Machine Learning 1: Assignment 1

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Exercise 1

a)

$$P(error) = \int P(error \mid x)p(x)dx \tag{1}$$

$$P(error \mid x) = \min(P(w_1 \mid x), P(w_2 \mid x)) \tag{2}$$

With these equations, we want to show that

$$P(error) \leq \int \frac{2}{\frac{1}{P(w_1|x)} + \frac{1}{P(w_2|x)}} p(x)dx \tag{3}$$

At first, without restricting the general case, we assume that $P(w_1 \mid x) \geq P(w_2 \mid x)$, that is the function $P(error \mid x) = P(w_2 \mid x)$. Now with (**??**), (**??**) and (**??**) we have:

$$\int P(w_2 \mid x)p(x)dx \leq \int \frac{2}{\frac{1}{P(w_1|x)} + \frac{1}{P(w_2|x)}} p(x)dx$$

Because both sides are integrating over the same variable we can simplify the term to:

$$\begin{aligned}
&P(w_2 \mid x)p(x) \leq \frac{2}{\frac{1}{P(w_1|x)} + \frac{1}{P(w_2|x)}} p(x) \\
\Leftrightarrow &\frac{1}{P(w_2 \mid x)p(x)} \geq \frac{\frac{1}{P(w_1|x)} + \frac{1}{P(w_2|x)}}{2} p(x) \\
\Leftrightarrow &\frac{1}{P(w_2 \mid x)} \geq \frac{\frac{1}{P(w_1|x)} + \frac{1}{P(w_2|x)}}{2} \\
\Leftrightarrow &\frac{2}{P(w_2 \mid x)} \geq \frac{1}{P(w_1 \mid x)} + \frac{1}{P(w_2 \mid x)} \\
\Leftrightarrow &\frac{1}{P(w_2 \mid x)} \geq \frac{1}{P(w_1 \mid x)} \\
\Leftrightarrow &P(w_2 \mid x) \leq P(w_1 \mid x)
\end{aligned}$$

This holds true with the assumptions we made earlier.

b)

With this result, we now show that:

$$P(error) \leq \frac{2P(w_1)P(w_2)}{\sqrt{P(w_1)^2 + (4\mu^2 + 2)P(w_1)P(w_2) + P(w_2)^2}}$$

While using the univariate probability distribution:

$$p(x \mid w_1) = \frac{\pi^{-1}}{1 + (x-\mu)^2} \text{ and } p(x \mid w_2) = \frac{\pi^{-1}}{1 + (x+\mu)^2}$$

With the rule of bayes we have $P(w_1 \mid x) = \frac{p(x|w_1)P(w_1)}{p(x)}$:

$$P(error) \leq \int \frac{2}{\frac{1}{P(w_1|x)} + \frac{1}{P(w_2|x)}} p(x)dx \tag{4}$$

$$\Leftrightarrow P(error) \leq \int \frac{2}{\frac{1}{\frac{p(x|w_1)P(w_1)}{p(x)}} + \frac{1}{\frac{p(x|w_2)P(w_2)}{p(x)}}} p(x)dx \tag{5}$$

$$\Leftrightarrow P(error) \leq \int \frac{2}{\frac{p(x)}{p(x|w_1)P(w_1)} + \frac{p(x)}{p(x|w_2)P(w_2)}} p(x)dx \tag{6}$$

$$\Leftrightarrow P(error) \leq \int \frac{2}{\frac{1}{p(x|w_1)P(w_1)} + \frac{1}{p(x|w_2)P(w_2)}} dx \tag{7}$$

$$\Leftrightarrow P(error) \leq \int \frac{2}{\frac{1}{\frac{\pi^{-1}}{1+(x-\mu)^2}P(w_1)} + \frac{1}{\frac{\pi^{-1}}{1+(x+\mu)^2}P(w_2)}} dx \tag{8}$$

$$\Leftrightarrow P(error) \leq \int \frac{2}{\frac{1}{\frac{\pi^{-1}P(w_1)}{1+(x-\mu)^2}} + \frac{1}{\frac{\pi^{-1}P(w_2)}{1+(x+\mu)^2}}} dx \tag{9}$$

$$\Leftrightarrow P(error) \leq \int \frac{2}{\frac{1+(x-\mu)^2}{\pi^{-1}P(w_1)} + \frac{1+(x+\mu)^2}{\pi^{-1}P(w_2)}} dx \tag{10}$$

$$\tag{11}$$