NOTES ON CATEGORY THEORY

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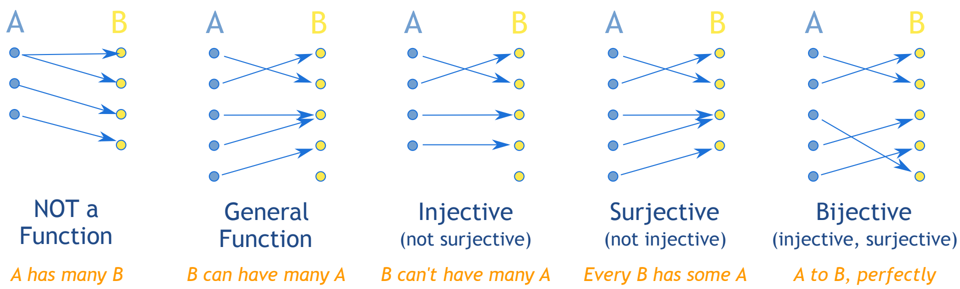
This note is based on the book "Basic Category Theory" by Tom Leinster [1].

1. Chapter 0: Basic concepts

I noticed that the book frequently uses the following objects as examples: sets, groups, fields, rings and topological spaces. Here a brief review of these objects is given.

1.1. Sets.

A set is a collection of different objects, which are called elements. The mapping between sets is called a function.



1.2. Groups.

A group is a non-empty set G equipped with a binary operation $\cdot$ that satisfies the following properties:

- Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- Identity: $e \cdot a = a \cdot e = a$
- Inverses: $a \cdot a^{\{-1\}} = a^{\{-1\}} \cdot a = e$

The set is called the **underlying set** of the group, and the binary operation is called the **group operation**.

Group homomorphism are functions that respect group structure: a map $f : (G, \cdot) \to (H, *)$ between two groups is a homomorphism if $f(a \cdot b) = f(a) * f(b)$ for all $a, b \in G$. An isomorphism is a homomorphism that has an inverse homomorphism; equivalently, it is a bijective homomorphism.

G-sets: a G-set is a set S equipped with an action of a group G. S is called a left G-set if there exists a map $\varphi : G \times S \to S$ that satisfies $\varphi(g, \varphi(h, s)) = \varphi(g \cdot h, s)$ and $\varphi(e, s) = s$ for all $g, h \in G$ and $s \in S$, then φ is called a left action. Similarly, a right G-set is defined by a map $\varphi : S \times G \to S$ that satisfies $\varphi(\varphi(s, g), h) = \varphi(s, g \cdot h)$ and $\varphi(s, e) = s$ for all $g, h \in G$ and $s \in S$, then φ is called a right action.

A **quotient group** or factor group is a mathematical group obtained by aggregating similar elements of a larger group using an equivalence relation that preserves some of the group structure (the rest of the structure is "factored out"). For example, the cyclic group of addition modulo n can be obtained from the group of integers under addition by identifying elements that differ by a multiple of n and defining a group structure that operates on each such class (known as a congruence class) as a single entity.

1.3. **Fields.**

A field is a non-empty set F equipped with two binary operations, addition and multiplication, that satisfies the following properties:

- Associativity: $(a+b)+c = a+(b+c)$
- Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- Identity: $0+a = a+0 = a$
- Identity: $1 \cdot a = a \cdot 1 = a$
- Inverses: $a+(-a) = (-a)+a = 0$
- Inverses: for any $a \neq 0$, $a \cdot a^{\{-1\}} = a^{\{-1\}} \cdot a = 1$
- Distributivity: $a \cdot (b+c) = (a \cdot b)+(a \cdot c)$

Here 0 and 1 are the additive and multiplicative identities, respectively.

An example: $\mathbb{R}$ is a set, $(\mathbb{R}, +)$ is a group, and $(\mathbb{R}, +, \cdot)$ is a field.

1.4. **Rings.**

A ring is a set R equipped with two binary operations, addition and multiplication, and satisfies the following properties:

- R is an abelian group under addition, (addition is commutative)
- R is a monoid under multiplication, where monoids are semigroups with identity
- Distributivity: $a \cdot (b+c) = (a \cdot b)+(a \cdot c)$

A simple example is the 2 by 2 matrices with integer entries.

1.5. **Topological spaces.**

A topological space is a set whose elements are called points, along with an additional structure called a topology, which can be defined as a set of neighbourhoods for each point that satisfy some axioms formalizing the concept of closeness.

2. Chapter 1: Categories, functors and natural transformations

A category is a system of related objects, such as group or topological spaces, and are connected by relations such as homeomorphisms or continuous maps.

