Numerical Methods for Partial Differential Equations
A.Y. 2025/2026

Written exam - January 20, 2026

Maximum score: 26. Duration: 2h 15m.

Answers to questions must be provided on paper (handwritten). Source code and picture files should be uploaded to the dedicated WeBeep folder. **All uploaded files (including source code) must be appropriately referenced when answering to questions on paper** (e.g. "The solution is depicted in `exercise-1-plot.png`").

Questions marked with $(*)$ should be skipped by multichance students who have requested a 30 % reduction in the exam length.

Exercise 1.

Consider the problem

$$\begin{cases} -\mathrm{div}(\alpha \boldsymbol{\nabla} u) + \gamma u = f & \text{in } \Omega \,, \\ u = 0 & \text{on } \partial\Omega \,, \end{cases} \tag{1}$$

where $\alpha > 0$ and $\gamma \geq 0$ are two given constant coefficients, while f is a given function and $\Omega \subset \mathbb{R}^2$.

1.1. [*1 pt*] Write the weak formulation of problem (1).

1.2. [*2 pt*] Prove existence and uniqueness of the solution of the problem obtained at point 1.1.

1.3. [*2 pt*] Approximate (1) by the finite element method, using polynomials of degree 2 and provide the corresponding stability and error estimate.

1.4. [*2 pt*] $(*)$ Partition the domain Ω into two disjoint subdomains Ω_1 and Ω_2 and write one iteration of the Dirichlet-Neumann method. Is this method convergent?

1.5. [*2 pt*] Using `deal.II`, implement a finite element solver for problem (1) that uses the Dirichlet-Neumann method with relaxation, with the following settings:

$$\Omega = (0,3) \times (0,1) \,, \qquad \Omega_1 = (0,1) \times (0,1) \,, \qquad \Omega_2 = (1,3) \times (0,1) \,,$$
$$\alpha = 1 \,, \qquad \gamma = 1 \,, \qquad f(\mathbf{x}) = 1 \,.$$

The meshes for the two subdomains can be downloaded at `https://github.com/michelebucelli/nmpde-labs-aa-25-26/tree/main/lab-08/mesh`. Use linear finite elements. Upload the source code of the solver.

1.6. [*2 pt*] $(*)$ Set the maximum number of Dirichlet-Neumann iterations to 15. For the two relaxation values $\lambda = 1.0$ and $\lambda = 0.25$, upload a plot of the solution at the final iteration and a plot of the solution along the line $y = 0.5$.

Exercise 2.

Consider the problem

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \mu\Delta\mathbf{u} + \boldsymbol{\nabla} p = \mathbf{0} & \text{in } \Omega,\ t \in (0,T)\ , \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega,\ t \in (0,T)\ , \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_{\mathrm{D}},\ t \in (0,T)\ , \\ \mu\dfrac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} = \boldsymbol{\psi} & \text{on } \Gamma_{\mathrm{N}},\ t \in (0,T)\ , \\ \mathbf{u} = \mathbf{0} & \text{in } \Omega,\ t = 0\ , \end{cases} \tag{2}$$

where $\mu > 0$, $\boldsymbol{\psi}$ is a function depending on both $\mathbf{x}$ and t, $\Omega \subset \mathbb{R}^2$, $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$, $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}} = \partial\Omega$, and $T > 0$.

2.1. [*3 pt*] (*) Write the weak formulation of (2).

2.2. [*1 pt*] Write the spatial approximation of (2) using the Taylor-Hood finite element spaces of degree 3 for the velocity $\mathbf{u}$ and 2 for the pressure p.

2.3. [*2 pt*] Approximate the problem obtained at point 2.2 by the implicit backward Euler method in time. Then explain why the problem obtained at every timestep has a unique solution.

2.4. [*2 pt*] Write the expected error estimate for the problem obtained in point 2.3 in terms of both Δt and h.

2.5. [*4 pt*] Using `deal.II`, implement a finite element solver for problem (2). Use Taylor-Hood finite elements of degree 2 for the velocity and 1 for the pressure. Use the mesh found at `https://github.com/michelebucelli/nmpde-labs-aa-24-25/blob/main/examples/gmsh/mesh-pipe.msh`, and depicted in Figure 1. Consider the following data:

$$\Gamma_{\mathrm{D}} = \Gamma_2 \cup \Gamma_3\ , \qquad \Gamma_{\mathrm{N}} = \Gamma_0 \cup \Gamma_1\ ,$$

$$\boldsymbol{\psi}(\mathbf{x},t) = -p_0(\mathbf{x})\mathbf{n}\ , \qquad p_0(\mathbf{x}) = \begin{cases} 1 & \text{on } \Gamma_0\ , \\ 2 & \text{on } \Gamma_1\ , \end{cases}$$

$$\mu = 1\ , \qquad T = 1\ .$$

Upload the source code of the solver.

2.6. [*3 pt*] Compute the solution to the problem, and upload:

1. a plot of the solution at the final time T;

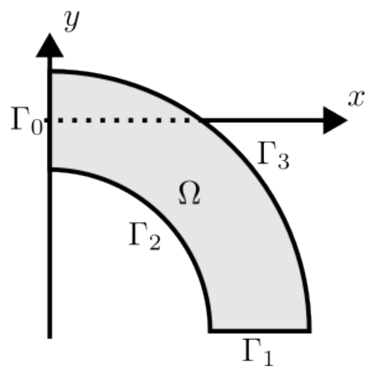

Figure 1: Domain for Exercise 1. Each boundary Γ_i has tag i in the mesh file.

2. a plot of the velocity magnitude profile along $R = \Omega \cap \{y = x\}$, that is along the line $y = x$ (**Hint:** use the Paraview filter "Plot over line");
3. $(*)$ a plot of the velocity magnitude over time at the point $\mathbf{x} = (1.25, -0.4)^T$ (**Hint:** combine the Paraview filters "Probe location" and "Plot data over time").