

# MACHINE LEARNING 1: ASSIGNMENT 1

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## Exercise 1

a)

$$P(\text{error}) = \int P(\text{error} | x) p(x) dx \quad (1)$$

$$P(\text{error} | x) = \min(P(w_1 | x), P(w_2 | x)) \quad (2)$$

With these equations, we want to show that

$$P(\text{error}) \leq \int \frac{2}{\frac{1}{P(w_1|x)} + \frac{1}{P(w_2|x)}} p(x) dx \quad (3)$$

At first, without restricting the general case, we assume that  $P(w_1 | x) \geq P(w_2 | x)$ , that is the function  $P(\text{error} | x) = P(w_2 | x)$ . Now with (??), (??) and (??) we have:

$$\int P(w_2 | x) p(x) dx \leq \int \frac{2}{\frac{1}{P(w_1|x)} + \frac{1}{P(w_2|x)}} p(x) dx$$

Because both sides are integrating over the same variable we can simplify the term to:

$$\begin{aligned} P(w_2 | x) p(x) &\leq \frac{2}{\frac{1}{P(w_1|x)} + \frac{1}{P(w_2|x)}} p(x) \\ \Leftrightarrow P(w_2 | x) &\leq \frac{2}{\frac{1}{P(w_1|x)} + \frac{1}{P(w_2|x)}} \\ \Leftrightarrow \left( \frac{1}{P(w_1 | x)} + \frac{1}{P(w_2 | x)} \right) P(w_2 | x) &\leq 2 \\ \Leftrightarrow \frac{1}{P(w_1 | x)} + \frac{1}{P(w_2 | x)} &\leq \frac{2}{P(w_2 | x)} \\ \Leftrightarrow \frac{1}{P(w_1 | x)} &\leq \frac{1}{P(w_2 | x)} \\ \Leftrightarrow P(w_1 | x) &\geq P(w_2 | x) \end{aligned}$$

This holds true with the assumptions we made earlier.

b)

With this result, we now show that:

$$P(\text{error}) \leq \frac{2P(w_1)P(w_2)}{\sqrt{P(w_1)^2 + (4\mu^2 + 2)P(w_1)P(w_2) + P(w_2)^2}}$$

While using the univariate probability distribution:

$$p(x | w_1) = \frac{\pi^{-1}}{1 + (x - \mu)^2} \text{ and } p(x | w_2) = \frac{\pi^{-1}}{1 + (x + \mu)^2}$$

With the rule of bayes we have  $P(w_1 | x) = \frac{p(x|w_1)P(w_1)}{p(x)}$ :

$$\begin{aligned}
P(\text{error}) &\leq \int \frac{2}{\frac{1}{P(w_1|x)} + \frac{1}{P(w_2|x)}} p(x) dx \\
\Leftrightarrow P(\text{error}) &\leq \int \frac{2}{\frac{1}{\frac{p(x|w_1)P(w_1)}{p(x)}} + \frac{1}{\frac{p(x|w_2)P(w_2)}{p(x)}}} p(x) dx \\
\Leftrightarrow P(\text{error}) &\leq \int \frac{2}{\frac{p(x)}{p(x|w_1)P(w_1)} + \frac{p(x)}{p(x|w_2)P(w_2)}} p(x) dx \\
\Leftrightarrow P(\text{error}) &\leq \int \frac{2}{\frac{1}{p(x|w_1)P(w_1)} + \frac{1}{p(x|w_2)P(w_2)}} dx \\
\Leftrightarrow P(\text{error}) &\leq \int \frac{2}{\frac{1}{\frac{\pi^{-1}}{1+(x-\mu)^2}P(w_1)} + \frac{1}{\frac{\pi^{-1}}{1+(x+\mu)^2}P(w_2)}} dx \\
\Leftrightarrow P(\text{error}) &\leq \int \frac{2}{\frac{1}{\frac{\pi^{-1}P(w_1)}{1+(x-\mu)^2}} + \frac{1}{\frac{\pi^{-1}P(w_2)}{1+(x+\mu)^2}}} dx \\
\Leftrightarrow P(\text{error}) &\leq \int \frac{2}{\frac{1+(x-\mu)^2}{\pi^{-1}P(w_1)} + \frac{1+(x+\mu)^2}{\pi^{-1}P(w_2)}} dx \\
\Leftrightarrow P(\text{error}) &\leq \int \frac{2}{\frac{(1+(x-\mu)^2)P(w_2)}{\pi^{-1}P(w_1)P(w_2)} + \frac{(1+(x+\mu)^2)P(w_1)}{\pi^{-1}P(w_2)P(w_1)}} dx \\
\Leftrightarrow P(\text{error}) &\leq \int \frac{2\pi^{-1}P(w_2)P(w_1)}{(1+(x+\mu)^2)P(w_1) + (1+(x-\mu)^2)P(w_2)} dx \\
\Leftrightarrow P(\text{error}) &\leq \int \frac{2\pi^{-1}P(w_2)P(w_1)}{(x^2+2x\mu+\mu^2+1)P(w_1) + (x^2-2x\mu+\mu^2+1)P(w_2)} dx \\
\Leftrightarrow P(\text{error}) &\leq \int \frac{2\pi^{-1}P(w_2)P(w_1)}{(P(w_1)+P(w_2))x^2 + (P(w_1)-P(w_2))2\mu x + (P(w_1)\mu^2+P(w_2)\mu^2+P(w_1)+P(w_2))} dx
\end{aligned}$$

