Numerical Methods for Partial Differential Equations
A.Y. 2024/2025

Written exam - June 4, 2025

Maximum score: 26. Duration: 2h 15m.

Answers to questions must be provided on paper (handwritten). Source code and picture files should be uploaded to the dedicated WeBeep folder. **All uploaded files (including source code) must be appropriately referenced when answering to questions on paper** (e.g. "The solution is depicted in `exercise-1-plot.png`").

Questions marked with (∗) should be skipped by multichance students.

Exercise 1.

Consider the problem

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\varepsilon(1+x^2)\dfrac{\mathrm{d}u}{\mathrm{d}x}\right) + \dfrac{\mathrm{d}u}{\mathrm{d}x} = 0 \quad 0 < x < 1, \quad 0 < \varepsilon \ll 1 \\ u(0) = 0 \ , \\ u(1) = 1 \ . \end{cases} \tag{1}$$

1.1. [*2 pt*] Write problem (1) under the weak form:

$$\text{find } u \in V \text{ such that } a(u,v) = F(v) \qquad \forall v \in V \ , \tag{2}$$

and

- provide the expression of V, $a(\cdot,\cdot)$ and $F(\cdot)$;
- find the coercivity constant α;
- find the continuity constant M;
- find the norm $\|F\|_{V'}$.

1.2. [*1 pt*] Using the *a priori* bound

$$\|u\|_V \leq \frac{1}{\alpha}\|F\|_{V'} \ ,$$

which is a consequence of the Lax-Milgram lemma, provide an upper bound of $\|u\|_V$ (that is, $\|u\|_V \leq \dots$).

1.3. [*3 pt*] Approximate problem (2) by the Galerkin method using piecewise linear finite elements. Then:

- using the stability estimate of the Céa lemma, find an upper bound of $\|u_h\|_V$ (that is, $\|u_h\|_V \leq \dots$);
- do you expect that this approximation is stable when ε is very small?

1.4. [*1 pt*] For the approximation considered at point 1.3, provide the expected error estimate (that is, $\|u - u_h\|_V \leq \dots$).

1.5. [*3 pt*] Using `deal.II`, implement a finite element solver for (2). Use piecewise linear finite elements. Use the internal functions of `deal.II` to generate the mesh. Upload the source code of the solver.

1.6. [*1 pt*] Compute the solution setting $\varepsilon = 1$ and $h = 0.1$, and upload a plot of the solution. **Hint:** use the Paraview filter "Plot over line".

1.7. [*2 pt*] Repeat point 1.6 setting $\varepsilon = 10^{-2}$. What do you observe?

1.8. [*3 pt*] Transport-dominated problems can be stabilized through the *Generalized Least-Squares* method. In this case, it corresponds to adding the following term to the weak formulation:

$$\int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\varepsilon(1+x^2)\frac{\mathrm{d}u}{\mathrm{d}x}\right) + \frac{\mathrm{d}u}{\mathrm{d}x}\right)\left(h\frac{\mathrm{d}}{\mathrm{d}x}\left(\varepsilon(1+x^2)\frac{\mathrm{d}v}{\mathrm{d}x}\right) + h\frac{\mathrm{d}v}{\mathrm{d}x}\right)\mathrm{d}x \ .$$

Modify the solver to implement this term, compute the solution setting $\varepsilon = 10^{-2}$ and $h = 0.1$, and upload the source code and a plot of the solution. Compare the results with the solution of previous points. **Hint:** you can compute $\frac{\mathrm{d}}{\mathrm{d}x}\left(\varepsilon(1+x^2)\frac{\mathrm{d}u}{\mathrm{d}x}\right)$ and exploit the fact that the elements are linear to simplify the implementation.

Exercise 2.

Consider the problem

$$\begin{cases} -\mu\Delta\mathbf{u} + \boldsymbol{\nabla} p = \mathbf{f} & (x,y) \in \Omega = (-1,1)^2 \ , \\ \operatorname{div}\mathbf{u} = 0 & (x,y) \in \Omega \ , \\ \mathbf{u} = 0 & \text{if } y = -1 \text{ or } y = 1 \ , \\ \mu\dfrac{\partial u}{\partial \mathbf{n}} - p\mathbf{n} = 0 & \text{if } x = -1 \text{ or } x = 1 \ , \end{cases}$$

with $\mu = 2$ and $\mathbf{f} = [1 + \sin^2(x), 0]^T$.

2.1. [*2 pt*] Write its weak formulation.

2.2. [*1 pt*] Approximate it by the Taylor-Hood finite element method of degree 3 for the velocity and degree 2 for the pressure.

2.3. [*2 pt*] Is the approximation at point 2.2 stable? Motivate your answer.

2.4. [*1 pt*] Provide the error estimate of the approximation at point 2.2.

2.5. [*2 pt*] Provide the algebraic form (i.e., the linear system) associated with the approximation at point 2.2. Is this system non-singular? Explain why.

2.6. [*2 pt*] Using `deal.II`, implement a finite element solver for problem (2). Use finite elements of degrees 2 and 1 for velocity and pressure, respectively. Use the mesh of triangular elements which can be found at the link `https://github.com/michelebucelli/nmpde-labs-aa-24-25/tree/main/examples/gmsh/mesh-square-2.msh`. Upload the source code of the solver and a plot of the solution.