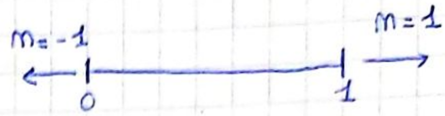
2025-06-04

EX 1

1.1

$$\begin{cases} -\dfrac{d}{dx}\left(\varepsilon(1+x^2)\dfrac{du}{dx}\right)+\dfrac{du}{dx}=0 & 0<x<1 \quad 0<\varepsilon<<1 \\ u(0)=0 & \to \Gamma_D \\ u(1)=1 & \to \Gamma_D \neq 0 \Rightarrow \text{(LIFTING)} \end{cases}$$



WEAK FORMULATION:

Spazio funzioni test: $V_0 = H^1_0((0,1))$

Spazio soluzioni: $V = \{ v \in H^1((0,1)) \text{ s.t. } \underbrace{v(1)=1}_{\text{Dirichlet boundary condition}} \}$

Let $v \in V_0$:

$$\boxed{-\int_0^1 \frac{d}{dx}\Big(\underbrace{\varepsilon(1+x^2)}_{=\mu(x)}\frac{du}{dx}\Big)\cdot v\,dx} + \int_0^1 \frac{du}{dx}\cdot v\,dx = 0$$

$$\int_0^1 \varepsilon(1+x^2)\frac{du}{dx}\frac{dv}{dx}\,dx - \cancel{\left[\varepsilon(1+x^2)\frac{du}{dx}\cdot n \cdot v\right]_{x=0}^{x=1}}^{\,0} + \int_0^1 \frac{du}{dx}\cdot v\,dx = 0$$

because $v(x) \in V_0$, and $v(0) = v(1) = 0$

$$\underbrace{\int_0^1 \varepsilon(1+x^2)\frac{du}{dx}\frac{dv}{dx}\,dx + \int_0^1 \frac{du}{dx}\,v\,dx}_{a(u,v)} = \underbrace{0}_{F(v)}$$

$$\boxed{\text{find } u \in V \text{ s.t. } a(u,v) = F(v) \quad \forall v \in V_0} \quad \text{(WP)}$$

LIFTING: I want to have solutions in V_0; suppose $\exists R_g \in V$ s.t. $u = \hat{u} + R_g$ and $\hat{u} \in V_0$

$$a(\hat{u}+R_g, v) = F(v) = 0$$

$$a(\hat{u}, v) + a(R_g, v) = 0$$

$$a(\hat{u}, v) = \underbrace{-a(R_g, v)}_{G(v)}$$

nel testo ci viene chiesto $F(v)$, che dovrebbe essere il termine noto $G(v)$

- $a(\hat{u}, v) = \displaystyle\int_0^1 \varepsilon(1+x^2)\frac{d\hat{u}}{dx}\frac{dv}{dx}\,dx + \int_0^1 \frac{d\hat{u}}{dx}\,v\,dx$

- $G(v) = -a(R_g, v)$

Final (WP):

$$\boxed{\begin{array}{l}\text{Find } \hat{u} \in V_0 \text{ s.t.:} \\ a(\hat{u}, v) = G(v) \qquad \forall v \in V_0\end{array}}$$

GG:

$$-\frac{1}{2}\int_0^1 v^2\,dx + \cancel{\frac{1}{2}\left[v^2\right]_0^1} = \qquad v\Big|_{x=0,\,x=1} = 0$$

$$= -\frac{1}{2}\int_0^1 v^2\,dx$$

COERCIVITY:

$$a(v,v) = \int_0^1 \varepsilon(1+x^2)\frac{dv}{dx}\frac{dv}{dx}\,dx + \int_0^1 \frac{dv}{dx}\,v\,dx =$$

$$= \int_0^1 \varepsilon(1+x^2)\left(\frac{dv}{dx}\right)^2 dx + \int_0^1 \frac{dv}{dx}\,v\,dx \geq$$

$$\geq \underbrace{\varepsilon\cdot\left(1+\min_x(x^2)\right)}_{\geq 0}\cdot\int_0^1\left(\frac{dv}{dx}\right)^2 dx + \int_0^1 \frac{dv}{dx}\,v\,dx =$$

$$= \mu_{min}\,\|v'\|^2_{L^2(0,1)} - \frac{1}{2}\int_0^1 v^2\,dx \qquad ?? \qquad -\frac{1}{2} < 0$$

disuguaglianza triangolare

CONTINUITY:

$$|a(u,v)| \leq \left|\int_0^1 \varepsilon(1+x^2)\frac{du}{dx}\frac{dv}{dx}\,dx\right| + \left|\int_0^1 \frac{du}{dx}\cdot v\,dx\right| \leq$$