We can now take out the numerator of the integral and use the following equation:

$$\int \frac{1}{ax^2 + bx + c} dx = \frac{2\pi}{\sqrt{4ac - b^2}} \quad (4)$$

with:

$$\begin{aligned}
a &= P(w_1) + P(w_2) \\
b &= (P(w_1) - P(w_2))2\mu \\
c &= P(w_1)\mu^2 + P(w_2)\mu^2 + P(w_1) + P(w_2)
\end{aligned}$$

because

$$\begin{aligned}
b^2 &< 4ac \\
\Leftrightarrow 0 &< 4ac - b^2 \\
\Leftrightarrow 0 &< 4(P(w_1) + P(w_2))(P(w_1)\mu^2 + P(w_2)\mu^2 + P(w_1) + P(w_2)) - ((P(w_1) - P(w_2))2\mu)^2 \\
\Leftrightarrow 0 &< 4P(w_1)^2\mu^2 + 4P(w_1)P(w_2)\mu^2 + 4P(w_1)^2 + 4P(w_1)P(w_2) \\
&\quad + 4P(w_1)P(w_2)\mu^2 + 4P(w_2)^2\mu^2 + 4P(w_2)P(w_1) + 4P(w_2)^2 \\
&\quad - (4P(w_1)^2\mu^2 - 8P(w_1)P(w_2)\mu^2 + 4P(w_2)^2\mu^2) \\
\Leftrightarrow 0 &< 16P(w_1)P(w_2)\mu^2 + 8P(w_1)P(w_2) + 4P(w_1)^2 + 4P(w_2)^2
\end{aligned}$$

and this holds since  $P(w_1), P(w_2) \in [0, 1]$  and  $P(w_1) + P(w_2) = 1$ .

We now already calculated  $4ac - b^2$  in the steps before and just need to use (4) to proceed where we stopped before introducing equation 4:

$$\begin{aligned}
\Leftrightarrow P(\text{error}) &\leq \int \frac{2\pi^{-1}P(w_2)P(w_1)}{(P(w_1) + P(w_2))x^2 + (P(w_1) - P(w_2))2\mu x + (P(w_1)\mu^2 + P(w_2)\mu^2 + P(w_1) + P(w_2))} dx \\
\Leftrightarrow P(\text{error}) &\leq 2\pi^{-1}P(w_2)P(w_1) \frac{2\pi}{\sqrt{16P(w_1)P(w_2)\mu^2 + 8P(w_1)P(w_2) + 4P(w_1)^2 + 4P(w_2)^2}} \\
\Leftrightarrow P(\text{error}) &\leq \frac{4P(w_1)P(w_2)}{\sqrt{4((4\mu^2 + 2)P(w_1)P(w_2) + P(w_1)^2 + P(w_2)^2)}} \\
\Leftrightarrow P(\text{error}) &\leq \frac{4P(w_1)P(w_2)}{\sqrt{4((4\mu^2 + 2)P(w_1)P(w_2) + P(w_1)^2 + P(w_2)^2)}} \\
\Leftrightarrow P(\text{error}) &\leq \frac{2P(w_1)P(w_2)}{\sqrt{P(w_1)^2 + (4\mu^2 + 2)P(w_1)P(w_2) + P(w_2)^2}}
\end{aligned}$$

c)

