

Bayesian statistics course (Vasishth/Nicenboim)

Date: 25-29 March, 2019

1 Deriving Bayes' rule

Let A and B be two observable events. $P(A)$ is the probability that A occurs, and $P(B)$ is the probability that B occurs. $P(A|B)$ is the conditional probability that A occurs given that B has happened. $P(A, B)$ is the joint probability of A and B both occurring.

You are given the definition of conditional probability:

$$P(A|B) = \frac{P(A, B)}{P(B)} \text{ where } P(B) > 0 \quad (1)$$

Using the above definition, and using the fact that $P(A, B) = P(B, A)$ (i.e., the probability of A and B both occurring is the same as the probability of B and A both occurring), derive an expression for $P(B|A)$. Show the steps clearly in the derivation.

2 Conjugate forms 1

1. Suppose you are given a vector of data x consisting of 1's and 0's, coming from a Binomial(n, θ) distribution. 1 represents success, and 0 failure. Example data are shown below, generated with probability of success $\theta = 0.5$, just for illustration:

```
## data:
x<-rbinom(n=10,size=1,prob=0.5)
x

## [1] 0 1 1 0 0 1 0 1 1 0

## k:
sum(x)

## [1] 5
```

Here, n represents the number of trials, and k the number of successes. **The above code and output is just an example, and is no longer relevant for the question below.**

Given k successes in n trials coming from a Binomial distribution, we define a $\text{Beta}(a,b)$ prior on the parameter θ .

Write down the Beta distribution that represents the posterior, in terms of a, b, n , and k .

2. We ask 10 yes/no questions from a participant, and the participant returns 0 correct answers. We assume a Binomial likelihood function for these data. Also assume a $\text{Beta}(1,1)$ prior on the parameter θ , which represents the probability of success. Use the result you derived in the preceding part of this question to write down the posterior distribution of the θ parameter.

3 Conjugate forms 2

Suppose you have n independent and identically distributed data points from a distribution that has the likelihood function $f(x|\theta) = \theta(1 - \theta)^{\sum_{i=1}^n x_i}$, where the data points x can have values $0, 1, 2, \dots$. Let the prior on θ be $\text{Beta}(a,b)$, a Beta distribution with parameters a, b . The posterior distribution is a Beta distribution with parameters a^* and b^* . Determine these parameters in terms of a, b , and $\sum_{i=1}^n x_i$.

4 Conjugate forms 3

The Gamma distribution is defined in terms of the parameters a, b : $\text{Ga}(a,b)$. The probability density function is:

$$\text{Ga}(a, b) = \frac{b^a \lambda^{a-1} \exp\{-b\lambda\}}{\Gamma(a)} \quad (2)$$

We have data x_1, \dots, x_n , with sample size n that is exponentially distributed. The exponential likelihood function is:

$$f(x_1, \dots, x_n; \lambda) = \lambda^n \exp\left\{-\lambda \sum_{i=1}^n x_i\right\} \quad (3)$$

It turns out that if we assume a $\text{Ga}(a,b)$ prior distribution and the above likelihood, the posterior distribution is a Gamma distribution. Find the parameters a' and b' of the posterior distribution.