MACHINE LEARNING 1: ASSIGNMENT 1

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Exercise 1

a)

$$P(error) = \int P(error \mid x) p(x) dx \tag{1}$$

$$P(error \mid x) = \min(P(w_1 \mid x), P(w_2 \mid x)) \tag{2}$$

With these equations, we want to show that

$$P(error) \leq \int \frac{2}{\frac{1}{P(w_1|x)} + \frac{1}{P(w_2|x)}} p(x) dx \tag{3}$$

At first, without restricting the general case, we assume that $P(w_1 \mid x) \geq P(w_2 \mid x)$, that is the function $P(error \mid x) = P(w_2 \mid x)$. Now with (**??**), (**??**) and (**??**) we have:

$$\int P(w_2 \mid x) p(x) dx \leq \int \frac{2}{\frac{1}{P(w_1|x)} + \frac{1}{P(w_2|x)}} p(x) dx$$

Because both sides are integrating over the same variable we can simplify the term to:

$$\begin{aligned}
& P(w_2 \mid x) p(x) \leq \frac{2}{\frac{1}{P(w_1|x)} + \frac{1}{P(w_2|x)}} p(x) \\
& \Leftrightarrow P(w_2 \mid x) \leq \frac{2}{\frac{1}{P(w_1|x)} + \frac{1}{P(w_2|x)}} \\
& \Leftrightarrow \left(\frac{1}{P(w_1 \mid x)} + \frac{1}{P(w_2 \mid x)}\right) P(w_2 \mid x) \leq 2 \\
& \Leftrightarrow \frac{1}{P(w_1 \mid x)} + \frac{1}{P(w_2 \mid x)} \leq \frac{2}{P(w_2 \mid x)} \\
& \Leftrightarrow \frac{1}{P(w_1 \mid x)} \leq \frac{1}{P(w_2 \mid x)} \\
& \Leftrightarrow P(w_1 \mid x) \geq P(w_2 \mid x)
\end{aligned}$$

This holds true with the assumptions we made earlier.

b)

With this result, we now show that:

$$P(error) \leq \frac{2P(w_1)P(w_2)}{\sqrt{P(w_1)^2 + (4\mu^2 + 2)P(w_1)P(w_2) + P(w_2)^2}}$$

While using the univariate probability distribution:

$$p(x \mid w_1) = \frac{\pi^{-1}}{1 + (x - \mu)^2} \text{ and } p(x \mid w_2) = \frac{\pi^{-1}}{1 + (x + \mu)^2}$$

With the rule of bayes we have $P(w_1 \mid x) = \frac{p(x|w_1)P(w_1)}{p(x)}$:

$$
\begin{aligned}
& P(error) \leq \int \frac{2}{\frac{1}{P(w_1|x)} + \frac{1}{P(w_2|x)}} p(x)dx \\
\Leftrightarrow\ & P(error) \leq \int \frac{2}{\frac{1}{\frac{p(x|w_1)P(w_1)}{p(x)}} + \frac{1}{\frac{p(x|w_2)P(w_2)}{p(x)}}} p(x)dx \\
\Leftrightarrow\ & P(error) \leq \int \frac{2}{\frac{p(x)}{p(x|w_1)P(w_1)} + \frac{p(x)}{p(x|w_2)P(w_2)}} p(x)dx \\
\Leftrightarrow\ & P(error) \leq \int \frac{2}{\frac{1}{p(x|w_1)P(w_1)} + \frac{1}{p(x|w_2)P(w_2)}} dx \\
\Leftrightarrow\ & P(error) \leq \int \frac{2}{\frac{1}{\frac{\pi^{-1}}{1+(x-\mu)^2}P(w_1)} + \frac{1}{\frac{\pi^{-1}}{1+(x+\mu)^2}P(w_2)}} dx \\
\Leftrightarrow\ & P(error) \leq \int \frac{2}{\frac{1}{\frac{\pi^{-1}P(w_1)}{1+(x-\mu)^2}} + \frac{1}{\frac{\pi^{-1}P(w_2)}{1+(x+\mu)^2}}} dx \\
\Leftrightarrow\ & P(error) \leq \int \frac{2}{\frac{1+(x-\mu)^2}{\pi^{-1}P(w_1)} + \frac{1+(x+\mu)^2}{\pi^{-1}P(w_2)}} dx \\
\Leftrightarrow\ & P(error) \leq \int \frac{2}{\frac{(1+(x-\mu)^2)P(w_2)}{\pi^{-1}P(w_1)P(w_2)} + \frac{(1+(x+\mu)^2)P(w_1)}{\pi^{-1}P(w_2)P(w_1)}} dx \\
\Leftrightarrow\ & P(error) \leq \int \frac{2\pi^{-1}P(w_2)P(w_1)}{(1+(x+\mu)^2)P(w_1) + (1+(x-\mu)^2)P(w_2)} dx \\
\Leftrightarrow\ & P(error) \leq \int \frac{2\pi^{-1}P(w_2)P(w_1)}{(x^2+2x\mu+\mu^2+1)P(w_1) + (x^2-2x\mu+\mu^2+1)P(w_2)} dx \\
\Leftrightarrow\ & P(error) \leq \int \frac{2\pi^{-1}P(w_2)P(w_1)}{(P(w_1)+P(w_2))x^2 + (P(w_1)-P(w_2))2\mu x + (P(w_1)\mu^2 + P(w_2)\mu^2 + P(w1) + P(w_2))} dx
\end{aligned}
$$

