Simplification Rules for Poka Yoke Activities

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1 Introduction

This document defines simplification rules for (simple UML-like) activities. These activities and their semantics are slightly different from other UML semantics, e.g. fUML, activities. For example, every action has an associated guard and effects that specify how action execution influences the system state. Edges have two guards, named incoming and outgoing guard, that define when a token can be placed on an edge, and when it can be removed, respectively. This document aims at finding patterns of action, nodes, edges that can be transformed into simpler patterns, while maintaining the same operational semantic of the activity. We call these transformations *simplifications.*

2 Preliminaries and Notation

We largely refer to the Operation Semantics of Poka Yoke Activities document for the static structure and dynamics behaviors of activities. We recall here only the elements that are useful for this document.

2.1 Statics

State. Activities are defined in the presence of *state*, for example a set of variables and their current valuation. We keep the notion of state more abstract for the purpose of defining the semantics, and let State be the set of all states. Users of this semantics could later instantiate State as desired, e.g., as variable valuation mappings. Let $\sigma \in \mathsf{State}$ be a typical state.

Guards and effects. Guards are defined to be state predicates, while effects are defined to be state transformers, i.e., functions that map states to any number of 'successor states', to be able to model nondeterministic action behavior.

Let $\mathsf{Guard} = \mathsf{State} \to \mathbb{B}$ be the set of all *guards* and $\mathsf{Effects} = \mathsf{State} \to 2^{\mathsf{State}}$ be the set of all *effects functions* over states, with 2^{State} the powerset of State. We use $g \in \mathsf{Guard}$ and $u \in \mathsf{Effects}$ to denote a typical guard and effects function, respectively.

Core nodes. Now that actions have been defined, we can define the *nodes* of core activities, which carry out these actions during executions of the activity. Core activity nodes consist of two parts: a node identifier, and a node type that also contains (wraps) the action to perform. The node identifiers are used to give identity to nodes, which are needed to allow having multiple nodes in an activity that perform the same action. In core activities, actions can be executed in one of two ways: in and-style or in or-style, as indicated by the node type. An and-style execution of an action node requires that *all* its incoming edges must have a token, and after having performed the action, *all* outgoing edges will get a token (under some additional conditions that are explained later). In contrast, an or-style execution of an action node requires just *a single* incoming edge to have a token, and after having performed the action, *a single* outgoing edge will receive a token (again, under some extra conditions). This distinction between and-type and or-type nodes is also made in our semantics document with respect to a token firing semantics, explaining that most UML node types can be divided into these two groups. For example, UML nodes like 'fork' and 'join' can be represented as and-nodes, while 'decision' and 'merge' can be represented in terms of or-nodes.

More formally, let NodeID be a set of all *node identifiers*, used to give identity to activity nodes, and let $\ell \in \mathsf{NodeID}$ be a typical node identifier. Then the set $\mathsf{CoreNode}$ of all possible *core activity nodes* is defined as follows:

$$\mathsf{CoreNode} = \mathsf{NodeID} \times \mathsf{CoreNodeType} \qquad \mathsf{CoreNodeType} ::= \mathsf{and}\langle\alpha\rangle \mid \mathsf{or}\langle\alpha\rangle$$

Core activity nodes are pairs consisting of a node identifier ℓ and an action α to execute, either in and-style or in or-style. We write $\eta \in \mathsf{CoreNode}$ to denote a typical node. Let $\mathsf{action} : \mathsf{CoreNode} \to \mathsf{Action}$ be a projection function for obtaining the actions of core nodes, so that:

$$\mathsf{action}((\ell, \mathsf{and}\langle\alpha\rangle)) = \alpha \qquad \mathsf{action}((\ell, \mathsf{or}\langle\alpha\rangle)) = \alpha$$

2.2 Dynamics

This section defines the dynamic behavior of core activities, by means of a token passing style operational semantics. Its rules describe how to go from one configuration to another, where a configuration is essentially a set of edges holding a token together with a 'current' state. There are two token-passing rules: one for and-style and one for or-style action execution.

$\mathsf{Configurations}$. The semantics of core activities is essentially defined as a relation between *configurations*, in the sense that an execution step in an activity gets you from one configuration to another configuration. A configuration describes which edges currently hold a token, and what the current state is.

More formally, let $\mathsf{Config} = 2^{\mathcal{E}} \times \mathsf{State}$ be the set of all configurations. A configuration $(\Sigma, \sigma) \in \mathsf{Config}$ is a pair with $\Sigma \subseteq \mathcal{E}$ a set of edges—the ones currently holding a token—and σ a 'current' state. We write $c \in \mathsf{Config}$ to denote a typical configuration, such that $c = (\Sigma, \sigma)$.

