Numerical Methods for Partial Differential Equations
A.Y. 2024/2025

Written exam - January 13, 2025

Maximum score: 26. Duration: 2h 15m.

Answers to questions must be provided on paper (handwritten). Source code and picture files should be uploaded to the dedicated WeBeep folder. **All uploaded files (including source code) must be appropriately referenced when answering to questions on paper** (e.g. "The solution is depicted in `exercise-1-plot.png`").

Questions marked with ($*$) should be skipped by multichance students.

Exercise 1.

Consider the parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + Lu = 0 & 0 < x < 1,\ t > 0\ , \\ u(x = 0, t) = \alpha & t > 0\ , \\ u(x = 1, t) = \beta & t > 0\ , \\ u(x, t = 0) = u_0(x) & 0 < x < 1\ , \end{cases} \tag{1}$$

where

$$Lu = -\frac{\partial^2 u}{\partial x^2} + \kappa \frac{\partial u}{\partial x}\ ,$$

$\kappa = 1$ and α and β are two constants.

1.1. [*2 pt*] Write the weak formulation of (1).

1.2. [*2 pt*] Discuss the coercivity properties of the bilinear form associated with the weak formulation.

1.3. [*1 pt*] Write the fully discrete approximation of (1) by using piecewise linear finite elements in space and the implicit backward Euler method in time.

1.4. [*3 pt*] When $\alpha = 0$ and $\beta = 0$, prove the stability of the scheme at point 1.3 and provide (without proof) the expected error estimate.

1.5. [*2 pt*] Using `deal.II`, implement a finite element solver for problem (1) using the data and discretization scheme described at previous points. Set $\alpha = 0$, $\beta = 1$, $u_0(x) = x$. Generate a mesh with 10 elements (i.e. with $h = 0.1$) using `deal.II`'s internal functions. Set the time step to $\Delta t = 0.01$ and use piecewise linear finite elements. Upload the source code and a plot of the solution at time $t = 1.0$.

1.6. [*1 pt*] ($*$) Upload a plot of the solution over time at point $x = 0.5$. **Hint:** combine the filters `Probe location` and `Plot data over time`.

1.7. [*2 pt*] Diffusion-advection problems are prone to instabilities (such as spurious oscillations) when the advection term dominates the diffusion. Repeat previous point setting $\kappa = 25$. Upload a plot of the solution at time $t = 1.0$ and discuss the results.

1.8. [*2 pt*] ($*$) Instabilities due to the dominating advection can be corrected through stabilization methods such as Streamline-Upwind Petrov-Galerkin (SUPG), that, for problem (1) and using linear elements, corresponds to adding the following term to the weak formulation:

$$\int_\Omega \tau \left(\frac{\partial u}{\partial t} + \kappa \frac{\partial u}{\partial x} \right) \kappa \frac{\partial v}{\partial x} \, dx \; ,$$

$$\tau = \frac{h}{2\kappa} \, \xi \left(\frac{h\kappa}{2} \right) \; ,$$

$$\xi(\theta) = \frac{1}{\tanh(\theta)} - \frac{1}{\theta} \; .$$

Implement this stabilization term in the solver, upload the plot of the solution with $\kappa = 25$ at time $t = 1.0$ and discuss the results.

Exercise 2.

Consider the problem

$$\begin{cases} -\mu\Delta\mathbf{u} + \alpha\mathbf{u} + \boldsymbol{\nabla} p = 0 & \text{in } \Omega \; , \\ \mathrm{div}\mathbf{u} = 0 & \text{in } \Omega \subset \mathbb{R}^2 \; , \\ \mu \dfrac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} = \mathbf{h} & \text{on } \partial\Omega \; , \end{cases} \tag{2}$$

where μ and α are two positive constants, $\mathbf{h}$ is a given vector and $\mathbf{n}$ is the outward unit vector normal to the boundary.

2.1. [*2 pt*] Provide the weak formulation of problem (2).

2.2. [*1 pt*] Approximate the weak formulation of problem (2) by using the Taylor-Hood finite elements of degree $k \geq 2$ for the velocity component $\mathbf{u}$ and degree $k - 1$ for the pressure component p.

2.3. [*1 pt*] ($*$) Is the approximation at point 2.2 stable? Explain why.

2.4. [*1 pt*] ($*$) Provide at least an example of finite element spaces that are not stable.

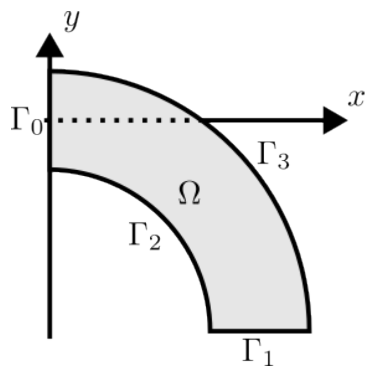

Figure 1: Domain for Exercise 2. Each boundary Γ_i has tag i in the mesh file.

2.5. [*2 pt*] ($*$) Write the expected error estimate for both velocity and pressure of the approximation at point 2.2. Do you expect spurious modes for this approximation? Motivate your answer, explaining what a spurious mode is.

2.6. [*3 pt*] Using `deal.II`, implement a finite element solver for the Stokes problem (2). Consider the domain of Figure 1. Set $\mu = 1$, $\alpha = 50$, and consider the following boundary conditions (in place of the ones of (2)):

$$\begin{cases} \mathbf{u} = \left[\dfrac{1}{4} - y^2, 0\right]^T & \text{on } \Gamma_0 \, , \\ \mu \dfrac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} = \mathbf{0} & \text{on } \Gamma_1 \, , \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_2 \cup \Gamma_3 \, . \end{cases}$$

Use finite elements of degrees 2 and 1 for velocity and pressure, respectively. Use the mesh of triangular elements which can be found at the link `https://github.com/michelebucelli/nmpde-labs-aa-24-25/tree/main/examples/gmsh/mesh-pipe.msh` (the boundary Γ_i has tag i in the mesh file). Upload the source code of the solver.

2.7. [*1 pt*] Plot and upload the solution. Use the filter `Glyph` to visualize the velocity field.