N MPDE @ HPC 20 January 2026

$\boxed{\text{Ex 1}}$

1.1 (1 PT)

$V = H^1_0(\Omega)$ Find $u \in V$: $a(u,v) = F(v)$ $\forall v \in V$

with $a(u,v) = \int_\Omega \alpha \nabla u \nabla v + \gamma u v$ $\quad F(v) = \int_\Omega f v$

1.2 (2 PTS)

Choose $\|v\| = |v|_{H^1(\Omega)} = \sqrt{\int_\Omega |\nabla v|^2} = \|\nabla v\|_{L^2}$

Then:

$$|a(u,v)| \leq \alpha \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \gamma \|u\|_{L^2} \|v\|_{L^2}$$

$$\leq \gamma C_p^2 \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} \quad \text{(Poincaré)}$$

$$\leq \underbrace{(\alpha + \gamma C_p^2)}_{M} \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} \quad \text{continuity}$$

$$a(v,v) = \alpha \|\nabla v\|^2_{L^2(\Omega)} + \gamma \|u\|_{L^2} \|v\|_{L^2} \geq \alpha \|\nabla v\|^2_{L^2}$$

(coercivity)

$(\circ)$ $|F(v)| \leq \|f\|_{L^2} \|v\|_{L^2} \leq C_p \|f\|_{L^2} \|\nabla v\|_{L^2}$

continuity of F

$\Rightarrow$ thanks to Lax-Milgram Lemma:

there exists a unique solution of problem in (1.1)

1.3 (2 PTS)

$V_h = \{ v_h \in C^0(\bar{\Omega}) \mid v_h|_K \in \mathbb{P}^2(K) \ \forall K \in \mathcal{T}_h, \ v_h|_{\partial\Omega} = 0 \}$

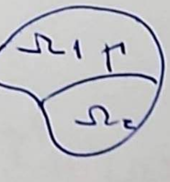
Find $u_h \in V_h$: $a(u_h, v_h) = F(v_h)$ $\forall v_h \in V_h$

$\|u_h\| \leq \frac{1}{\alpha} \|F\|_{V'} \leq \frac{C_p \|f\|_{L^2}}{\alpha}$ (from $(\circ)$) stability

$\|u - u_h\| \leq C h^2 |u|_{H^3(\Omega)}$ error estimate

1.4 (2 pts)

$$\begin{cases} -\mathrm{div}(\alpha \nabla u_1^{k+1}) + \gamma u_1^{k+1} = f & \text{in } \Omega_1 \\ u_1^{k+1} = \theta u_2^k + (1-\theta) u_1^k & \text{on } \Gamma \end{cases} \qquad \begin{cases} -\mathrm{div}(\alpha \nabla u_2^{k+1}) + \gamma u_2^{k+1} = f & \text{in } \Omega_2 \\ \alpha \frac{\partial u_2^{k+1}}{\partial n} = \alpha \frac{\partial u_1^{k+1}}{\partial n} & \text{on } \Gamma \end{cases}$$



There exists $\theta_{max} > 0$ s.t.

if $0 < \theta < \theta_{max}$ the DN method converges.

$\boxed{\text{Ex. 2}}$

2.1 (3 PTS)

$$V = [H^1_{\Gamma_D}(\Omega)]^2 = \{ \vec{v} \in [H^1(\Omega)]^2 : \vec{v}|_{\Gamma_D} = \vec{0} \}$$

$$Q = L^2(\Omega)$$

Find $\vec{u} \in V$, $p \in Q$: $\forall t > 0$

$$\begin{cases} \int_\Omega \frac{\partial \vec{u}}{\partial t} \vec{v} + a(\vec{u}, \vec{v}) + b(\vec{v}, p) = \int_{\Gamma_N} \underbrace{(\mu \frac{\partial \vec{u}}{\partial n} - p\vec{n})}_{\psi} \vec{v} \quad \forall \vec{v} \in V \\ b(\vec{u}, q) = 0 \quad \forall q \in Q \\ \vec{u}|_{t=0} = \vec{0} \end{cases}$$

$$a(u,v) = \int_\Omega \mu \nabla \vec{u} \nabla \vec{v} \qquad b(\vec{v}, p) = -\int_\Omega p \, \mathrm{div}\, \vec{v}$$

2.2 (1 PT)

$$V_h = \{ \vec{v}_h \in V : \vec{v}_h|_K \in [\mathbb{P}^3(K)]^2 \ \forall K \in \mathcal{T}_h \}$$

$$Q_h = \{ q_h \in Q : q_h|_K \in \mathbb{P}^2(K), \ q_h \text{ continuous}, \ \forall K \in \mathcal{T}_h \}$$

Find $\vec{u}_h \in V_h$, $p_h \in Q_h$: $\forall t > 0$

$$\begin{cases} \int_\Omega \frac{\partial \vec{u}_h}{\partial t} v_h + a(u_h, v_h) + b(v_h, p_h) = \int_{\Gamma_N} \psi \, v_h \quad \forall v_h \in V_h \\ b(\vec{u}_h, q_h) = 0 \quad \forall q_h \in Q_h \\ \vec{u}_h|_{t=0} = \vec{0} \end{cases}$$

2.3 (2 PT) $t^k = k\Delta t$, $k = 0, \dots$

$\forall k \geq 0$, Find $\vec{u}^{k+1} \in V_h$, $p^{k+1} \in Q_h$:

$$\begin{cases} \int_\Omega \frac{1}{\Delta t} (\vec{u}_h^{k+1} - u_h^k) \vec{v}_h + a(\vec{u}_h^{k+1}, \vec{v}_h) + b(\vec{v}_h, p_h^{k+1}) = \int_\Omega \psi^{k+1} v_h \quad \forall \vec{v}_h \in V_h \\ b(\vec{u}_h^{k+1}, q_h) = 0 \quad \forall q_h \in Q_h \end{cases}$$

$$\vec{u}_h^0 = \vec{0}$$

2.4 (2 PT)

$$\|\vec{u}^k - u_h^k\|_{H^1(\Omega)} + \|p^k - p_h^k\|_{L^2(\Omega)} \leq C(\Delta t + h^3)$$

C is a constant depending on $\|\vec{u}\|_{H^4(\Omega)}$ and $\|p\|_{H^3(\Omega)}$.