Parabolic
+ LIFTING

EXERCISE 1
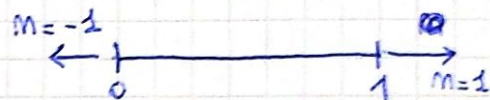
(1.1) Parabolic:

$$\begin{cases} \frac{\partial u}{\partial t} + Lu = 0 & 0 < x < 1,\ t > 0 \\ u(x=0, t) = \alpha & t > 0 \\ u(x=1, t) = \beta & t > 0 \\ u(x, t=0) = u_0(x) & 0 < x < 1 \end{cases} \Bigg] \rightarrow \Gamma_D$$

$d = 1$

$\boxed{dim = 1}$ (per codice)

where $Lu = -\frac{\partial^2 u}{\partial x^2} + K\frac{\partial u}{\partial x}$, $K=1$, α and β are 2 constants.

WEAK:

~~(scribbled out text)~~

$M = -1 \longleftarrow \overset{0}{\vert} \cdots \overset{1}{\vert} \longrightarrow M=1$

$$V = \{ v \in H^1(\Omega):\ \underbrace{v(0,t) = \alpha,\ v(1,t) = \beta}_{\text{Dirichlet boundary conditions}} \}$$

$$V_0 = \{ v \in H^1(\Omega):\ v|_{\Gamma_D} = 0 \} = \boxed{H^1_{\Gamma_D}(\Omega)}$$

Let $v \in V_0$:

$$\int_\Omega \frac{\partial u}{\partial t} \cdot v\, dx - \underbrace{\int_\Omega \frac{\partial^2 u}{\partial x^2} \cdot v\, dx}_{(A)} + \underbrace{\int_\Omega \frac{\partial u}{\partial x} v\, dx}_{} = 0$$

$$(A) = +\int_\Omega \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}\, dx - \left[\frac{\partial u}{\partial x}\cdot\vec{n}\cdot v \right]_{x=0}^{x=1} =$$

$$= \int_0^1 \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}\, dx - \left(\frac{\partial u}{\partial x}\cdot 1\cdot v\Big|_{\Gamma_D}^{\ 0} - \frac{\partial u}{\partial x}(-1)\cdot v\Big|_{\Gamma_D}^{\ 0} \right) = \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\, dx$$

$$\Rightarrow \int_0^1 \frac{\partial u}{\partial t} v\, dx + \underbrace{\int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\, dx + \int_0^1 \frac{\partial u}{\partial x} v\, dx}_{a(u,v)} = 0$$

$$\boxed{\begin{array}{l} \forall t > 0,\ \text{find } u \in V \text{ s.t.:} \\[4pt] \quad \int_0^1 \frac{\partial u}{\partial t} v\, dx + a(u,v) = 0 \qquad \forall v \in V_0 \\[4pt] \text{with } u(x, t=0) = u_0(x) \qquad 0 < x < 1 \end{array}} \quad \text{(WP)}$$

LIFTING OPERATOR: I want the solution to be in V_0; suppose

$\exists R_g \in V$ s.t. $u = \hat{u} + R_g$, and $\hat{u} \in V_0$

$\Rightarrow$ ~~$a(\hat{u}+...)$~~ $\int_0^1 \frac{\partial u}{\partial t} \cdot v \, dx + a(\hat{u} + R_g, v) = 0$

$\int_0^1 \frac{\partial \hat{u}}{\partial t} \cdot v \, dx + \boxed{\int_0^1 \frac{\partial R_g}{\partial t} \cdot v \, dx} + a(\hat{u}, v) + \boxed{a(R_g, v)} = 0$

since R_g is α in $x=0$
and β in $x=1$, doesn't
depend upon time $\Rightarrow \frac{\partial R_g}{\partial t} = 0$
($\Rightarrow$ ~~I should be putting it to zero~~
BUT for generality ~~I'll~~ keep it)

$\bullet$ $F(v) = -a(R_g, v) - \int_0^1 \frac{\partial R_g}{\partial t} v \, dx$

$\Rightarrow$ Final (WP) with lifting:

$\boxed{\begin{array}{l} \forall t > 0, \text{ find } \hat{u} \in V_0 \text{ s.t.} \\ \int_0^1 \frac{\partial \hat{u}}{\partial x} \cdot v \, dx + a(\hat{u}, v) = F(v) \qquad \forall v \in V_0 \\ \text{with } \boxed{u(x,0) = u_0(x)} \qquad\qquad 0 < x < 1 \end{array}}$ (WP + lifting)

↳ lascio così anche se
ho il lifting

(1.2) Coercivity:

"$\exists \alpha > 0 : a(v,v) \geq \alpha \|v\|^2 \quad \forall v \in V_0$" $\rightarrow$ this by definition

Let's try to prove it:

$\boxed{a(v,v)} = \int_0^1 \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} \, dx + \int_0^1 \frac{\partial u}{\partial x} v \, dx = \begin{cases} u = v \end{cases}$

$= \int_0^1 (v')^2 \, dx + \int_0^1 \frac{1}{2} (v^2)' \, dx = (*)$

$= \frac{1}{2} \left[v^2(x) \right]_{x=0}^{x=1} = \frac{1}{2} \left(v(1)^2 - v(0)^2 \right)$