C-S

$$\leq \max_{x\in(0,1)}|\varepsilon(1+x^2)|\int_0^1 \frac{du}{dx}\frac{dv}{dx}\,dx + \left|\int_0^1 u'\,v\,dx\right| \leq$$

$$\leq \mu_{max}\cdot\|u'\|_{L^2(0,1)}\cdot\|v'\|_{L^2(0,1)} + \|u'\|_{L^2(0,1)}\cdot\|v\|_{L^2(0,1)} \leq$$

$$\leq \mu_{max}\|u\|_{H^1}\|v\|_{H^1} + \|u\|_{H^1}\cdot\|v\|_{H^1} =$$

per $x=1$: 2ε

$$= \underbrace{(2\varepsilon+1)}_{=M}\cdot\|u\|_{H^1}\cdot\|v\|_{H^1}$$

NORM $\|F\|_{V'}$:

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EXERCISE 2

TIME - DEPENDENT STOKES

$$\begin{cases} -\mu \Delta \vec{u} + \nabla p = \vec{f} & (x,y) \in \Omega = (-1,1)^2 \\ \operatorname{div} \vec{u} = 0 & (x,y) \in \Omega \\ \vec{u} = 0 & \text{if } y=-1 \text{ or } y=1 \,]\ \Gamma_D \\ \mu \dfrac{\partial \vec{u}}{\partial \vec{n}} - p\vec{n} = 0 \ (*) & \text{if } x=-1 \text{ or } x=1 \,]\ \Gamma_N \end{cases}$$

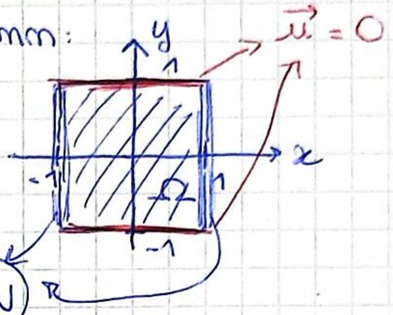
$d=2$

With $\mu = 2$ and $f = [1+\sin^2(x), 0]^T$.

2.1 Mixed dirichlet - neumann:

$V = \left[H^1_{\Gamma_D}(\Omega)\right]^2$

$Q = L^2(\Omega)$



Let $\vec{v} \in V$, $q \in Q$:

(a) $\displaystyle\int_\Omega -\mu \Delta \vec{u} \cdot \vec{v}\, dx + \int_\Omega \nabla p \cdot \vec{v}\, dx = \int_\Omega \vec{f} \cdot \vec{v}\, dx$

$\displaystyle\int_\Omega \mu \nabla \vec{u}\, \nabla \vec{v}\, dx - \int_{\partial\Omega} \mu \underbrace{(\nabla \vec{u} \cdot \vec{n})}_{\frac{\partial \vec{u}}{\partial \vec{n}}} \cdot \vec{v}\, d\gamma - \int_\Omega \operatorname{div} \vec{v} \cdot p\, dx + \int_{\partial\Omega} p \cdot \vec{n} \cdot \vec{v}\, d\gamma = \int_\Omega \vec{f}\, \vec{v}\, dx$

$\displaystyle\int_\Omega \mu \nabla \vec{u} \cdot \nabla \vec{v}\, dx - \int_\Omega \nabla \cdot \vec{v}\, p\, dx - \int_{\partial\Omega} \left(\mu \frac{\partial \vec{u}}{\partial \vec{n}} - p\vec{n}\right) \cdot \vec{v}\, d\gamma = \int_\Omega \vec{f} \cdot \vec{v}\, dx$

because $v \in V$ and $v|_{\Gamma_D} = 0$

$\displaystyle\cancel{\int_{\Gamma_D} (\dots)\, \vec{v}\, d\gamma}^{\,0} + \cancel{\int_{\Gamma_N} \left(\mu \frac{\partial \vec{u}}{\partial \vec{n}} - p\vec{n}\right) \cdot \vec{v}\, d\gamma}^{\,0\ (*)}$

$\displaystyle\Rightarrow \underbrace{\int_\Omega \mu \nabla \vec{u} \cdot \nabla \vec{v}\, dx}_{=a(\vec{u},\vec{v})} - \underbrace{\int_\Omega \nabla \cdot \vec{v} \cdot p\, dx}_{=b(\vec{v},p)} = \underbrace{\int_\Omega \vec{f} \cdot \vec{v}\, dx}_{F(\vec{v})}$

(b) $\displaystyle\underbrace{-\int_\Omega \operatorname{div} \vec{u} \cdot q}_{b(\vec{u},q)} = 0$

Find $\vec{u} \in V$, $p \in Q$ s.t.	
(a) $a(\vec{u},\vec{v}) + b(\vec{v},p) = F(\vec{v})$	$\forall \vec{v} \in V$
(b) $b(\vec{u},q) = 0$	$\forall q \in Q$

(WP)

2.2 MESH-TRIANGULATION:

① Partition of Ω: K_c; $c = 1, \dots, N_{el}$ s.t. $\nearrow \bigcup_{i=1}^{N_{el}} K_i = \Omega$; $\searrow K_i \cap K_j = \emptyset \quad \forall i \neq j$

② $X_h^{r,d}(\Omega) = \{ v \in [C^0(\Omega)]^d,\ v_h|_{K_j} \in [\mathbb{P}^r(K_j)]^d \ \forall j \}$