The maps between categories are called functors, and the maps between functors are called natural transformations.

2.1. **Categories.**

A category $\mathscr{A}$ consists of:

- $\mathrm{ob}(\mathscr{A})$: objects
- $\mathscr{A}(A, B)$: maps between the objects
- composition: $\mathscr{A}(B, C) \times \mathscr{A}(A, B) \to \mathscr{A}(A, C)$
- identity: $\mathscr{A}(A, A)$

In a discrete category, all objects are not connected to each other.

Group as a category: A group G can be regarded as a category with a single object, each element of G is considered as a map.

Preorder as a category: $A \leq B$ and $B \leq C$ implies $A \leq C$. Order: if $A \leq B$ and $B \leq A$ implies $A = C$.

The **principle of duality** is fundamental to category theory. Informally, it states that every categorical definition, theorem and proof has a dual, obtained by reversing all the arrows.

2.2. **Functors.**

Functors describe how categories are related to each other. Two parts: mapping objects and mapping morphisms.

Definition 1.2.1 Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A **functor** $F : \mathscr{A} \to \mathscr{B}$ consists of:

- a function
$$\mathrm{ob}(\mathscr{A}) \to \mathrm{ob}(\mathscr{B}),$$
written as $A \mapsto F(A)$;
- for each $A, A' \in \mathscr{A}$, a function
$$\mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A')),$$
written as $f \mapsto F(f)$,

satisfying the following axioms:

- $F(f' \circ f) = F(f') \circ F(f)$ whenever $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathscr{A}$;
- $F(1_A) = 1_{F(A)}$ whenever $A \in \mathscr{A}$.

Forgetful functors: when mapping categories, it forgets some structure or properties.

Free functors: a dual of forgetful functors, it adds structure to the objects.

Contravariant functor: Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A contravariant functor $F : \mathscr{A}^{\mathrm{op}} \to \mathscr{B}$. An ordinary functor is also called a covariant functor.

An example: Map from a vector space of row vectors to scalars is a column vector.

A **presheaf** on a category $\mathscr{A}$ is a contravariant functor from $\mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$.

Faithful and full functors: A functor $F : \mathscr{A} \to \mathscr{B}$ is faithful if it is injective on morphisms, and full if it is surjective on morphisms.

2.3. **Natural transformations.**

When functors have the same domain and codomain, mapping between them is called a natural transformation.

Definition 1.3.1 Let $\mathscr{A}$ and $\mathscr{B}$ be categories and let $\mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{B}$ be functors. A **natural transformation** $\alpha\colon F \to G$ is a family $\left(F(A) \xrightarrow{\alpha_A} G(A)\right)_{A \in \mathscr{A}}$ of maps in $\mathscr{B}$ such that for every map $A \xrightarrow{f} A'$ in $\mathscr{A}$, the square

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(A') \arrow[d, "\alpha_{A'}"] \\ G(A) \arrow[r, "G(f)"'] & G(A') \end{tikzcd} \tag{1.3}$$

commutes. The maps α_A are called the **components** of α.

the following notation is used to indicate a natural transformation $\alpha : F \to G$:

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} \end{tikzcd}$$

The concept of **natural isomorphism** is defined as follows:

Definition 1.3.10 Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A **natural isomorphism** between functors from $\mathscr{A}$ to $\mathscr{B}$ is an isomorphism in $[\mathscr{A}, \mathscr{B}]$.

An equivalent form of the definition is often useful:

Lemma 1.3.11 *Let* $\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} \end{tikzcd}$ *be a natural transformation. Then* α *is a natural isomorphism if and only if* $\alpha_A\colon F(A) \to G(A)$ *is an isomorphism for all* $A \in \mathscr{A}$.

If such a natural isomorphism exists, we say that F and G are **naturally isomorphic**.

Equivalence of categories

Two categories $\mathscr{A}$ and $\mathscr{B}$ are isomorphic if there exists a pair of functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ such that $F \circ G = 1_{\{\mathscr{B}\}}$ and $G \circ F = 1_{\{\mathscr{A}\}}$.

Definition 1.3.15 An **equivalence** between categories $\mathscr{A}$ and $\mathscr{B}$ consists of a pair (1.4) of functors together with natural isomorphisms

$$\eta\colon 1_{\mathscr{A}} \to G \circ F, \qquad \varepsilon\colon F \circ G \to 1_{\mathscr{B}}.$$

If there exists an equivalence between $\mathscr{A}$ and $\mathscr{B}$, we say that $\mathscr{A}$ and $\mathscr{B}$ are **equivalent**, and write $\mathscr{A} \simeq \mathscr{B}$. We also say that the functors F and G are **equivalences**.