$(*) = \int_0^1 (v')^2 \, dx = \boxed{\|v'\|^2_{L^2(0,1)}}$ → $a(\cdot,\cdot)$ is coercive
with respect to the
seminorm $\|v\|_V := \|v_x\|_{L^2}$
with $\alpha = 1$

$v \in V_0$
$\Rightarrow v(0) = v(1) = 0$

If we want coercivity with respect to the COMPLETE NORM
$\|v\|_{H_1}$, we use Poincaré:

$\|v\|_{L^2} \leq C_p \|v_x\|_{L^2} \implies \|v\|^2_{H_1} = \|v\|^2 + \|v_x\|^2 \leq (C_p^2 + 1)\|v_x\|^2$

$\implies a(v,v) = \|v_x\|^2 \geq \frac{1}{1+C_p^2} \|v\|^2_{H_1}$

(1.3) DISCRETE APPROXIMATION (Piecewise finite elements)?

PARTITION of Ω: K_c, $c = 1, \ldots, N_{ee}$ s.t.
$$\begin{cases} \bigcup_{i=1}^{N_{ee}} K_i = \Omega \\ K_i \cap K_j = \emptyset \quad \forall i \neq j \end{cases}$$

$$X_{R}^1 ((0,1)) = \{ v \in \mathcal{C}^0(\overline{(0,1)}), \; v_R|_{K_i} \in P^1(K_j), \; \forall_j \}$$

$$\Rightarrow V_{R,0} = V_0 \cap X_R^1((0,1)) \longrightarrow \text{fem space for velocity} \qquad ! \; \boxed{N_{V_{R,0}} \leq +\infty}$$

Therefore:

$$\boxed{\begin{array}{l} \forall t > 0, \; \text{find } \hat{u}_R \in V_{0,R} \text{ s.t.:} \\[4pt] \int_0^1 \dfrac{\partial \hat{u}_R}{\partial x} \cdot v_R \, dx + a(\hat{u}_R, v_R) = F(v_R) \qquad \forall v_R \in V_{0,R} \\[6pt] \text{with } \; u_R(x,0) = u_0(x) \qquad\qquad 0 < x < 1 \end{array}}$$

Galerkin semi-discretization in space

$$\{\varphi_j(x)\}_{j=1}^{N_{V_{R,0}}} \longrightarrow \text{basis for } V_{0,R}$$

Expansions:

$$\hat{u}_R(x,t) = \sum_{j=1}^{N_{V_{R,0}}} \hat{u}_j(t) \cdot \varphi_j(x) \qquad\qquad v_R(x) = \varphi_i(x)$$

with: $\vec{u}(t) = [u_1(t), \ldots, u_{N_{V_{R,0}}}(t)]$; $\vec{F} = [F_1(\varphi_i(x)), \ldots, F_{N_{V_{R,0}}}]$

so:

$$\int_0^1 \dfrac{\partial \sum_j \hat{u}_j(t) \cdot \varphi_j(x)}{\partial t} \varphi_i(x) \, dx + a\left(\sum_j \hat{u}_j(t) \varphi_j(x), \varphi_i(x)\right) = F(\varphi_i(x))$$

$$\sum_{j=1}^{N_{V_{R,0}}} \hat{u}_j \underbrace{\int_0^1 \varphi_j(x) \cdot \varphi_i(x) \, dx}_{M_{ij}} + \sum_j \hat{u}_j(t) \cdot \underbrace{a(\varphi_j(x), \varphi_i(x))}_{A_{ij}} = F(\varphi_i(x)) \; \forall i = 1, \ldots, N_{V_{R,0}}$$

$$\Rightarrow \boxed{M\dot{\vec{u}} + A\vec{u} = \vec{F}}$$ algebraic system for semidiscret. in space

FULL DISCRETIZATION $\begin{cases} \text{space} \\ \text{time (BE, } \theta = 1) \end{cases}$

(FD) $\begin{cases} M\dfrac{\vec{u}^{m+1} - \vec{u}^m}{\Delta t} + A\vec{u}^{m+1} = \vec{F}^{m+1} \qquad m = 0, \ldots, M-1 \\ \vec{u}^0 = \vec{u}_0 \qquad \text{I.C.} \end{cases}$

$M = \#$ timesteps we do

$\xleftrightarrow{\Delta t}$ time axis: $0 \; \cdots \; t^m \; t^{m+1} \; T \to t$ (G)

(1.4)

~~(if (1.9) is satisfied, then (G) has unique solution and $\|u\| \leq \frac{1}{\alpha}\|F\|$ STABILITY we have to prove LM there~~

(1.4) Stability for $\alpha = \beta = 0$:

Take in the discrete formula $u_A = u_B$: $\left(\dfrac{u_A^m - u_A^{m-1}}{\Delta t}, u_A^m \right) + a(u_A^m, u_B^m) = 0$

We use the identity in L^2:

$(a - b, a) = \dfrac{1}{2} \left(\|a^o\|_o^2 - \|b\|^2 + \|a - b\|^2 \right)$

With $a = u_A^m$, $b = u_A^{m-1}$

$\underset{time}{\underline{Stability}}$ EI è unconditionally stable $\rightarrow$ $\theta = 1 : Err \underset{\sim \sigma(\Delta t)}{\sim}$

$\underline{Space}$: we don't have any result for stability, in space.

EXERCISE 2

(2.1)