2025-07-08

EXERCISE 1

1.1 • $V = \left[H^1_{\Gamma_D}(\Omega)\right]^2$ = funzioni che sono nulle su Γ_D

• $Q = L^2(\Omega)$ } time-dependent Stokes

$$
\begin{cases}
\dfrac{\partial \vec{u}}{\partial t} + \alpha \vec{u} - \mu \Delta \vec{u} + \nabla p = 0 & \text{in } \Omega,\ t \in (0,T) \\
\operatorname{div} \vec{u} = 0 & \text{in } \Omega,\ t \in (0,T) \\
\vec{u} = 0 & \text{on } \Gamma_D,\ t \in (0,T) \\
\mu \dfrac{\partial \vec{u}}{\partial \vec{n}} - p\vec{n} = \vec{\psi} & \text{on } \Gamma_N,\ t \in (0,T) \\
\vec{u} = 0 & \text{in } \Omega,\ t \in (0,T)
\end{cases}
$$

$\alpha \geq 0,\ \psi = \psi(x,t),\ \Omega \subset \mathbb{R}^2,\ \Gamma_D \cap \Gamma_N = \emptyset,\ \Gamma_D \cup \Gamma_N = \partial\Omega,\ T > 0$

WEAK FORMULATION:

Let $(v,q) \in V \times Q$:

$$
(a)\ \int_\Omega \frac{\partial \vec{u}}{\partial t} \cdot \vec{v}\, dx + \int_\Omega \alpha \vec{u} \cdot \vec{v}\, dx \underbrace{\boxed{- \int_\Omega \overbrace{\operatorname{div}(\mu \cdot \nabla \vec{u})}^{\mu \Delta \vec{u}} \cdot \vec{v}\, dx}}_{(1)} \underbrace{\boxed{+ \int_\Omega \nabla p \cdot \vec{v}\, dx}}_{(2)} = 0
$$

$$
(b)\ \underbrace{-\int_\Omega \nabla \cdot \vec{u} \cdot q = 0}_{b(u,q)}
$$

$$
\bullet\ (1) = \int_\Omega \mu \nabla \vec{u} \cdot \nabla \vec{v}\, dx - \int_{\Gamma_N} \mu \big(\overbrace{\nabla \vec{u} \cdot \vec{n}}^{\frac{\partial \vec{u}}{\partial \vec{n}}}\big) \cdot \vec{v}\, dx - \cancel{\int_{\Gamma_D} \mu \cdot \nabla \vec{u}\, \vec{n} \cdot \vec{v}\, dx}
$$

0 since $u|_{\Gamma_D} = 0$

$$
\bullet\ (2) = -\int_\Omega \operatorname{div}(\vec{v}) \cdot p\, dx + \int_{\Gamma_N} (p \cdot \vec{n}) \cdot \vec{v}\, dx + \cancel{\int_{\Gamma_D} p \cdot \vec{n} \cdot \vec{v}\, dx}
$$

0 since $u|_{\Gamma_D} = 0$

we can choose $v|_{\Gamma_D} = 0$

$$
\Rightarrow (a):\ \int_\Omega \frac{\partial \vec{u}}{\partial t} \cdot \vec{v}\, dx + \int_\Omega \alpha \vec{u} \cdot \vec{v}\, dx + \int_\Omega \mu \nabla \vec{u} \cdot \nabla \vec{v}\, dx -
$$

$$
- \int_{\Gamma_N} \underbrace{\left(\mu \frac{\partial \vec{u}}{\partial n} - p\vec{n}\right)}_{\vec{\psi}} \cdot \vec{v}\, dx - \int_\Omega \operatorname{div}(\vec{v}) \cdot p\, dx = 0
$$

$$
(a)\ \int_\Omega \frac{\partial \vec{u}}{\partial t} \cdot \vec{v}\, dx + \underbrace{\int_\Omega \alpha \vec{u} \cdot \vec{v}\, dx + \int_\Omega \mu \nabla \vec{u} \cdot \nabla \vec{v}\, dx}_{a(u,v)} - \overbrace{\int \vec{\psi} \cdot \vec{v}\, dx}^{F(\vec{v})} - \underbrace{\int \operatorname{div}(\vec{v}) \cdot p\, dx}_{b(\vec{v},p)} = 0
$$

So the weak formulation reads:

$$\boxed{\begin{aligned}&\forall t>0, \text{ find } u\in V \text{ and } p\in Q \text{ s.t.:}\\ &(a)\ \int_\Omega \frac{\partial \vec{u}}{\partial t}\cdot\vec{v}\,dx + a(\vec{u},\vec{v}) + b(\vec{v},p) = F(\vec{v}) \\ &\qquad\qquad\qquad\qquad\qquad \forall v\in V,\ \forall q\in Q\\ &(b)\ b(\vec{u},q)=0\\ &\vec{u}(x,t=0)=0\end{aligned}}$$