According to informations from chapter 2.8 in Pattern Classification:

For both case we can find the upper-bound numerically via the Chernoff Bound or the Bhattacharyya Bound, this may lead to a tighter upper bound than the analytical version, which is even harder to approximate due to the discontinues nature of the integrals. For the high-dimensional space the Bhattacharyya Bound might be better, because it is computationally less expensive that the Chernoff Bound.

## Exercise 2

a)

The data is generated by the univariate Laplacian Distrubution:

$$p(x | w_1) = \frac{1}{2\sigma} \exp\left(-\frac{|x - \mu|}{\sigma}\right) \text{ and } p(x | w_2) = \frac{1}{2\sigma} \exp\left(-\frac{|x + \mu|}{\sigma}\right)$$

To get the optimal decision boundary we have to solve  $P(w_1 | x) = P(w_2 | x)$  for  $x$ .

$$\begin{aligned}
& P(w_1 | x) = P(w_2 | x) \\
\Leftrightarrow & \frac{p(x | w_1)P(w_1)}{p(x)} = \frac{p(x | w_2)P(w_2)}{p(x)} \\
\Leftrightarrow & p(x | w_1)P(w_1) = p(x | w_2)P(w_2) \\
\Leftrightarrow & \frac{1}{2\sigma} \exp\left(-\frac{|x - \mu|}{\sigma}\right) P(w_1) = \frac{1}{2\sigma} \exp\left(-\frac{|x + \mu|}{\sigma}\right) P(w_2) \\
\Leftrightarrow & \exp\left(-\frac{|x - \mu|}{\sigma}\right) P(w_1) = \exp\left(-\frac{|x + \mu|}{\sigma}\right) P(w_2) \\
\Leftrightarrow & \ln(\exp\left(-\frac{|x - \mu|}{\sigma}\right) P(w_1)) = \ln(\exp\left(-\frac{|x + \mu|}{\sigma}\right) P(w_2)) \\
\Leftrightarrow & -\frac{|x - \mu|}{\sigma} + \ln(P(w_1)) = -\frac{|x + \mu|}{\sigma} + \ln(P(w_2)) \\
\Leftrightarrow & \ln(P(w_1)) - \ln(P(w_2)) = \frac{|x - \mu|}{\sigma} - \frac{|x + \mu|}{\sigma} \\
\Leftrightarrow & \ln\left(\frac{P(w_1)}{P(w_2)}\right) = \frac{|x - \mu| - |x + \mu|}{\sigma} \\
\Leftrightarrow & \sigma \ln\left(\frac{P(w_1)}{P(w_2)}\right) = |x - \mu| - |x + \mu|
\end{aligned}$$

There are now 3 cases to look at:

1.  $x \leq \mu$  and  $-x \leq \mu$

$$\begin{aligned}
\sigma \ln\left(\frac{P(w_1)}{P(w_2)}\right) &= -(x - \mu) - (x + \mu) \\
&= 2(\mu - x) \\
\mu - \frac{\sigma}{2} \ln\left(\frac{P(w_1)}{P(w_2)}\right) &= x
\end{aligned}$$

2.  $x \leq \mu$  and  $-x > \mu$

$$\begin{aligned}
\sigma \ln\left(\frac{P(w_1)}{P(w_2)}\right) &= -(x - \mu) + (x + \mu) \\
&= 2\mu
\end{aligned}$$

3.  $x > \mu$

$$\begin{aligned}
\sigma \ln\left(\frac{P(w_1)}{P(w_2)}\right) &= (x - \mu) - (x + \mu) \\
&= -2\mu
\end{aligned}$$

**b)**

To find out for which values of  $P(w_1), P(w_2), \mu, \sigma$  the optimal decision is to always predict the first class, which means  $P(\text{error} | x) = P(w_2 | x)$ , so we have to show  $P(w_2 | x) \leq P(w_1 | x)$ :