Any edge ε is said to be *enabled in* c if $\varepsilon \in \Sigma$ and $\mathsf{guard}_{\mathsf{out}}(\varepsilon)(\sigma)$ holds, i.e., if ε has a token in c and if the outgoing guard of ε holds in c. If c is clear from the context, we may just say that ε is *enabled.* A (firing) schedule for core activity $\mathcal{A}$ is a *finite* [1] sequence of edges e_i such that e_1 is fireable in the initial configuration of $\mathcal{A}$, and each e_i is fireable from the configuration reached by starting in the initial configuration and firing e_j for $1 \leq j \leq i$ in the schedule.

2.3 Additional Notation

A core activity $\mathcal{A} = (\mathcal{N}, \mathcal{E})$ is fundamentally a graph with nodes $\mathcal{N} \subseteq \mathsf{CoreNode}$ and edges $\mathcal{E} \subseteq \mathcal{N} \times \mathsf{Guard} \times \mathsf{Guard} \times \mathcal{N}$. Let us define F as the *flow relation* of $\mathcal{A}$, defined as $F \subseteq (\mathcal{N} \times \mathcal{E}) \cup (\mathcal{E} \times \mathcal{N})$. We use the following symbols and notation for the set of predecessors and successors of a node $n \in \mathcal{N}$ and edge $e \in \mathcal{E}$:

$$^{\bullet}e = \{n \mid (n, e) \in F\} = \text{the set of input nodes of } e$$

$$e^{\bullet} = \{n \mid (e, n) \in F\} = \text{the set of output nodes of } e$$

$$^{\bullet}n = \{e \mid (e, n) \in F\} = \text{the set of input edges of } n$$

$$n^{\bullet} = \{e \mid (n, e) \in F\} = \text{the set of output edges of } n$$

[1] do we need infinite?

With slight abuse of notation, we extend this notation to sets of nodes and edges.

A core activity $\mathcal{A}$ is *safe* if all configurations reachable by legal sequences of edge firings from the initial configuration have either zero or one tokens in every edge. Note that edges already have at most one token by the definition of configuration: every core activity is safe by definition. Further, the core activity is ordinary in the Petri net sense: all edges have a weight equal to 1; for this reason, we largely disregard edge weights.

For every core activity $\mathcal{A}$, the language of $\mathcal{A}$, denoted $L(\mathcal{A})$, is the set of all legal sequences $\omega \in \mathcal{N}^*$ of edges firings starting from the initial configuration of tokens c_0. Let X be a set of nodes of core activity $\mathcal{A}$. We use $\mathcal{A}|X$ for the core activity obtained by restricting the underlying graph to X, and $\mathcal{A} - X$ to denote $\mathcal{A}|\bar{X}$, where $\bar{X}$ denotes the complement of X. This notation is also used for core activity states and firing schedules. For example, $\mathcal{A} - \{e_1, e_2\}$ refers to what is left of activity $\mathcal{A}$ once one removes the edges e_1 and e_2. Finally, for a core activity $\mathcal{A}$ and $X \subset \mathcal{N}$, we define $L(\mathcal{A}) \mid X$ to be the set of all schedules of $\mathcal{A}$ with all pairs containing a transition *not* from X deleted. Further, $L(\mathcal{A}) - X = L(\mathcal{A}) \mid (\mathcal{N} \setminus X)$. Note that this simply means that $L(\mathcal{A}) \mid X$ contains the same language of $L(\mathcal{A})$ where each occurrence of nodes in X and every edge e whose source is in X have been projected away from each sequence ω. The firings, and the overall dynamics of the activity remains the same: just its sequence output is "cleaned" from the elements of X and from any edge whose source is within X.

We draw inspiration from [4] for the following definition of equivalence of core activities.

Equivalence of Core Activities The goal of this manuscript is to take a core activity $\mathcal{A}$ and to reduce it to a smaller activity $\mathcal{A}'$ that is more amenable to analysis. Of course, we want the activity $\mathcal{A}'$ to be in some sense the "same" as the original activity $\mathcal{A}$. We will focus on language preservation (among unmodified edges).

Definition 2.1. *Fix some rule for transforming a core activity $\mathcal{A}$ to a core activity $\mathcal{A}'$. Let U be the set of edges of $\mathcal{A}$ which are left completely unmodified by this transformation. We say that the (simplified) core activity $\mathcal{A}'$ is equivalent to activity $\mathcal{A}$ after the transformation if*

$$L(\mathcal{A})|U = L(\mathcal{A}')|U.$$

Deadlock Preserving Equivalence prove that this notion is dealock preserving.

