Numerical Methods for Partial Differential Equations
A.Y. 2024/2025

Written exam - July 8, 2025

Maximum score: 26. Duration: 2h 15m.

Answers to questions must be provided on paper (handwritten). Source code and picture files should be uploaded to the dedicated WeBeep folder. **All uploaded files (including source code) must be appropriately referenced when answering to questions on paper** (e.g. "The solution is depicted in `exercise-1-plot.png`").

Exercise 1.

Consider the problem

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \alpha \mathbf{u} - \mu \Delta \mathbf{u} + \boldsymbol{\nabla} p = \mathbf{0} & \text{in } \Omega,\ t \in (0,T)\ , \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega,\ t \in (0,T)\ , \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_{\mathrm{D}},\ t \in (0,T)\ , \\ \mu \dfrac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} = \boldsymbol{\psi} & \text{on } \Gamma_{\mathrm{N}},\ t \in (0,T)\ , \\ \mathbf{u} = \mathbf{0} & \text{in } \Omega,\ t \in (0,T)\ , \end{cases} \tag{1}$$

with $\alpha \geq 0$, $\boldsymbol{\psi}$ is a function depending on both $\mathbf{x}$ and t, $\Omega \subset \mathbb{R}^2$, $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$, $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}} = \partial\Omega$, and $T > 0$.

1.1. [*3 pt*] Write the weak formulation of (1).

1.2. [*1 pt*] Write the spatial approximation of (1) using the Taylor-Hood finite element spaces of degree 3 for the velocity $\mathbf{u}$ and 2 for the pressure p.

1.3. [*2 pt*] Approximate the problem obtained at point 1.2 by the implicit backward Euler method in time.

1.4. [*2 pt*] Write the expected error estimate for the problem obtained in point 1.3 in terms of both Δt and h.

1.5. [*4 pt*] Using `deal.II`, implement a finite element solver for problem (1). Use Taylor-Hood finite elements of degree 2 for the velocity and 1 for the pressure. Use the mesh found at https://github.com/michelebucelli/nmpde-labs-aa-24-25/

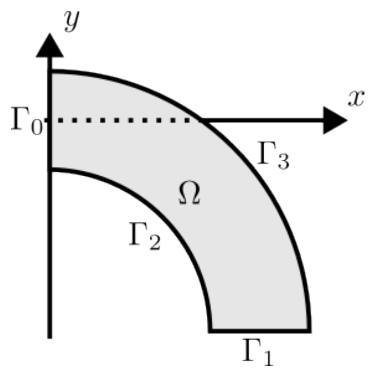

Figure 1: Domain for Exercise 1. Each boundary Γ_i has tag i in the mesh file.

`blob/main/examples/gmsh/mesh-pipe.msh`, and depicted in Figure 1. Consider the following data:

$$\Gamma_{\mathrm{D}} = \Gamma_2 \cup \Gamma_3 \;, \qquad \Gamma_{\mathrm{N}} = \Gamma_0 \cup \Gamma_1 \;,$$
$$\boldsymbol{\psi}(\mathbf{x}, t) = -p_0(\mathbf{x})\mathbf{n} \;, \qquad p_0(\mathbf{x}) = \begin{cases} 2 & \text{on } \Gamma_0 \;, \\ 1 & \text{on } \Gamma_1 \;, \end{cases}$$
$$\alpha = 100 \;, \qquad \mu = 1 \;, \qquad T = 1 \;.$$

Upload the source code of the solver.

1.6. [*3 pt*] Compute the solution to the problem, and upload:

1. a plot of the solution at the final time T;
2. a plot of the velocity magnitude profile along $R = \Omega \cap \{y = x\}$, that is along the line $y = x$ (**Hint:** use the Paraview filter "Plot over line");
3. a plot of the velocity magnitude over time at the point $\mathbf{x} = (1.25, -0.4)^T$ (**Hint:** combine the Paraview filters "Probe location" and "Plot data over time").

1.7. [*2 pt*] Now set $\alpha = 0$ and solve the problem again, using finite elements of degree 1 for both the velocity and the pressure. Upload a plot of the solution, and discuss the results in light of the theory.

1.8. [*2 pt*] Describe the preconditioning strategy you used to solve the linear system at each time step. How does the number of GMRES iterations change between timesteps? Justify it in light of the observed solution.

Exercise 2.

Consider the problem

$$\begin{cases} -\mathrm{div}(\alpha \boldsymbol{\nabla} u) + \boldsymbol{\beta} \cdot \boldsymbol{\nabla} u + \gamma u = f & \text{in } \Omega \ , \\ \alpha \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_{\mathrm{N}} \ , \\ u = \phi & \text{on } \Gamma_{\mathrm{D}} \ , \end{cases} \tag{2}$$

where $\alpha > 0$, $\boldsymbol{\beta} \in \mathbb{R}^2$ and $\gamma \geq 0$ are three given constant coefficients, while f and ϕ are two given functions, $\Omega \subset \mathbb{R}^2$ and $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$, $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}} = \partial\Omega$.

2.1. [*2 pt*] Write the weak formulation of problem (2).

2.2. [*3 pt*] Discuss the properties of existence and uniqueness of the solution.

2.3. [*2 pt*] Approximate (2) by the finite element method, using polynomials of order 2. Discuss its stability, convergence and error estimates.