2024-07-15

EXERCISE 1

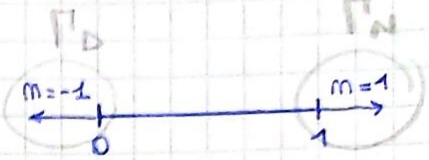
1.1 Consider the parabolic problem:

$$\begin{array}{rl} & \left\{\begin{array}{ll} \dfrac{\partial u}{\partial t} + Lu = f & 0<x<1,\ t>0 \\ u(x=0,t)=0 & t>0 \\ \varepsilon \dfrac{\partial u}{\partial x}(x=1,t)=0 & t>0 \\ u(x,t=0)=u_0(x) & 0<x<1 \end{array}\right. \end{array}$$

$\Gamma_D \leftarrow u(x=0,t)=0$

$\Gamma_N \leftarrow \varepsilon \frac{\partial u}{\partial x}(x=1,t)=0$

(*) $u(x,t=0)=u_0(x)$



in $x=1$: $\varepsilon \frac{\partial u}{\partial x}\cdot \vec{n} = \varepsilon \frac{\partial u}{\partial x}$

where $Lu = -\varepsilon \frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} - Ku$ $\quad \underbrace{(\varepsilon>0,\ K>0,\ b)}_{\in \mathbb{R},\ \text{constants}}$

$\Omega=(0,1)$; generally speaking, we take $V=H^1_0(\Omega)$ if $\Gamma_D=\partial\Omega$

But in this case since the boundary is not made by just Γ_D but also by Γ_N, then:

$$\cancel{V=H^1_0(\Omega)} \longrightarrow V=H^1_{\Gamma_N}(\Omega)$$

Moreover: $\Gamma_D=\{x=0\}$

$\Gamma_N=\{x=1\}$

Let's take the generic function $v\in V$:

$$\int_0^1 \frac{\partial u}{\partial t}\cdot v\,dx - \int_0^1 \varepsilon \frac{\partial^2 u}{\partial x^2}\cdot v\,dx + \int_0^1 b\frac{\partial u}{\partial x} v\,dx - \int_0^1 Kuv\,dx = \int_0^1 fv\,dx$$

$\frac{\partial^2 u}{\partial x^2} = \nabla^2 u = \Delta u$

to this term here I can apply Gauss-Green

$$-\int_\Omega \operatorname{div}(\mu\nabla u)\cdot v\,dx = \int_\Omega \mu\nabla u\nabla v\,dx - \int_{\partial\Omega}\mu \underbrace{(\nabla u\cdot\vec{n})}_{\text{scalar}} v\,d\gamma$$

$$\rightarrow \int_0^1 \frac{\partial u}{\partial t}\cdot v\,dx + \int_0^1 \varepsilon \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx - \cancel{\left[\varepsilon\nabla u\cdot n\, v\right]_{x=0}^{x=1}}\rightarrow 0 \; +$$

$$+\int_0^1 b\frac{\partial u}{\partial x} v\,dx - \int_0^1 Kuv\,dx = \underbrace{\int_0^1 fv\,dx}_{F(v)}$$

$$\Rightarrow \int_0^1 \frac{\partial u}{\partial t}\cdot v\,dx + a(u,v) = F(v)$$

The weak formulation reads as: "$\forall t>0$, find $u\in V$ s.t.

(WP) $\displaystyle\int_0^1 \frac{\partial u}{\partial t} v\,dx + a(u,v) = F(v) \quad \forall v\in V$

(*) AND $u(x,t=0)=u_0(x)$ $\quad \forall\ 0<x<1$"

condizioni iniziali nel tempo

1.2 Well-posedness is guaranteed provided $a(\cdot,\cdot)$ is only <u>weakly coercive</u>:

$$\boxed{\exists \lambda \geq 0 \quad s.t. \quad a(v,v) + \lambda \|v\|^2_{L^2(\Omega)} \geq \alpha \|v\|^2_V \qquad (\text{if } \lambda = 0 \rightarrow \text{coercivity})}$$

Let $v \in H^1_{\Gamma_D}$:

$$\varepsilon \int_0^1 \left(\frac{\partial v}{\partial x}\right)^2 dx + b \int_0^1 \frac{\partial v}{\partial x} v - K \underbrace{\int_0^1 v^2 dx}_{\|v\|^2_{L^2}} + \lambda \|v\|^2_{L^2} =$$

$$= \varepsilon \left\|\frac{\partial}{\partial x} v\right\|^2_{L^2} + \frac{1}{2} b \int_0^1 \frac{\partial}{\partial x}(v^2)\, dx + (\lambda - K)\|v\|^2_{L^2} =$$

$$= \varepsilon \left\|\frac{\partial}{\partial x} v\right\|^2_{L^2} + \frac{1}{2} b (v^2)\Big|_{x=1} + (\lambda - K)\|v\|^2_{L^2} =$$

$$= \varepsilon \left\|\frac{\partial}{\partial x} v\right\|^2_{L^2} + (\lambda - K)\|v\|^2_{L^2} + \frac{1}{2} b (v^2)\Big|_{\gamma=1}$$

Now we can assume $b \geq 0$, so $\frac{1}{2} b (v^2)\Big|_{x=1} = c \geq 0$

for $\lambda = K + \varepsilon \longmapsto a(v,v) + \lambda \|v\|^2_{L^2} =$

$$= \varepsilon \|v\|^2_{H^1} + c \underset{c \geq 0}{\geq} \varepsilon \|v\|^2_{H^1} \underset{\text{obv}}{\geq} \varepsilon \|v\|^2_{H^1_{\Gamma_D}}$$