1.2 SPATIAL APPROXIMATION: let's consider K_c, $c=1,\dots,N_{el}$ UNIFORM PARTITION/MESH of Ω

- $V_h = V\cap X_h^3(\Omega) = [H^1_{\Gamma_D}(\Omega)]^2 \cap X_h^3(\Omega)$
- $Q_h = Q\cap X_h^2(\Omega) = L^2(\Omega)\cap X_h^2(\Omega)$

Discretization in space:

$$\boxed{\begin{aligned}&\forall t>0, \text{ find } u_h\in V_h \text{ and } p_h\in Q_h \text{ s.t.:}\\ &(a)\ \int_\Omega \frac{\partial \vec{u}_h}{\partial t}\cdot\vec{v}_h\,dx + a(\vec{u}_h,\vec{v}_h) + b(\vec{v}_h,p_h) = F(\vec{v}_h) \\ &\qquad\qquad\qquad\qquad\qquad \forall v_h\in V_h,\ \forall q_h\in Q_h\\ &(b)\ b(\vec{u}_h,q_h)=0\\ &\vec{u}_h(x,t=0)=0\end{aligned}}$$

Let's choose the FEM basis as such:

$\{\vec{\varphi}_i\}_{i=1}^{N_{V_h}}$ → basis of V_h, with $N_{V_h}<+\infty$

$\{\psi_j\}_{j=1}^{N_{Q_h}}$ → basis for Q_h, with $N_{Q_h}<+\infty$

We can write the following expansions:

$$\vec{u}_h(x,t)=\sum_{j=1}^{N_{V_h}} U_j(t)\cdot\vec{\varphi}_j(x) \qquad p_h(x,t)=\sum_{j=1}^{N_{Q_h}} P_j(t)\cdot\psi_j(x)$$

$$(a)\ \int_\Omega \frac{\partial \sum_{j=1}^{N_{V_h}} U_j(t)\cdot\vec{\varphi}_j(x)}{\partial t}\cdot\vec{\varphi}_i(x)\,dx + a\Big(\sum_{j=1}^{N_{V_h}} U_j(t)\cdot\vec{\varphi}_j(x),\ \vec{\varphi}_i(x)\Big) + b\Big(\vec{\varphi}_i(x),\ \sum_{j=1}^{N_{Q_h}} P_j(t)\cdot\psi_j(x)\Big) = F(\vec{\varphi}_i(x)) \quad \forall i$$

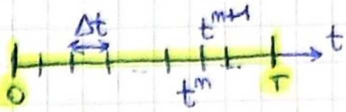
$$\sum_{j=1}^{N_{V_h}} \dot{U}_j(t)\cdot\int_\Omega \vec{\varphi}_j(x)\cdot\vec{\varphi}_i(x)\,dx + \sum_{j=1}^{N_{V_h}} U_j(t)\cdot a(\vec{\varphi}_j(x),\vec{\varphi}_i(x)) + \sum_{j=1}^{N_{Q_h}} P_j(t)\cdot b(\vec{\varphi}_i(x),\psi_j(x)) = F(\vec{\varphi}_i(x)) \quad \forall i$$

$$\sum_{j=1}^{N_{V_h}} \dot{U}_j(t)\cdot\underbrace{\int_\Omega \vec{\varphi}_j(x)\cdot\vec{\varphi}_i(x)\,dx}_{\substack{M_{ij}\\ \text{(mass matrix)}}} + \sum_{j=1}^{N_{V_h}} U_j(t)\cdot\underbrace{\int_\Omega \nabla\vec{\varphi}_j\cdot\nabla\vec{\varphi}_i\,dx}_{A_{ij}\text{(stiffness matrix)}} + \sum_{j=1}^{N_{Q_h}} P_j(t)\underbrace{\int_\Omega \psi_j(\nabla\cdot\vec{\varphi}_i)\,dx}_{B_{ji}} = \underbrace{\int_{\Gamma_N} \vec{g}(x,t)\cdot\vec{\varphi}_i\,d\gamma}_{F_i} \quad \forall i$$

$$\to M\dot{\vec{U}} + AU + B^T p = F$$

1.3 Semidiscretization in time

Backward implicit Euler (BE), $\theta = 1$: consider $t^n = n \cdot \Delta t$ and



$$\begin{cases} \vec{u}^{(n)} \approx u(t^n) \\ p^{(n)} \approx p(t^n) \end{cases}$$