A functor is an **equivalence** if it is full, faithful and essentially surjective.

3. Chapter 2: Adjoints

3.1. Definition of adjoint functors.

Definition 2.1.1 Let $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ be categories and functors. We say that F is **left adjoint** to G, and G is **right adjoint** to F, and write $F \dashv G$, if

$$\mathscr{B}(F(A), B) \cong \mathscr{A}(A, G(B)) \tag{2.1}$$

naturally in $A \in \mathscr{A}$ and $B \in \mathscr{B}$. The meaning of 'naturally' is defined below. An **adjunction** between F and G is a choice of natural isomorphism (2.1).

'Naturally in $A \in \mathscr{A}$ and $B \in \mathscr{B}$' means that there is a specified bijection (2.1) for each $A \in \mathscr{A}$ and $B \in \mathscr{B}$, and that it satisfies a naturality axiom. To state it, we need some notation. Given objects $A \in \mathscr{A}$ and $B \in \mathscr{B}$, the correspondence (2.1) between maps $F(A) \to B$ and $A \to G(B)$ is denoted by a horizontal bar, in both directions:

$$\begin{aligned} \left(F(A) \xrightarrow{g} B\right) &\quad\mapsto\quad \left(A \xrightarrow{\bar{g}} G(B)\right), \\ \left(F(A) \xrightarrow{\bar{f}} B\right) &\quad\leftarrow\!\shortmid\quad \left(A \xrightarrow{f} G(B)\right). \end{aligned}$$

So $\bar{\bar{f}} = f$ and $\bar{\bar{g}} = g$. We call $\bar{f}$ the **transpose** of f, and similarly for g. The naturality axiom has two parts:

$$\overline{\left(F(A) \xrightarrow{g} B \xrightarrow{q} B'\right)} \quad = \quad \left(A \xrightarrow{\bar{g}} G(B) \xrightarrow{G(q)} G(B')\right) \tag{2.2}$$

(that is, $\overline{q \circ g} = G(q) \circ \bar{g}$) for all g and q, and

$$\overline{\left(A' \xrightarrow{p} A \xrightarrow{f} G(B)\right)} \quad = \quad \left(F(A') \xrightarrow{F(p)} F(A) \xrightarrow{\bar{f}} B\right) \tag{2.3}$$

for all p and f. It makes no difference whether we put the long bar over the left or the right of these equations, since bar is self-inverse.

Question: why there is a direction? Shouldn't F and G be symmetric? Because the direction of the functors?

In general, a forgetful functor does not usually has a right adjoint. An exception is the functors from groups to monoids.

$$\begin{tikzcd} \mathbf{Grp} \arrow[d, "U" description] \\ \mathbf{Mon} \arrow[u, bend left=60, "F"] \arrow[u, bend right=60, "R"'] \end{tikzcd}$$

$F \dashv U \dashv R$

An algebraic theory consists of two things: first, a collection of operations, each with a specified arity (number of inputs), and second, a collection of equations. In a nutshell, the main property of algebras for an algebraic theory is that the operations are defined everywhere on the set, and the equations hold everywhere too.

Example 2.1.5 There are adjunctions

$$\begin{tikzcd} \mathbf{Top} \arrow[d, "U" description] \\ \mathbf{Set} \arrow[u, bend left=50, "D"] \arrow[u, bend right=50, "I"'] \end{tikzcd}$$

$$D \dashv U \dashv I$$

where U sends a space to its set of points, D equips a set with the discrete topology, and I equips a set with the indiscrete topology.

Indiscrete topology: only contains the empty set and the whole space, very trivial.

Definition 2.1.7 Let $\mathscr{A}$ be a category. An object $I \in \mathscr{A}$ is **initial** if for every $A \in \mathscr{A}$, there is exactly one map $I \to A$. An object $T \in \mathscr{A}$ is **terminal** if for every $A \in \mathscr{A}$, there is exactly one map $A \to T$.

3.2. **Adjunctions via units and counits.**

Units and counits are natural transformations that are related to each other by an adjunction.

For each $A \in \mathscr{A}$, we have a map

$$\left(A \xrightarrow{\eta_A} GF(A)\right) = \overline{\left(F(A) \xrightarrow{1} F(A)\right)}.$$

Dually, for each $B \in \mathscr{B}$, we have a map

$$\left(FG(B) \xrightarrow{\varepsilon_B} B\right) = \overline{\left(G(B) \xrightarrow{1} G(B)\right)}.$$

(We have begun to omit brackets, writing $GF(A)$ instead of $G(F(A))$, etc.) These define natural transformations

$$\eta\colon 1_{\mathscr{A}} \to G \circ F, \qquad \varepsilon\colon F \circ G \to 1_{\mathscr{B}},$$

called the **unit** and **counit** of the adjunction, respectively.