- $V_h = V \cap X_h^{3,2}(\Omega)$ → fem space for vel ! $N_{V_h} < +\infty$
- $Q_h = Q \cap X_h^{2,1}(\Omega)$ → fem space for press ! $N_{Q_h} < +\infty$

(WP) is:

$$\boxed{\begin{array}{ll} \text{Find } \vec{u}_h \in V_h,\ p_h \in Q_h \text{ s.t.:} & \\ (a)\ a(\vec{u}_h, \vec{v}_h) + b(\vec{v}_h, p_h) = F(\vec{v}_h) & \forall \vec{v}_h \in V_h \\ (b)\ b(\vec{u}_h, q_h) = 0 & \forall q_h \in Q_h \end{array}} \quad (G)$$

$\{ \vec{\varphi}_j(x) \}_{j=1}^{N_{V_h}}$ → basis for V_h

$\{ \psi_j(x) \}_{j=1}^{N_{Q_h}}$ → basis for Q_h

Expansions:

$\vec{u}_h(x) = \sum_{j=1}^{N_{V_h}} u_j \cdot \vec{\varphi}_j(x)$ $\qquad \vec{v}_h(x) = \vec{\varphi}_i(x)$

$p_h(x) = \sum_{j=1}^{N_{Q_h}} p_j \cdot \psi_j(x)$ $\qquad q_h(x) = \psi_k(x)$

$\vec{u} = [u_1, \dots, u_{N_{V_h}}]$; $\qquad \vec{p} = [p_1, \dots, p_{N_{Q_h}}]$

Therefore:

(a) $a\left(\sum_{j=1}^{N_{V_h}} u_j \cdot \vec{\varphi}_j(x), \vec{\varphi}_i(x)\right) + b\left(\vec{\varphi}_i(x), \sum_{j=1}^{N_{Q_h}} p_j \cdot \psi_j(x)\right) = F(\vec{\varphi}_i(x)) \quad \forall i$

$\sum_{j=1}^{N_{V_h}} u_j \cdot \underbrace{a(\vec{\varphi}_j(x), \vec{\varphi}_i(x))}_{A_{ij}} + \sum_{j=1}^{N_{Q_h}} p_j \cdot \underbrace{b(\vec{\varphi}_i(x), \psi_k(x))}_{B_{ji}} = F(\vec{\varphi}_i(x)) \quad \forall i$

⇒ (a): $A\vec{u} + B^T\vec{p} = \vec{F}$

(b) $b\left(\sum_{j=1}^{N_{V_h}} u_j \vec{\varphi}_j(x), \psi_k(x)\right) = 0 \quad \forall i$

$\sum_{j=1}^{N_{V_h}} u_j \underbrace{b(\vec{\varphi}_j(x), \psi_k(x))}_{B_{ij}} = 0 \quad \forall i$

⇒ (b): $B\vec{u} = 0$

2.3 STABILITY for TAYLOR-HOODS:

TR: $\exists!$ of $\vec{u}_h, p_h$

Assume that:

① $a: V \times V \to \mathbb{R}$ is $\nearrow$ bilinear, → continuous, $\searrow$ coercive

② Assume that $b : V \times Q \to \mathbb{R}$ is: bilinear, continuous, LBB satisfied

$\Rightarrow \exists!\ \vec{u}_h, p_h$ solution to (G); moreover:

- $\| \vec{u}_h \|_V \leq \frac{1}{\alpha} \| F \|_{V'}$
- $\| p_h \|_Q \leq \underbrace{\boxed{\frac{1}{\beta}\left(1 + \frac{\gamma}{\alpha}\right)}}_{= C_1} \| F \|_{V'}$

} Stability properties $\Rightarrow$ we have (G) stable

2.4 ERROR ESTIMATE for T-H elements.

$$\underbrace{\| \vec{u} - \vec{u}_h \|_V}_{\text{error on velocity}} + \underbrace{\| p - p_h \|_Q}_{\text{error on pressure}} \leq C_1\, h^{\overset{3}{k+1}} \left(\| \vec{u} \|_{[H^{\overset{4}{k+2}}(\Omega)]^2} + \| p \|_{H^{\overset{3}{k+1}}(\Omega)} \right)$$

$k+1$: degree

2.5 NON-SINGULARITY of the system: $\to \begin{cases} A\vec{u} + B^T \vec{p} = \vec{F} \\ B\vec{u} = 0 \end{cases}$

S is non-singular because LBB is satisfied:

$\exists \beta > 0$ s.t. $\forall q_h \in Q_h\ \exists \vec{v}_h \in V_h : b(\vec{v}_h, q_h) \geq \beta \| \vec{v}_h \|_V \| q_h \|_Q$

$$\Leftrightarrow \exists \beta \text{ s.t. } \inf_{\substack{q_h \in Q_h \\ q_h = \emptyset}} \sup_{\substack{\vec{v}_h \in V_h \\ \vec{v}_h \neq \emptyset}} \frac{b(\vec{v}_h, q_h)}{\| \vec{v}_h \|_V \cdot \| q_h \|_Q} \geq \beta$$