Numerical Methods for Partial Differential Equations
A.Y. 2024/2025

Written exam - February 10, 2025

Maximum score: 26. Duration: 2h 15m.

Answers to questions must be provided on paper (handwritten). Source code and picture files should be uploaded to the dedicated WeBeep folder. **All uploaded files (including source code) must be appropriately referenced when answering to questions on paper** (e.g. "The solution is depicted in `exercise-1-plot.png`").

Questions marked with $(*)$ should be skipped by multichance students.

Exercise 1.

Consider the parabolic problem

$$\begin{cases} -u'' - xu' + u = f & 0 < x < 1\ , \\ u(0) = u(1) = 0\ , \end{cases} \tag{1}$$

where $\boldsymbol{f(x) = 0}$.

1.1. [*2 pt*] Write the weak formulation of (1), under the form:

$$\text{find } u \in V \text{ such that } a(u,v) = F(v) \text{ for all } v \in V\ ,$$

and provide the definitions of V, $a(u,v)$ and $F(v)$.

1.2. [*3 pt*] Prove that the bilinear form $a(\cdot,\cdot)$ is continuous, coercive and find its continuity constant M and its coercivity constant α.

1.3. [*1 pt*] Find the expression of the exact solution, that is $u(x) = \dots$

1.4. [*2 pt*] Approximate the weak problem in Exercise 1.1 by piecewise quadratic finite elements and provide the error estimate for both $H^1(0,1)$ and $L^2(0,1)$ norms.

1.5. [*1 pt*] Consider now the case where f is different from zero, and derive the expression for f such that the exact solution to (1) is $u_{\text{ex}} = x(1-x)$.

1.6. [*3 pt*] Using `deal.II`, implement a finite element solver for (1). Consider the definition of f computed at previous point. The program must solve the problem for different values of the mesh size h and compute the L^2 and H^1 norms of the error against the exact solution. Use the internal functions of `deal.II` to generate the mesh. Upload the source code of the solver.

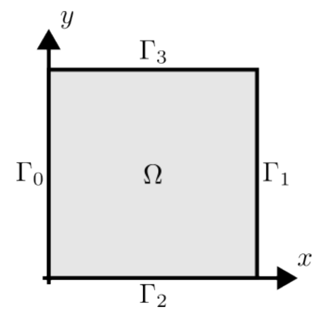

Figure 1: Domain for Exercise 2. Each boundary subset Γ_i corresponds to the tag i in the provided mesh files.

1.7. [*1 pt*] Compute the solution setting $h = 0.1$ and using finite elements of degree 1. Upload a plot of the solution.

1.8. [*2 pt*] Using finite elements of degree 1, perform a convergence study with respect to h. Repeat the convergence test using finite elements of degree 2. Report the results and discuss them in light of the theory.

Exercise 2.

Consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \boldsymbol{\Delta} \mathbf{u} + \boldsymbol{\nabla} p = 0 & \text{in } \Omega \subset \mathbb{R}^2, t > 0 \ , \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, t > 0 \ , \\ \mathbf{u}(\mathbf{x}, t = 0) = \mathbf{u}_0 & \text{in } \Omega \ , \\ \left(\dfrac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} \right)(\mathbf{x}, t) = \boldsymbol{\varphi}(\mathbf{x}, t) & \text{on } \partial\Omega, t > 0 \ , \end{cases} \tag{2}$$

where $\mathbf{u}_0$ and $\boldsymbol{\varphi}$ are two given functions, sufficiently regular.

2.1. [*2 pt*] Find the weak formulation of problem (2).

2.2. [*1 pt*] Approximate the weak formulation by the Taylor-Hood finite elements in space of degree 3 for the velocity and 2 for the pressure, and by the backward Euler method in time.

2.3. [*1 pt*] At any time step $t^n = n\Delta t$, discuss the well-posedness of the corresponding problem.

2.4. [*3 pt*] Is the approximation of Exercise 2.2 stable in time? Provide the corresponding error estimate.

2.5. [*3 pt*] Using `deal.II`, implement a finite element solver for (2). Consider the square domain $\Omega = (0,1)^2$ depicted in Figure 1. Consider the following problem data:

$$\begin{aligned}
\mathbf{u}_0(\mathbf{x}) &= \mathbf{0} && \text{in } \Omega \,, \\
\boldsymbol{\varphi}(\mathbf{x}, t) &= -g(\mathbf{x})\mathbf{n} \,, && \text{on } \partial\Omega, t > 0 \,, \\
g(\mathbf{x}) &= \begin{cases} 0 & \text{on } \Gamma_0 \,, \\ 1 & \text{on } \Gamma_1 \,, \\ 2 & \text{on } \Gamma_2 \,, \\ 3 & \text{on } \Gamma_3 \end{cases} && \text{on } \partial\Omega \,.
\end{aligned}$$

Use the mesh $\mathcal{T}_h$ of $h = 0.1$ that can be found at `https://github.com/michelebucelli/nmpde-labs-aa-24-25/tree/main/examples/gmsh`. Use finite elements of degree 2 and 1 for velocity and pressure respectively. Set the time step to $\Delta t = 0.1$ and the final time to $T = 1$. Upload the source code of the solver.

Hint: for simplicity, you can reassemble every time step the whole system (that is, you are not required to store matrices that do not change between time steps).

Hint: to evaluate the velocity on quadrature nodes of the current cell, you can use the following code snippet:
`fe_values[velocities].get_function_values(solution, velocity_loc)`
Refer to `deal.II`'s documentation for `FEValuesViews::Vector` for more information.

2.6. [*1 pt*] Upload a plot of the solution at time $t = 1$.