3 Simplifications

3.1 Serial Fusion

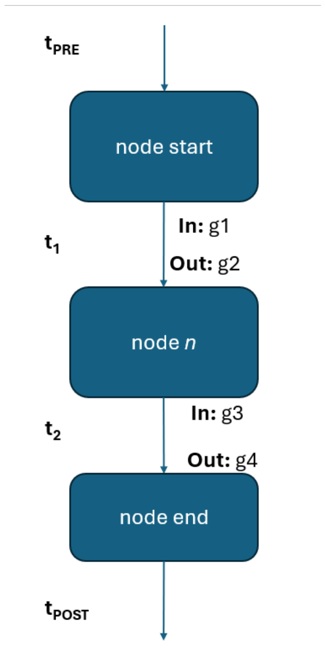
Simplification Rule 3.1. *If there exist two edges e_1 with $\mathsf{guard}_{in}(e_1) = g_1$, $\mathsf{guard}_{out}(e_1) = g_2$, and e_2 with $\mathsf{guard}_{in}(e_2) = g_3$, $\mathsf{guard}_{out}(e_2) = g_4$, and a node n satisfying:*

1. *The initial configuration c_0 does not place any token on e_2*
2. *$e_1{}^\bullet = \{n\} = {}^\bullet e_2$, i.e. node n is the unique output node of e_1 and the unique input node of e_2*
3. *Node n is disconnected from all other edges, and has no effects*
4. *$({}^\bullet e_1)^\bullet = \{e_1\}$, i.e. edge e_1 does not share any of its input place with any other edges*

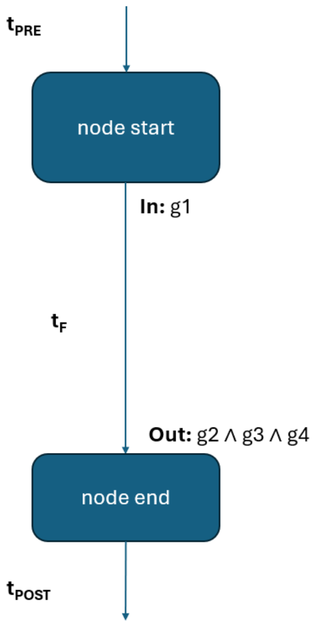
then edges e_1 and e_2 can be replaced by a single edge e_F with input set ${}^\bullet e_F = {}^\bullet e_1$, with output set $e_F{}^\bullet = e_2{}^\bullet$ and guards $\mathsf{guard}_{in}(e_F) = g_1$, $\mathsf{guard}_{out}(e_F) = g_2 \wedge g_3 \wedge g_4$.

Proof. To satisfy Definition 2.1, we must show that $L(\mathcal{A}) - \{e_1, e_2\} \subseteq L(\mathcal{A}') - \{e_F\}$ and that $L(\mathcal{A}) - \{e_1, e_2\} \supseteq L(\mathcal{A}') - \{e_F\}$. For simplicity, let us assume that node n does not have a guard. In presence of a guard, it is sufficient to conjunct it with either the outgoing guard g_2 or the incoming guard g_3.

Let us first show that $L(\mathcal{A}) - \{e_1, e_2\} \subseteq L(\mathcal{A}') - \{e_F\}$; we do so by induction on the number of firings of e_2. Note that we are interested in *non-blocking* sequences, so we should always be able to take edge e_2 and execute its outgoing node (which might be a final node or not). First, note that $\mathcal{A} - \{e_1, e_2\} = \mathcal{A}' - \{e_F\}$. For the base case, let ς be a schedule that contains no firing of e_2. If also $e_1 \notin \varsigma$, then $\varsigma \in L(\mathcal{A} - \{e_1, e_2, n\})$, and we can use $\varsigma' = \varsigma$ in $\mathcal{A}'$ to satisfy our claim. If instead $e_1 \in \varsigma$, note that there must be exactly one firing of e_1 in ς since $\mathcal{A}$ is safe. Let $\varsigma = \varsigma_1(e_1)\varsigma_2$ where neither e_1 or e_2 appear in ς_1, ς_2. By definition, g_1 must be true at some point of ς_1 in order to place a token in e_1. Similarly, both guards g_2 and g_3 must be true at the end of ς_1 for the firing of e_1. Note that e_1 cannot be enabled at the end of ς_2 – since e_2 has not fired yet – because the activity is safe and node n has no other outgoing edges. Then, we can use $\varsigma' = \varsigma_1(\cdot)\varsigma_2$ as a schedule for $\mathcal{A}'$ – note that we can put a token on e_F if and only if g_1



(a) Original edge scenario.



(b) Edge simplification.

Figure 1: The series edge merging scenario. Pictures to be improved.

holds at some point, but we cannot remove it until also g_4 holds – and our claim is satisfied.