Therefore, the Backward Euler is: $\forall n / \forall t^{n+1}$

$$\theta = 1: \begin{cases} \dfrac{\vec{u}^{(n+1)} - \vec{u}^{(n)}}{\Delta t} - \mu \Delta \vec{u}^{n+1} + (\vec{u}^{n+1} \cdot \nabla)\vec{u}^{n+1} + \nabla p^{n+1} = \vec{f}^{n+1} \\ \text{div} \cdot \vec{u}^{n+1} = 0 & \text{in } \Omega \\ \vec{u}^n = 0 & \text{on } \Gamma_D \\ \mu \dfrac{\partial \vec{u}^n}{\partial \vec{n}} - p \cdot \vec{n} = \vec{\xi} & \text{on } \Gamma_N \\ \vec{u}^0 = \vec{u}(x, t=0) & \text{in } \Omega, \end{cases}$$

↓ PER IL CODICE MI SERVE AVERE TUTTO n+1 a sinistra, e tutto n a destra

$$\frac{1}{\Delta t}\vec{u}^{(n+1)} - \mu \Delta \vec{u}^{n+1} + (\vec{u}^{n+1} \cdot \nabla)\vec{u}^{n+1} + \nabla p^{n+1} - \vec{f}^{n+1} = \frac{\vec{u}^{(n)}}{\Delta t}$$

1.4 Expected error estimate for the problem at 1.3 in terms of both Δt and h

TH: Under the conditions that make Stokes well-posed (LBB is necessary), then (BE+S) has a unique solution, which is UNCONDITIONALLY STABLE (stable $\forall t > 0$):

$$\forall t^{n+1}, \underbrace{\| \vec{u}^{n+1} - \vec{u}_h^{n+1} \|_V}_{H^1} + \underbrace{\| p^{n+1} - p_h^{n+1} \|_Q}_{L^2} \leq C(\Delta t + h^{k+1}) \cdot \| \vec{u}^{n+1} \|_V \cdot \| p^{n+1} \|_Q$$

EXERCISE 2

2.1

$$\begin{cases} \underbrace{-\operatorname{div}(\mu \cdot \nabla u)}_{\text{DIFFUSION}} + \underbrace{\vec{\beta}\cdot\nabla u}_{\text{ADVECTION}} + \underbrace{\gamma u}_{\text{REACTION}} = f & \text{in } \Omega \\ \mu \dfrac{\partial u}{\partial \vec{n}} = 0 & \text{on } \Gamma_N \\ u = \phi & \text{on } \Gamma_D \end{cascases}$$

→ Elliptic completa

where $\mu > 0$, $\vec{\beta} \in \mathbb{R}^2$ and $\gamma \geq 0$ are three given constants coefficients, while f and ϕ are two given functions, $\Omega \subset \mathbb{R}^2$ and $\Gamma_D \cap \Gamma_N = \emptyset$ and $\Gamma_D \cup \Gamma_N = \partial\Omega$

WEAK FORMULATION:

$V_0 = \{ v \in H^1_{\Gamma_D}(\Omega) : v = 0 \text{ on } \Gamma_D \}$ ← SPAZIO VETTORIALE

$V = \{ v \in H^1_{\Gamma_D}(\Omega) : v = \phi \text{ on } \Gamma_D \}$ ← SPAZIO AFFINE

Let's take $v \in V_0$, as a test function:

$$\cancel{-\int_\Omega \operatorname{div}(\alpha \nabla u) + \int_\Omega \beta \cdot \nabla u \, dx + \int_\Omega \gamma u \, dx = \int_\Omega f}$$

$$\boxed{-\int_\Omega \operatorname{div}(\mu \cdot \nabla u) \cdot v \, dx} + \int_\Omega \beta \nabla u \cdot v \, dx + \int_\Omega \gamma u \cdot v \, dx = \int_\Omega f \cdot v \, dx$$

$$\boxed{\int_\Omega \mu \nabla u \nabla v \, dx - \int_{\partial\Omega} \mu \nabla u \cdot \vec{n} v \, dx} + \int_\Omega \beta \nabla u \cdot v \, dx + \int_\Omega \gamma u \cdot v \, dx = \int_\Omega f \cdot v \, dx$$

$$\mu \int_\Omega \nabla u \nabla v \, dx - \int_{\Gamma_D} \mu \frac{\partial u}{\partial n} v \, dx - \cancel{\int_{\Gamma_N} \mu \frac{\partial u}{\partial n} v \, dx}^{\,0} + \int_\Omega \beta \nabla u \cdot v \, dx + \int_\Omega \gamma u \cdot v \, dx = \int_\Omega f v \, dx$$