About example 2.2.1: https://math.stackexchange.com/questions/3357852/whats-the-difference-between-a-linear-sum-and-its-value

The main idea should be that a formal linear map is a set?

Lemma 2.2.2 *Given an adjunction $F \dashv G$ with unit η and counit ε, the triangles*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd}$$

commute.

Let $\mathscr{A}$ and $\mathscr{B}$ be an adjunction $F \dashv G$, which means with units and counits,

$$\bar{g} = G(g) \circ \eta_A, \ \ \bar{f} = \varepsilon_B \circ F(f)$$

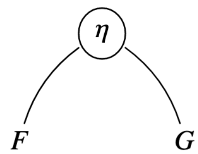
Theorem 2.2.5 *Take categories and functors $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$. There is a one-to-one correspondence between:*

(a) *adjunctions between F and G (with F on the left and G on the right);*
(b) *pairs $\left(1_{\mathscr{A}} \xrightarrow{\eta} GF, \ FG \xrightarrow{\varepsilon} 1_{\mathscr{B}}\right)$ of natural transformations satisfying the triangle identities.*

(Recall that by definition, an adjunction between F and G is a choice of isomorphism (2.1) for each A and B, satisfying the naturality equations (2.2) and (2.3).)

A great way to represent the triangle identities is to use the following diagram:

Now let us apply this notation to adjunctions. The unit and counit are drawn as



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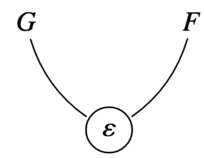
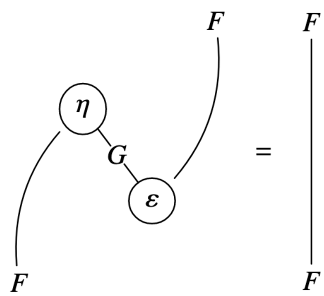


The triangle identities now become the topologically plausible equations



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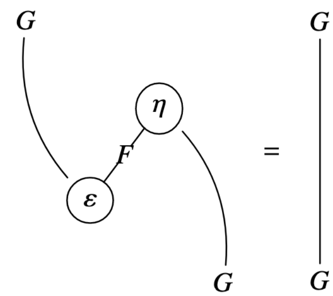


In both equations, the right-hand side is obtained from the left by simply pulling the string straight.

The comma category:

Definition 2.3.1 Given categories and functors

$$\begin{tikzcd} & \mathscr{B} \arrow[d, "Q"] \\ \mathscr{A} \arrow[r, "P"'] & \mathscr{C}, \end{tikzcd}$$

the **comma category** $(P \Rightarrow Q)$ (often written as $(P \downarrow Q)$) is the category defined as follows:

- objects are triples (A, h, B) with $A \in \mathscr{A}$, $B \in \mathscr{B}$, and $h\colon P(A) \to Q(B)$ in $\mathscr{C}$;
- maps $(A, h, B) \to (A', h', B')$ are pairs $(f\colon A \to A',\ g\colon B \to B')$ of maps such that the square

$$\begin{tikzcd} P(A) \arrow[r, "P(f)"] \arrow[d, "h"'] & P(A') \arrow[d, "h'"] \\ Q(B) \arrow[r, "Q(g)"'] & Q(B') \end{tikzcd}$$

commutes.

Example 2.3.4 Let $G\colon \mathscr{B} \to \mathscr{A}$ be a functor and let $A \in \mathscr{A}$. We can form the comma category $(A \Rightarrow G)$, as in the diagram

$$\begin{tikzcd} & \mathscr{B} \arrow[d, "G"] \\ \mathbf{1} \arrow[r, "A"'] & \mathscr{A}. \end{tikzcd}$$

Its objects are pairs $(B \in \mathscr{B},\ f\colon A \to G(B))$. A map $(B, f) \to (B', f')$ in $(A \Rightarrow G)$ is a map $q\colon B \to B'$ in $\mathscr{B}$ making the triangle

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f'"'] & G(B) \arrow[d, "G(q)"] \\ & G(B') \end{tikzcd}$$

commute.

Connection between comma categories and adjunctions:

Lemma 2.3.5 *Take an adjunction* $\begin{tikzcd} \mathscr{A} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, shift left, "G"] \end{tikzcd}$ *and an object* $A \in \mathscr{A}$. *Then the unit map* $\eta_A\colon A \to GF(A)$ *is an initial object of* $(A \Rightarrow G)$.

As a result, the adjoints can be characterized by the initial objects in the comma categories.

