Covenant University
Raising a new Generation of Leaders

CSC 241
LINEAR ALGEBRA II

BILINEAR AND QUADRATIC FORMS

Mrs G.O. Alao

BILINEAR FORMS

Let V be a vector space of finite dimension over a field K.

A bilinear form on V is a mapping $f : V \times V \to K$ such that for all $a, b \in K$ and for all $u_i, v_i \in V$:

i) $f\left(au_1 + bu_2, v\right) = af\left(u_1, v\right) + bf\left(u_2, v\right)$

ii) $f\left(u, av_1 + bv_2\right) = af\left(u, v_1\right) + bf\left(u, v_2\right)$

We express condition (i) and (ii) by saying f is linear in the first and second variables respectively.

Example 1

Let A be an $n \times n$ matrix over K. Show that the mapping f defined below is a bilinear form on K^n

$$f(X, Y) = X^T A Y$$

Solution

For any $a, b \in K$ and for any $X_i, Y_i \in K^n$

i) $f\left(aX_1 + bX_2, Y\right) = \left(aX_1 + bX_2\right)^T AY$

$$= \left(aX_1^T + bX_2^T\right) AY$$

$$= aX_1^T AY + bX_2^T AY$$

$$= af\left(X_1, Y\right) + bf\left(X_2, Y\right)$$

Hence f is linear in the first variable.

Solution (Cont.)

ii) $f\left(X, aY_1 + bY_2\right) = X^T A\left(aY_1 + bY_2\right)$

$$= aX^T AY_1 + bX^T AY_2$$

$$= af\left(X, Y_1\right) + bf\left(X, Y_2\right)$$

Hence f is linear in the second variable.

Therefore, f is a bilinear form on K^n.

Exercise

Let $u=\left(x_1,x_2\right)$ and $v=\left(y_1,y_2\right)$. Determine which of the following are bilinear forms on R^2.

i) $f\left(u,v\right)=2x_1y_2-3x_2y_1$

ii) $f\left(u,v\right)=x_1x_2+y_1y_2$

Example 2

Let $u=\left(x_1,x_2,x_3\right)$ and $v=\left(y_1,y_2,y_3\right)$. Express f in matrix notation, where

$$f\left(u,v\right)=3x_1y_1-2x_1y_3+5x_2y_1+7x_2y_2-8x_2y_3$$
$$+4x_3y_2-6x_3y_3$$

Solution

Let $A = \left[a_{ij} \right]$, where a_{ij} is the coefficient of $x_i y_i$.

Then

$$f\left(u,v\right) = X^T A Y = \left[x_1 \quad x_2 \quad x_3 \right] \begin{bmatrix} 3 & 0 & -2 \\ 5 & 7 & -8 \\ 0 & 4 & -6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Exercise

Express the following functions in matrix notations:

i) $f\left[(x_1, x_2), (y_1, y_2)\right] = 2x_1y_1 - 3x_1y_2 + 4x_2y_2$

ii) $f\left[(x_1, x_2), (y_1, y_2)\right] = 3x_1y_1 - 2x_1y_2 + 4x_2y_1 - x_2y_2$

SYMMETRIC BILINEAR FORMS

Let f be a linear form on V. Then f is said to be symmetric if, for every $u,v \in V$

$$f(u,v) = f(v,u)$$

QUADRATIC FORMS

A mapping $q:V \to K$ is a quadratic form if $q(v) = f(v,v)$ for some symmetric bilinear form f on V :

Let $X = [x_i]$ denote a column vector of variable, q can be represented in the form:

$$q(X) = f(X,X) = X^T AX$$

Example 3

Find the symmetric matrix that corresponds to the quadratic form below:

$$q(x,y,z)=3x^2+4xy-y^2+8xz-6yz+z^2$$

Hint

The symmetric matrix, $A=\left[a_{ij}\right]$ that represents the quadratic forms, $q\left(x_1,x_2,\ldots,x_n\right)$ has the diagonal entries, a_{ii} equal to the coefficients of the square terms, x_i^2 and the non-diagonal entries a_{ij} and a_{ji} each equal to half of the coefficients of the cross product terms $x_i x_j$

Solution

The symmetric matrix is thus:

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & -1 & -3 \\ 4 & -3 & 1 \end{bmatrix}$$

Exercise

Find the symmetric matrix that corresponds to each of the following quadratic forms:

i) $q(x,y,z) = 3x^2 + xz - 2yz$

ii) $q(x,y,z) = 2x^2 - 5y^2 - 7z^2$

Example 4

Find the quadratic form, $q(X)$ that corresponds to the following symmetric matrix:

$$A = \begin{bmatrix} 5 & -3 \\ -3 & 8 \end{bmatrix}$$

Hint

The quadratic matrix, $q(X)$ that corresponds to a symmetric matrix, A is defined by

$$q(X) = X^T AX$$

where $X = [x_i]$ is the column vectors of unknown.

Solution

$$q(x,y) = X^T A X = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 5x - 3y & -3x + 8y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= 5x^2 - 3xy - 3xy + 8y^2$$

$$= 5x^2 - 6xy + 8y^2$$

Exercise

Find the quadratic forms, $q\left(X\right)$ that correspond to each of the following symmetric matrices:

i) $M = \begin{bmatrix} 4 & -5 & 7 \\ -5 & -6 & 8 \\ 7 & 8 & -9 \end{bmatrix}$

ii) $P = \begin{bmatrix} 2 & 4 & -1 & 5 \\ 4 & -7 & -6 & 8 \\ -1 & -6 & 3 & 9 \\ 5 & 8 & 9 & 1 \end{bmatrix}$