a(u,v)} \underbrace{- \int_\Omega p\, \nabla\cdot\vec{v}\, dx}_{b(v,p)} - \int_{\partial\Omega} \vec{\varphi}\cdot\vec{v}\, ds = 0}$$

Final WEAK FORMULATION:

"Find $(\vec{u}(t), p(t)) \in V\times Q$ s.t. $u(x, t=0) = u_0$ and: $\forall t>0$

(a) $\displaystyle\int_\Omega \frac{\partial \vec{u}}{\partial t}\cdot\vec{v}\, dx + a(\vec{u},\vec{v}) + b(\vec{v},p) - \int_{\partial\Omega} \vec{\varphi}\cdot\vec{v}\, ds = 0$

(b) $b(\vec{u}, q) = 0 \qquad \forall(\vec{v}, q) \in V\times Q$

2.2 Approximation of the weak formulation:

SPACE: Taylor-Hood FEMs

① VELOCITY: degree $3 = r$
② PRESSURE: degree $2 = r$

Let • $V_h = [X_h^3]^d \cap V = [X_h^3]^2 \cap V =$

$= \{\vec{v}_h \in [C^0(\Omega)]^{r=3},\ \vec{v}_h|_K \in [\mathbb{P}^c_{K+1}]^{d=2}\ \forall K \in \mathcal{T}_h,\ \cancel{\vec{v}_h|_{\Gamma_D} = 0}\}$

continuous

questa cosa lunghissima è solo dalla teoria...

qui Dirichlet non c'è

$\mathcal{T}_h$ = triangulation of the mesh over Ω

$N_{V_h} < +\infty$

$N_{Q_h} < +\infty$

• $Q_h = X_h^2 \cap Q =$

$= \{q_h \in L^2(\Omega),\ q_h|_K \in \mathbb{P}^c_K,\ \forall K \in \mathcal{T}_h\}$

Approximation of weak formulation: $+\vec{u}_h(\vec{x}, t=0) = \vec{u}_0$

"Let $\vec{v}_h \in V_h$, $q_h \in Q_h$; $\forall t>0$ let's find $p_h \in Q_h$, $\vec{u}_h \in V_h$ s.t.

(a) $\int_\Omega \frac{\partial u_h}{\partial t} v_h \, dx + a(u_h, v_h) + b(v_h, p_h) - \int_\Omega \varphi(x,t) \cdot v_h = 0$

(b) $b(u_h, q_h) = 0$" $\rightarrow$ $a(u,v)$ e $b(v,p)$? sempre ?

Now let's suppose $\{\vec{\varphi}_i\}_{i=1}^{N_v}$ is a basis for V_h and $\{\psi_i\}_{i=1}^{N_q}$ is a basis for Q such that:

$\vec{u}_h(x,t) = \sum_{j=1}^{N_v} U_j(t) \cdot \vec{\varphi}_j(x)$; $p_h = \sum_{m=1}^{N_q} p_m \psi_m$

↓ substitute

$$\begin{cases} \int_\Omega \frac{\partial \sum_{j=1}^{N_v} U_j(t) \vec{\varphi}_j(x)}{\partial t} \vec{\varphi}_i(x)\, dx + a\left(\sum_{j=1}^{N_v} U_j(t) \cdot \vec{\varphi}_j(x), \vec{\varphi}_i(x)\right) + \\ \qquad + b\left(\vec{\varphi}_i(x), \sum_{m=1}^{N_q} p_m \psi_m\right) = \int_\Omega \varphi(x,t) \cdot \varphi_i(x)\, dx \\ b\left(\sum_{j=1}^{N_v} U_j(t) \cdot \vec{\varphi}_j(x), \psi_m\right) = 0 \\ \vec{u}_h(x, t=0) = u_0 \end{cases}$$

VEDI BENE LE DIMENSIONI

$$\begin{cases} \dot{\vec{U}} \underbrace{\int_\Omega \varphi_j(x) \cdot \varphi_i(x)\, dx}_{M_{ij}} + \vec{U} \underbrace{a(\varphi_j(x), \varphi_i(x))}_{A_{ij}} + p\, b(\varphi_i(x), \psi_m) = F_i \\ B\vec{u} = 0 \\ \vec{u}_h(x, t=0) = u_0 \end{cases}$$

So, in the end, the matrix approximation of WEAK FORMULATION is:

$$\begin{cases} M\dot{\vec{u}} + A\vec{u} + B^T \vec{p} = \vec{F} \\ B\vec{u} = 0 \\ \vec{u}(t=0, x) = \vec{u}_0 \end{cases}$$

← Find $\vec{u}, \vec{p}$: $\forall t > 0$

le basi delle velocità sono ~~scalari~~ vettoriali, le basi delle pressione sono scalari

semidiscretization in space

TIME: Backward Euler, $t^n = n \cdot \Delta t$, $\theta = 1$

$$\begin{cases} M \cdot \frac{u^{n+1} - u^n}{\Delta t} + A u^{n+1} + B^T p^{n+1} = F^{n+1} \\ B u^{n+1} = 0 \\ u^0 = u_0 \end{cases}$$

← semidiscretization in time

2.3 At any time step $t^n = n \cdot \Delta t$, discuss the well-posedness of the corresponding problem.

$\forall t^n = \Delta t$, the problem is well-posed because LBB holds since:

- a is bilinear, continuous and coercive
- b is bilinear and coercive

and LBB holds because we use th ($\forall t > 0 \; \exists! \; u_h, p_h, \dots$)

2.4 BE ($\theta = 1$), unconditionally stable; the error estimate is:

"$\forall t^{n+1} \quad \|\vec{u}^{n+1} - \vec{u}_h^{n+1}\|_V + \|p^{n+1} - p_h^{n+1}\|_Q \leq C(h^{k+1} + \Delta t) \cdot \|\vec{u}^{n+1}\|_V \cdot \|p^{n+1}\|_Q$