Theorem 2.3.6 *Take categories and functors* $\begin{tikzcd} \mathscr{A} \arrow[r, shift left, "F"] & \mathscr{B} \arrow[l, shift left, "G"] \end{tikzcd}$. *There is a one-to-one correspondence between:*

(a) *adjunctions between F and G (with F on the left and G on the right);*
(b) *natural transformations* $\eta\colon 1_{\mathscr{A}} \to GF$ *such that* $\eta_A\colon A \to GF(A)$ *is initial in* $(A \Rightarrow G)$ *for every* $A \in \mathscr{A}$.

An application of this characterization is the following:

Corollary 2.3.7 *Let $G\colon \mathscr{B} \to \mathscr{A}$ be a functor. Then G has a left adjoint if and only if for each $A \in \mathscr{A}$, the category $(A \Rightarrow G)$ has an initial object.*

Proof Lemma 2.3.5 proves 'only if'. To prove 'if', let us choose for each $A \in \mathscr{A}$ an initial object of $(A \Rightarrow G)$ and call it $(F(A), \eta_A\colon A \to GF(A))$. (Here $F(A)$ and η_A are just the names we choose to use.) For each map $f\colon A \to A'$ in $\mathscr{A}$, let $F(f)\colon F(A) \to F(A')$ be the unique map such that

$$\begin{tikzcd} A \arrow[rr, "\eta_A"] \arrow[dr, "f"'] & & G(F(A)) \arrow[d, "G(F(f))"] \\ & A' \arrow[r, "\eta_{A'}"'] & G(F(A')) \end{tikzcd}$$

commutes (in other words, the unique map $\eta_A \to \eta_{A'} \circ f$ in $(A \Rightarrow G)$). It is easily checked that F is a functor $\mathscr{A} \to \mathscr{B}$, and the diagram tells us that η is a natural transformation $1 \to GF$. So by Theorem 2.3.6, F is left adjoint to G. $\square$

4. Chapter 3: Representables

This chapter explores the theme of how each object sees and is seen by the category in which it lives. We are naturally led to the notion of representable functor, which (after adjunctions) provides our second approach to the idea of universal property.

Some related link:

- Category Theory For Beginners: Representable Functors

4.1. Definitions and examples.

A representable functor can be fully characterized by $\mathscr{A}(A, -)$.

Definition 4.1.1 Let $\mathscr{A}$ be a locally small category and $A \in \mathscr{A}$. We define a functor

$$H^A = \mathscr{A}(A, -) \colon \mathscr{A} \to \mathbf{Set}$$

as follows:

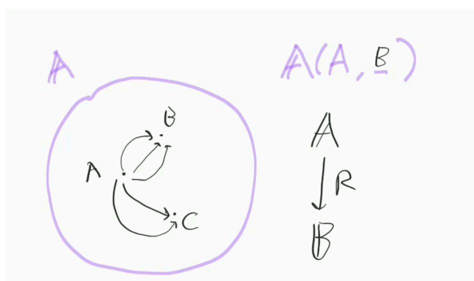
- for objects $B \in \mathscr{A}$, put $H^A(B) = \mathscr{A}(A, B)$;
- for maps $B \xrightarrow{g} B'$ in $\mathscr{A}$, define

  $$H^A(g) = \mathscr{A}(A, g) \colon \mathscr{A}(A, B) \to \mathscr{A}(A, B')$$

  by

  $$p \mapsto g \circ p$$

  for all $p \colon A \to B$.



Definition 4.1.3 Let $\mathscr{A}$ be a locally small category. A functor $X \colon \mathscr{A} \to \mathbf{Set}$ is **representable** if $X \cong H^A$ for some $A \in \mathscr{A}$. A **representation** of X is a choice of an object $A \in \mathscr{A}$ and an isomorphism between H^A and X.

Not all functors into **Set** are representable, but forgetful functors (more generally, functors with left adjoints) are.

Lemma 4.1.10 *Let* $\begin{tikzcd} \mathscr{A} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, shift left, "G"] \end{tikzcd}$ *be locally small categories, and let* $A \in \mathscr{A}$*. Then the functor*

$$\mathscr{A}(A, G(-)) \colon \mathscr{B} \to \mathbf{Set}$$

(that is, the composite $\mathscr{B} \xrightarrow{G} \mathscr{A} \xrightarrow{H^A} \mathbf{Set}$*) is representable.*