Let us now assume that the claim is true until k firings of e_2, and consider a schedule ς of $\mathcal{A}$ where there are $k+1$ firings of e_2. Since $e_1{}^\bullet = \{n\} = {}^\bullet e_2$, activity $\mathcal{A}$ is safe, and e_2 is not in the initial configuration, e_1 must fire exactly once before each firing of e_2. Hence, ς contains either $k+1$ or $k+2$ firings of e_1. Let consider a schedule with $k+1$ firings of e_1: we can write

$$\varsigma = \varsigma_1(e_1)v_1(e_2)\varsigma_2(e_1)v_2 \ldots \varsigma_{k+1}(e_1)v_{k+1}(e_2)\varsigma_{k+2},$$

where neither e_1 or e_2 appears in ς_i nor v_i. By inductive hypothesis, we can set $\varsigma' = \varsigma_1(e_1)v_1(e_2)\varsigma_2(e_1)v_2 \ldots \varsigma_{k+1}$ such that the two languages are equivalent (until step k). If edge e_1 does not share any input nodes, then then final firing in ς must happen when $g_2 \wedge g_3$ holds. Let us now analyze v_{k+1}: if during this sequence of firings the values of g_2, g_3, g_4, do not change – and in particular must all be true, since we have a firing of e_2 afterwards – we can simply adapt the final part of the schedule as $\varsigma_{k+1}(e_F)v_{k+1}\varsigma_{k+2}$ or $\varsigma_{k+1}v_{k+1}(e_F)\varsigma_{k+2}$ without loss of generality, and again our claim is satisfied. In general however, sequence v_{k+1} can change the truth values of g_2 or g_3 (Note that at the end of v_{k+1} guard g_4 must hold, since we have a firing of e_2). If that happens, we have a sequence in $\mathcal{A}$ that cannot be reproduced in $\mathcal{A}'$: to prevent this, we impose that if g_4 is true, then also $g_2 \wedge g_3$ must hold. In this way, we ensure that at the end of v_{k+1} all guards g_2, g_3, g_4 must hold, and we can use the adapted version of ς (see previous discussion) in $\mathcal{A}'$, and this concludes the proof.

Let us now show that $L(\mathcal{A}) - \{e_1, e_2\} \supseteq L(\mathcal{A}') - \{e_F\}$. Similarly to the previous case, we consider schedules ς' in $\mathcal{A}'$ and prove that there exists a schedule ς in $\mathcal{A}$ that ensure language equivalence. We proceed by induction in the number of steps of e_F. If there are no firings of e_F, the proof follows analogously to the previous case scenario. Let us now take the inductive hypothesis, and consider a schedule

$$\varsigma' = \varsigma'_1(e_F)\varsigma'_2(e_F) \ldots \varsigma'_{k+1}(e_F)\varsigma'_{k+2},$$

that contains $k+1$ firings of e_F, and the ς_i do not contain any firing of e_F. Note that ς_{k+1} enables the firing of e_F by definition: this means that, by inductive hypothesis, there exists a schedule ς in $\mathcal{A}$ where edge e_1 is enabled. The guards as well simply mirror this, as if $g_2 \wedge g_3 \wedge g_4$ holds, so does $g_2 \wedge g_3$ (the condition to enable e_1). Thus, edge e_1 can freely fire: we extend the schedule ς for $\mathcal{A}$ by appending the firing of e_1 at the end of ς. To that, we append also $(e_2)\varsigma'_{k+2}$: since ${}^\bullet e_F = {}^\bullet e_1$, if the last edge of ς'_{k+1} enables e_F, then it also enables e_1. $\square$

Simplification Rule 3.2. *If there exist two edges* e_1 *with* $\textsf{guard}_{in}(e_1) = g_1$, $\textsf{guard}_{out}(e_1) = g_2$, *and* e_2 *with* $\textsf{guard}_{in}(e_2) = g_3$, $\textsf{guard}_{out}(e_2) = g_4$, *and a node* n *satisfying:*

1. *The initial configuration* c_0 *does not place any token on* e_1
2. $e_1{}^\bullet = \{n\} = {}^\bullet e_2$, *i.e. node* n *is the unique output node of* e_1 *and the unique input node of* e_2
3. *Node* n *is disconnected from all other edges, and has no effects*
4. $({}^\bullet e_1)^\bullet = \{e_1\}$, *i.e. edge* e_1 *does not share any of its input place with any other edges*

then edges e_1 *and* e_2 *can be replaced by a single edge* e_F *with input set* ${}^\bullet e_F = {}^\bullet e_1$, *with output set* $e_F{}^\bullet = e_2{}^\bullet$ *and guards* $\textsf{guard}_{in}(e_F) = g_1 \wedge g_2 \wedge g_3$, $\textsf{guard}_{out}(e_F) = g_4$.

Proof. The proof *should* follow similarly, perhaps with induction on e_1 for the first part. #TODO $\square$

References

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