! $\boxed{\int_0^1 \frac{\partial v}{\partial x} \cdot v\, dx = \frac{1}{2} v^2}$

1.3 For the discretization:

① <u>Semidiscretization in space: Galerkin formulation</u>

We have to restrict the weak formulation to V_h, a finite dimensional ~~...~~ subspace of V. It reads: find $u_h \in V_h$: $a(u_h, v_h) = F(v_h)$ $\forall v_h \in V_h$

Let us introduce a uniform partition (= mesh) of Ω into N_{el} subintervals (elements) K_c, $c = 1, \dots, N_{el}$. Let $X_h^r(\Omega)$ be the space of piecewise polynomials over the mesh elements, that is:

$$X_h^r(\Omega) = \{ v_h \in \mathcal{C}^0(\overline{\Omega}) : v_h(x) \in \mathbb{P}^r \ \forall x \in K_c \quad \forall c = 1, \dots, N_{el} \}$$

$\Rightarrow$ We take $V_h = X_h^r(\Omega) \cap H_0^1(\Omega)$ = finite element approximation of V, using piecewise polynomials of order r, $N_h = \dim(V_h)$

$$\boxed{\begin{aligned} &\forall t>0 \quad \text{find } u_h \in V_h \text{ s.t. } \int_0^1 \frac{\partial u_h}{\partial t} v_h dx + a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h \\ &\text{AND} \quad u_h(x, t=0) = u_0(x) \qquad \forall 0<x<1 \end{aligned}}$$

I continue with the semidiscretization in space but I write it in the MATRIX form:

I have to define the new basis: let

$\varphi_i, \varphi_j(x)$ for $i = 1, \dots, N_h$ the lagrangian basis functions of the space V_h; we look for $u_h \in V_h$ s.t:

$$u_h(x) = \sum_{j=1}^{N_h} U_j \varphi_j(x) \qquad x \in \overline{\Omega}$$

$U_j \in \mathbb{R}$ are the (unknown) control variables or degrees of freedom

the discrete weak formulation rewrites as:

$$\boxed{\text{find } U_j \text{ for } j = 1, \dots, N_h \text{ s.t.: } \sum_{j=1}^{N_h} U_j \, a(\varphi_j, \varphi_i) = F(\varphi_i) \quad \text{for } i = 1, \dots, N_h}$$

In matrix formulation: $Au = f$

$u \in \mathbb{R}^n$ $\quad u = (U_1, U_2, \dots, U_{N_h})^T$

$A \in \mathbb{R}^{N_h \times N_h}$ $\quad A_{ij} = a(\varphi_j, \varphi_i) = \int_0^1 \varepsilon \frac{\partial \varphi_j}{\partial x} \frac{\partial \varphi_i}{\partial x} dx + \int_0^1 b \frac{\partial \varphi_j}{\partial x} \varphi_i \, dx - \int_0^1 K \varphi_j \varphi_i \, dx$

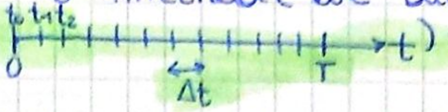
$f_i \in \mathbb{R}^{N_h}$ $\quad f_i = F(\varphi_i) = \int_0^1 f(x) \varphi_i(x) \, dx$

$\hookrightarrow \int_0^1 \frac{\partial u_h}{\partial t} \varphi_i(x) \, dx = \int_0^1 \frac{\partial}{\partial t} \left(\sum_{j=1}^{N_h} U_j(t) \cdot \varphi_j(x) \right) \varphi_i(x) \, dx =$

$$= \sum_{j=1}^{N_h} \dot{U}_j \int_0^1 \varphi_j \varphi_i \, dx = \sum_{j=1}^{N_h} \dot{U}_j M_{ij}$$

$\rightarrow$ with $u(x, t=0) = u_0(x)$ $\quad \forall 0<x<1$

② Semidiscretization in time: Crank-Nicholson

Considering the general θ-method (we divide the time in small intervals: 

):

$$\theta\text{-method}: \begin{cases} \dfrac{y^{n+1}-y^n}{\Delta t} = \theta \cdot f(t^{n+1}, y^{n+1}) + (1-\theta)\, f(t^n, y^n) \\ y^0 = y_0 \end{cases}$$

CN: $\theta = 1/2$

$$\text{CN}: \begin{cases} \dfrac{y^{n+1}-y^n}{\Delta t} = \dfrac{1}{2} f(t^{n+1}, y^{n+1}) + \dfrac{1}{2} f(t^n, y^n) \\ y^0 = y_0 \end{cases}$$

1.4

When $b=0$ and $f=0$, we can discuss the stability properties:

- TIME: we use CN $\rightarrow$ unconditionally stable. ($\theta \geq \frac{1}{2}$ we have unconditional stability), in particular we have Δt^2 convergence, convergence of order 2;
- SPAZIO: we didn't really demonstrate anything for space, but we can say:

$\forall t^n, \quad n = 1, \dots, N$

$$\| u(t^n) - u_h^n \|_{H^1_{\Gamma_D}(\Omega)} \leq C \left[h^2 + \Delta t^2 \right]$$

intervallo spaziale

intervallo temporale