Proof We have

$$\mathscr{A}(A, G(B)) \cong \mathscr{B}(F(A), B)$$

for each $B \in \mathscr{B}$. If we can show that this isomorphism is natural in B, then we will have proved that $\mathscr{A}(A, G(-))$ is isomorphic to $H^{F(A)}$ and is therefore representable. So, let $B \xrightarrow{q} B'$ be a map in $\mathscr{B}$. We must show that the square

$$\begin{tikzcd} \mathscr{A}(A, G(B)) \arrow[r] \arrow[d, "G(q)\circ-"'] & \mathscr{B}(F(A), B) \arrow[d, "q\circ-"] \\ \mathscr{A}(A, G(B')) \arrow[r] & \mathscr{B}(F(A), B') \end{tikzcd}$$

commutes, where the horizontal arrows are the bijections provided by the adjunction. For $f \colon A \to G(B)$, we have

$$\begin{tikzcd} f \arrow[r, mapsto] \arrow[d, mapsto] & \bar{f} \arrow[d, mapsto] \\ G(q) \circ f \arrow[r, mapsto] & \frac{q \circ \bar{f}}{G(q) \circ f}, \end{tikzcd}$$

so we must prove that $q \circ \bar{f} = \overline{G(q) \circ f}$. This follows immediately from the naturality condition (2.2) in the definition of adjunction (with $g = \bar{f}$). □

Definition 4.1.15 Let $\mathscr{A}$ be a locally small category. The functor

$$H^\bullet \colon \mathscr{A}^{\mathrm{op}} \to [\mathscr{A}, \mathbf{Set}]$$

is defined on objects A by $H^\bullet(A) = H^A$ and on maps f by $H^\bullet(f) = H^f$.

Definition 4.1.16 Let $\mathscr{A}$ be a locally small category and $A \in \mathscr{A}$. We define a functor

$$H_A = \mathscr{A}(-, A)\colon \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$$

as follows:

- for objects $B \in \mathscr{A}$, put $H_A(B) = \mathscr{A}(B, A)$;
- for maps $B' \xrightarrow{g} B$ in $\mathscr{A}$, define

  $$H_A(g) = \mathscr{A}(g, A) = g^* = - \circ g\colon \mathscr{A}(B, A) \to \mathscr{A}(B', A)$$

  by

  $$p \mapsto p \circ g$$

  for all $p\colon B \to A$.

Definition 4.1.17 Let $\mathscr{A}$ be a locally small category. A functor $X\colon \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$ is **representable** if $X \cong H_A$ for some $A \in \mathscr{A}$. A **representation** of X is a choice of an object $A \in \mathscr{A}$ and an isomorphism between H_A and X.

Definition 4.1.21 Let $\mathscr{A}$ be a locally small category. The **Yoneda embedding** of $\mathscr{A}$ is the functor

$$H_\bullet\colon \mathscr{A} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$$

defined on objects A by $H_\bullet(A) = H_A$ and on maps f by $H_\bullet(f) = H_f$.

Here is a summary of the definitions so far.

For each $A \in \mathscr{A}$, we have a functor	$\mathscr{A} \xrightarrow{H^A} \mathbf{Set}$.
Putting them all together gives a functor	$\mathscr{A}^{\mathrm{op}} \xrightarrow{H^\bullet} [\mathscr{A}, \mathbf{Set}]$.
For each $A \in \mathscr{A}$, we have a functor	$\mathscr{A}^{\mathrm{op}} \xrightarrow{H_A} \mathbf{Set}$.
Putting them all together gives a functor	$\mathscr{A} \xrightarrow{H_\bullet} [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$.

The second pair of functors is the dual of the first. Both involve contravariance; it cannot be avoided.

One more definition: (I don't understand this)

Definition 4.1.22 Let $\mathscr{A}$ be a locally small category. The functor

$$\mathrm{Hom}_{\mathscr{A}} : \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathbf{Set}$$

is defined by

$$\begin{array}{ccc} (A, B) & \mapsto & \mathscr{A}(A, B) \\ f \uparrow \downarrow g & \mapsto & \downarrow g \circ - \circ f \\ (A', B') & \mapsto & \mathscr{A}(A', B'). \end{array}$$

In other words, $\mathrm{Hom}_{\mathscr{A}}(A, B) = \mathscr{A}(A, B)$ and $(\mathrm{Hom}_{\mathscr{A}}(f, g))(p) = g \circ p \circ f$, whenever $A' \xrightarrow{f} A \xrightarrow{p} B \xrightarrow{g} B'$.

Generalized elements:

Objects of an arbitrary category do not have elements in any obvious sense. However, *sets* certainly have elements, and we have observed that an element of a set A is the same thing as a map $1 \to A$. This inspires the following definition.

Definition 4.1.25 Let A be an object of a category. A **generalized element** of A is a map with codomain A. A map $S \to A$ is a generalized element of A of **shape** S.

