NonLinear Diffusion 2/3D

$$R(u,v) = \int_\Omega (\mu_0+\mu_1 u^2)\nabla u \nabla v + \int_\Omega \bar{b}\cdot\nabla u\, v + \int_\Omega \sigma u v - F(v)$$

$$R(u+\delta,\ v) = \int_\Omega (\mu_0+\mu_1(u+\delta)^2)\nabla(u+\delta)\nabla v + \int_\Omega \bar{b}\cdot\nabla u v + \bar{b}\cdot\nabla\delta\, v + \int_\Omega \sigma u v + \sigma\delta v - F(v)$$

$$R(u+\delta,v) - R(u,v) \simeq \int_\Omega (2\mu_1 u\delta)\nabla u\nabla v + \int_\Omega (\mu_0+\mu_1 u^2)\nabla\delta\nabla v + \underbrace{\int_\Omega \bar{b}\,\nabla\delta\, v + \int_\Omega \sigma\, \delta\, v}_{\text{Normal addition to matrix as in standard elliptic.}} + o(\|\delta\|^2)$$

NOTE: other non-linearities lead to diff. result!

NonLinear Diffusion 1D

$$\begin{cases} -\left[(\mu_0+\mu_1 u^2)u'\right]' + bu' + \sigma u = F \\ u = 0 \qquad \text{on } \delta\Omega \end{cases}$$

$$R(u,v) = \int_\Omega (\mu_0+\mu_1 u^2)u'v' + \int_\Omega \bar{b}\cdot u' v + \int_\Omega \sigma u v - F(v)$$

$$R(u+\delta,\ v) = \int_\Omega (\mu_0+\mu_1(u+\delta)^2)(u+\delta)'v' + \int_\Omega \bar{b}\, u'v + \bar{b}\cdot\delta' v + \int_\Omega \sigma u v + \sigma\delta v - F(v)$$

$$R(u+\delta,v) - R(u,v) \simeq \int_\Omega (2\mu_1 u\delta)\ u'\ v' + \int_\Omega (\mu_0+\mu_1 u^2)\ \delta' v' + \int_\Omega \bar{b}\,\delta' v + \int_\Omega \sigma\, \delta\, v + o(\|\delta\|^2)$$

Exact:

$$-\left[(\mu_0+\mu_1 u^2)u'\right]' + bu' + \sigma u = F$$

$$-\mu_0 u'' - (\mu_1 u^2 u')' + bu' + \sigma u = F$$

$$\underbrace{-\mu_1 (u^2 u')'}\ + Lu = F$$

$$(u^2u')' = (u^2)'u' + u^2 u'' =$$

$$= 2uu'u' + u^2u'' = 2u(u')^2 + u^2u''$$

$$-\mu_1\cdot 2u(u')^2 - \mu_1 u^2 u'' + bu' + \sigma u = F$$

$u = x$ $\qquad F = -\mu_1\cdot 2x + b + \sigma x$

$u' = 1$

$u'' = 0$

$u = \sin(2\pi x)$

$u' = 2\pi\cos(2\pi x)$

$u'' = 4\pi^2(-\sin(2\pi x))$

$$\begin{aligned} F = &-\mu_1\, 2\sin(2\pi x)\, 4\pi^2\cos^2(2\pi x) \\ &+\mu_1 \sin^2(2\pi x)\, 4\pi^2 \sin(2\pi x) \\ &+ b\, 2\pi\cos(2\pi x) \\ &+ \sigma \sin(2\pi x) \end{aligned} = \begin{aligned} &-\mu_1 8\pi^2 \sin(2\pi x)\cos^2(2\pi x) \\ &+ \mu_1 4\pi^2 \sin^3(2\pi x) \\ &+ b\,2\pi\cos(2\pi x) \\ &+ \sigma \sin(2\pi x) \end{aligned}$$