Now we write $u = u_0 + R_g$, where $u_0 \in V_0$ and $R_g \in V$ is an arbitrarily lifting function s.t. $R_g = \phi$ on Γ_D:

$$\begin{cases} a(u,v) = \mu \int_\Omega \nabla u \cdot \nabla v \, dx + \beta \int_\Omega \nabla u \cdot v \, dx + \gamma \int_\Omega u \cdot v \, dx \\ F(v) = \int_\Omega f \cdot v \, dx + \mu \int_{\Gamma_D} \dfrac{\partial u}{\partial n} \cdot v \, dx \end{cases}$$

$a(u_0 + R_g, v) = F(v)$

$a(u_0, v) + a(R_g, v) = F(v)$

- $a(u_0, v) = \mu \int_\Omega \nabla u \cdot \nabla v \, dx +$

- $F(v) = \int_\Omega f v \, dx + \mu \int_{\Gamma_D} \frac{\partial u}{\partial n} \, dx = \quad u = u_0 + R_g$

$= \int_\Omega f v \, dx + \mu \int_{\Gamma_D} \cancel{\frac{\partial u_0}{\partial n} \, dx}^{\,0} + \mu \int_{\Gamma_D} \frac{\partial R_g}{\partial n} \, dx$

because we have $u_0 \in V_0$ which is 0 on the Dirichlet boundary

Weak:

$$\boxed{\begin{array}{l}\text{find } u_0 \in V_0 \text{ such that:} \\ a(u_0, v) = \underbrace{F(v) - a(R_g, v)}_{G(v)},\ \forall v \in V_0\end{array}}$$

2.2 LAX-MILGRAM lemma:

Assuming:
- V is an Hilbert space with defined norm $\|\cdot\|$
- F is linear functional on V ($F \in V'$) $\begin{cases} F \text{ linear} \\ F \text{ bounded}\end{cases}$
- a is a bilinear form, continuous and coercive:

$$\begin{cases} \text{continuous: } \exists M > 0 : |a(u,v)| \leq M \cdot \|u\| \cdot \|v\| & \forall u, v \in V \\ \text{coercivity: } \exists \alpha > 0 : a(v,v) \geq \alpha \cdot \|v\|^2 & \forall v \in V \end{cases}$$

Then:

$\exists!$ solution u to (PW)
Moreover: $\|u\| \leq \frac{1}{\alpha} \|f\|_{V'}$

2.3 $V_0^h = V_0 \cap X_h^2$; $V_h = V \cap X_h^2$

$N_{V_h} < +\infty$

$N_{V_{h,0}} < +\infty$

$$\boxed{\begin{array}{l}\text{Find } u_0^h \in V_0^h \text{ such that:} \\ a(u_{h,0}, v_h) = \underbrace{F(v_h) - a(R_g^h, v_h)}_{G(v_h)} \quad \forall v_h \in V_0^h\end{array}}$$

We have defined the discretized spaces over a UNIFORM PARTITION/MESH of Ω, for which we consider K_c, $c = 1, \dots, N_e$

Let's consider a basis over $V_{0,h}$ and V_h

$\{\varphi_i\}_{i=1}^{N_{V_h}} \rightarrow$ basis for

Let's write the expansion:

$$u_{h,0}(x) = \sum_{j=1}^{N_{V_h}} U_j \cdot \varphi_i(x)$$

$$\Rightarrow a(u_{h,0}, v_h) = \mu \int_\Omega \sum_{j=1}^{N_{V_h}} U_j \nabla\varphi_j(x) \nabla\varphi_i(x) \cdot dx +$$

$$+ \vec{b} \int_\Omega \sum_{j=1}^{N_{V_h}} U_j \nabla\varphi_j(x) \cdot \varphi_i(x)\, dx + \gamma \int_\Omega \varphi_i(x) \varphi_j(x)\, dx$$

$$G(v_h) = \left(\int_\Omega f \cdot \varphi_i(x)\, dx + \mu \int_{\Gamma_D} \frac{\partial R_g^h}{\partial n} dx \right)$$

Because of LM applies to (WP)

$\Rightarrow \exists!\ u_h \in V_h : \ \|u_h\| \le \frac{1}{\alpha} \|F\|_V$ $\rightarrow$ STABILITY!

$\Rightarrow \|u - u_h\| \le \frac{1}{\alpha} M \|u - w_h\| \quad \forall w_h \in V_h$

$\hookrightarrow \|u - u_h\| \le \frac{M}{\alpha} \inf_{u_h \in V_h} \|u - u_h\|$ $\rightarrow$ CONVERGENCE!

(you need $u \in H^{\overset{=3}{r+1}}(\Omega)$)