4.2. The Yoneda lemma.

Video: Category Theory For Beginners: Yoneda Lemma

Informally, then, the Yoneda lemma says that for any $A \in \mathscr{A}$ and presheaf X on $\mathscr{A}$:

A natural transformation $H_A \to X$ is an element of $X(A)$.

Here is the formal statement. The proof follows shortly.

Theorem 4.2.1 (Yoneda) *Let $\mathscr{A}$ be a locally small category. Then*

$$[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X) \cong X(A) \tag{4.3}$$

naturally in $A \in \mathscr{A}$ and $X \in [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$.

5. Chapter 5: Limits

Limits are about what goes on inside a category. Whenever you meet a method for taking some objects and maps in a category and constructing a new object out of them, there is a good chance that you are looking at either a limit or a colimit.

5.1. Definitions and examples.

Here are three types of limits: product, equalizer and pullback.

5.1.1. *Product.*

Definition 5.1.1 Let $\mathscr{A}$ be a category and $X, Y \in \mathscr{A}$. A **product** of X and Y consists of an object P and maps

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd}$$

with the property that for all objects and maps

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ X & & Y \end{tikzcd} \tag{5.1}$$

in $\mathscr{A}$, there exists a unique map $\bar{f}: A \to P$ such that

$$\begin{tikzcd} & A \arrow[d, dotted, "\bar{f}"] \arrow[ddl, "f_1"'] \arrow[ddr, "f_2"] & \\ & P \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\ X & & Y \end{tikzcd} \tag{5.2}$$

commutes. The maps p_1 and p_2 are called the **projections**.

Here the point is existence of P.

An example:

Example 5.1.5 Now let X and Y be vector spaces. We can form their direct sum, $X \oplus Y$, whose elements can be written as either (x, y) or $x + y$ (with $x \in X$ and $y \in Y$), according to taste. There are linear projection maps

$$\begin{tikzcd} & X \oplus Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} & (x, y) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ x & & y. \end{tikzcd}$$

It can be shown that $X \oplus Y$, together with p_1 and p_2, is the product of X and Y in the category of vector spaces (Exercise 5.1.33).

A generalized version of product:

Definition 5.1.7 Let $\mathscr{A}$ be a category, I a set, and $(X_i)_{i\in I}$ a family of objects of $\mathscr{A}$. A **product** of $(X_i)_{i\in I}$ consists of an object P and a family of maps

$$\left(P \xrightarrow{p_i} X_i\right)_{i\in I}$$

with the property that for all objects A and families of maps

$$\left(A \xrightarrow{f_i} X_i\right)_{i\in I} \tag{5.3}$$

there exists a unique map $\bar{f}\colon A \to P$ such that $p_i \circ \bar{f} = f_i$ for all $i \in I$.

5.1.2. *Equalizers.*

To define our second type of limit, we need a preliminary piece of terminology: a **fork** in a category consists of objects and maps

$$A \xrightarrow{f} X \underset{t}{\overset{s}{\rightrightarrows}} Y \tag{5.4}$$

such that $sf = tf$.

Definition 5.1.11 Let $\mathscr{A}$ be a category and let $X \underset{t}{\overset{s}{\rightrightarrows}} Y$ be objects and maps in $\mathscr{A}$. An **equalizer** of s and t is an object E together with a map $E \xrightarrow{i} X$ such that

$$E \xrightarrow{i} X \underset{t}{\overset{s}{\rightrightarrows}} Y$$

is a fork, and with the property that for any fork (5.4), there exists a unique map $\bar{f}\colon A \to E$ such that

$$\begin{tikzcd} A \arrow[d, dotted, "\bar{f}"'] \arrow[dr, "f"] & \\ E \arrow[r, "i"'] & X \end{tikzcd} \tag{5.5}$$

5.1.3. *Pullbacks.*

Definition 5.1.16 Let $\mathscr{A}$ be a category, and take objects and maps

$$\begin{tikzcd} & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd} \tag{5.6}$$

in $\mathscr{A}$. A **pullback** of this diagram is an object $P \in \mathscr{A}$ together with maps $p_1 : P \to X$ and $p_2 : P \to Y$ such that

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd} \tag{5.7}$$

commutes, and with the property that for any commutative square

$$\begin{tikzcd} A \arrow[r, "f_2"] \arrow[d, "f_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd} \tag{5.8}$$

in $\mathscr{A}$, there is a unique map $\bar{f}: A \to P$ such that

$$\begin{tikzcd} A \arrow[dr, dotted, "\bar{f}"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\ & X \arrow[r, "s"'] & Z \end{tikzcd} \tag{5.9}$$

commutes. (For (5.9) to commute means only that $p_1\bar{f} = f_1$ and $p_2\bar{f} = f_2$, since the commutativity of the square is already given.)