We can now take out the numerator of the integral and use the following equation:

$$\int \frac{1}{ax^2+bx+c} dx = \frac{2\pi}{\sqrt{4ac-b^2}} \tag{4}$$

with:

$$
\begin{aligned}
a &= P(w_1) + P(w_2) \\
b &= (P(w_1) - P(w_2))2\mu \\
c &= P(w_1)\mu^2 + P(w_2)\mu^2 + P(w1) + P(w_2)
\end{aligned}
$$

because

$$
\begin{aligned}
& b^2 < 4ac \\
\Leftrightarrow\ & 0 < 4ac - b^2 \\
\Leftrightarrow\ & 0 < 4(P(w_1) + P(w_2))(P(w_1)\mu^2 + P(w_2)\mu^2 + P(w1) + P(w_2)) - ((P(w_1) - P(w_2))2\mu)^2 \\
\Leftrightarrow\ & 0 < 4P(w_1)^2\mu^2 + 4P(w_1)P(w_2)\mu^2 + 4P(w_1)^2 + 4P(w_1)P(w_2) \\
& \quad + 4P(w_1)P(w_2)\mu^2 + 4P(w_2)^2\mu^2 + 4P(w_2)P(w_1) + 4P(w_2)^2 \\
& \quad - (4P(w_1)^2\mu^2 - 8P(w_1)P(w_2)\mu^2 + 4P(w_2)^2\mu^2) \\
\Leftrightarrow\ & 0 < 16P(w_1)P(w_2)\mu^2 + 8P(w_1)P(w_2) + 4P(w_1)^2 + 4P(w_2)^2
\end{aligned}
$$

and this holds since $P(w_1), P(w_2) \in [0,1]$ and $P(w_1) + P(w_2) = 1$.
We now already calculated $4ac - b^2$ in the steps before and just need to use (4) to proceed where we stopped before introducing equation 4:

$$
\begin{aligned}
\Leftrightarrow P(error) &\leq \int \frac{2\pi^{-1}P(w_2)P(w_1)}{(P(w_1) + P(w_2))x^2 + (P(w_1) - P(w_2))2\mu x + (P(w_1)\mu^2 + P(w_2)\mu^2 + P(w1) + P(w_2))} dx \\
\Leftrightarrow P(error) &\leq 2\pi^{-1}P(w_2)P(w_1)\frac{2\pi}{\sqrt{16P(w_1)P(w_2)\mu^2 + 8P(w_1)P(w_2) + 4P(w_1)^2 + 4P(w_2)^2}} \\
\Leftrightarrow P(error) &\leq \frac{4P(w_1)P(w_2)}{\sqrt{4((4\mu^2 + 2)P(w_1)P(w_2) + P(w_1)^2 + P(w_2)^2)}} \\
\Leftrightarrow P(error) &\leq \frac{4P(w_1)P(w_2)}{\sqrt{4}\sqrt{((4\mu^2 + 2)P(w_1)P(w_2) + P(w_1)^2 + P(w_2)^2)}} \\
\Leftrightarrow P(error) &\leq \frac{2P(w_1)P(w_2)}{\sqrt{P(w_1)^2 + (4\mu^2 + 2)P(w_1)P(w_2) + P(w_2)^2}}
\end{aligned}
$$