5.1.4. *Definition of limit.*

Definition 5.1.18 Let $\mathscr{A}$ be a category and $\mathbf{I}$ a small category. A functor $\mathbf{I} \to \mathscr{A}$ is called a **diagram** in $\mathscr{A}$ of **shape** $\mathbf{I}$.

Definition 5.1.19 Let $\mathscr{A}$ be a category, $\mathbf{I}$ a small category, and $D\colon \mathbf{I} \to \mathscr{A}$ a diagram in $\mathscr{A}$.

(a) A **cone** on D is an object $A \in \mathscr{A}$ (the **vertex** of the cone) together with a family

$$\left(A \xrightarrow{f_I} D(I)\right)_{I\in\mathbf{I}} \tag{5.15}$$

of maps in $\mathscr{A}$ such that for all maps $I \xrightarrow{u} J$ in $\mathbf{I}$, the triangle

$$\begin{tikzcd} & D(I) \arrow[d, "Du"] \\ A \arrow[ur, "f_I"] \arrow[r, "f_J"'] & D(J) \end{tikzcd}$$

commutes. (Here and later, we abbreviate $D(u)$ as Du.)

(b) A **limit** of D is a cone $\left(L \xrightarrow{p_I} D(I)\right)_{I\in\mathbf{I}}$ with the property that for any cone (5.15) on D, there exists a unique map $\bar{f}\colon A \to L$ such that $p_I \circ \bar{f} = f_I$ for all $I \in \mathbf{I}$. The maps p_I are called the **projections** of the limit.

5.1.5. *Monics.*

Definition 5.1.29 Let $\mathscr{A}$ be a category. A map $X \xrightarrow{f} Y$ in $\mathscr{A}$ is **monic** (or a **monomorphism**) if for all objects A and maps $A \overset{x}{\underset{x'}{\rightrightarrows}} X$,

$$f \circ x = f \circ x' \implies x = x'.$$

This can be rephrased suggestively in terms of generalized elements: f is monic if for all generalized elements x and x' of X (of the same shape), $fx = fx' \implies x = x'$. Being monic is, therefore, the generalized-element analogue of injectivity.

Lemma 5.1.32 *A map $X \xrightarrow{f} Y$ is monic if and only if the square*

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback.

The dual concepts of product, equalizer and pullback are coproduct, coequalizer and pushout. Dual concept of monic is epic.

5.2. **Interactions between functors and limits.**

Definition 5.3.1 (a) Let $\mathbf{I}$ be a small category. A functor $F\colon \mathscr{A} \to \mathscr{B}$ **preserves limits of shape I** if for all diagrams $D\colon \mathbf{I} \to \mathscr{A}$ and all cones $\left(A \xrightarrow{p_I} D(I)\right)_{I\in\mathbf{I}}$ on D,

$$\begin{aligned} &\left(A \xrightarrow{p_I} D(I)\right)_{I\in\mathbf{I}} \text{ is a limit cone on } D \text{ in } \mathscr{A} \\ \Longrightarrow &\left(F(A) \xrightarrow{Fp_I} FD(I)\right)_{I\in\mathbf{I}} \text{ is a limit cone on } F\circ D \text{ in } \mathscr{B}. \end{aligned}$$

(b) A functor $F\colon \mathscr{A} \to \mathscr{B}$ **preserves limits** if it preserves limits of shape $\mathbf{I}$ for all small categories $\mathbf{I}$.

(c) **Reflection** of limits is defined as in (a), but with $\Longleftarrow$ in place of $\Longrightarrow$.

Definition 5.3.5 A functor $F\colon \mathscr{A} \to \mathscr{B}$ **creates limits (of shape I)** if whenever $D\colon \mathbf{I} \to \mathscr{A}$ is a diagram in $\mathscr{A}$,

- for any limit cone $\left(B \xrightarrow{q_I} FD(I)\right)_{I\in\mathbf{I}}$ on the diagram $F\circ D$, there is a unique cone $\left(A \xrightarrow{p_I} D(I)\right)_{I\in\mathbf{I}}$ on D such that $F(A) = B$ and $F(p_I) = q_I$ for all $I \in \mathbf{I}$;
- this cone $\left(A \xrightarrow{p_I} D(I)\right)_{I\in\mathbf{I}}$ is a limit cone on D.

The forgetful functors from **Grp**, **Ring**, ... to **Set** all create limits (Exercise 5.3.11). The word *creates* is explained by the following result.

References

1. Leinster, T.: Basic Category Theory, https://arxiv.org/abs/1612.09375