Hashing

Initially prepared by Dr. İlyas Çiçekli; improved by various Bilkent CS202 instructors.

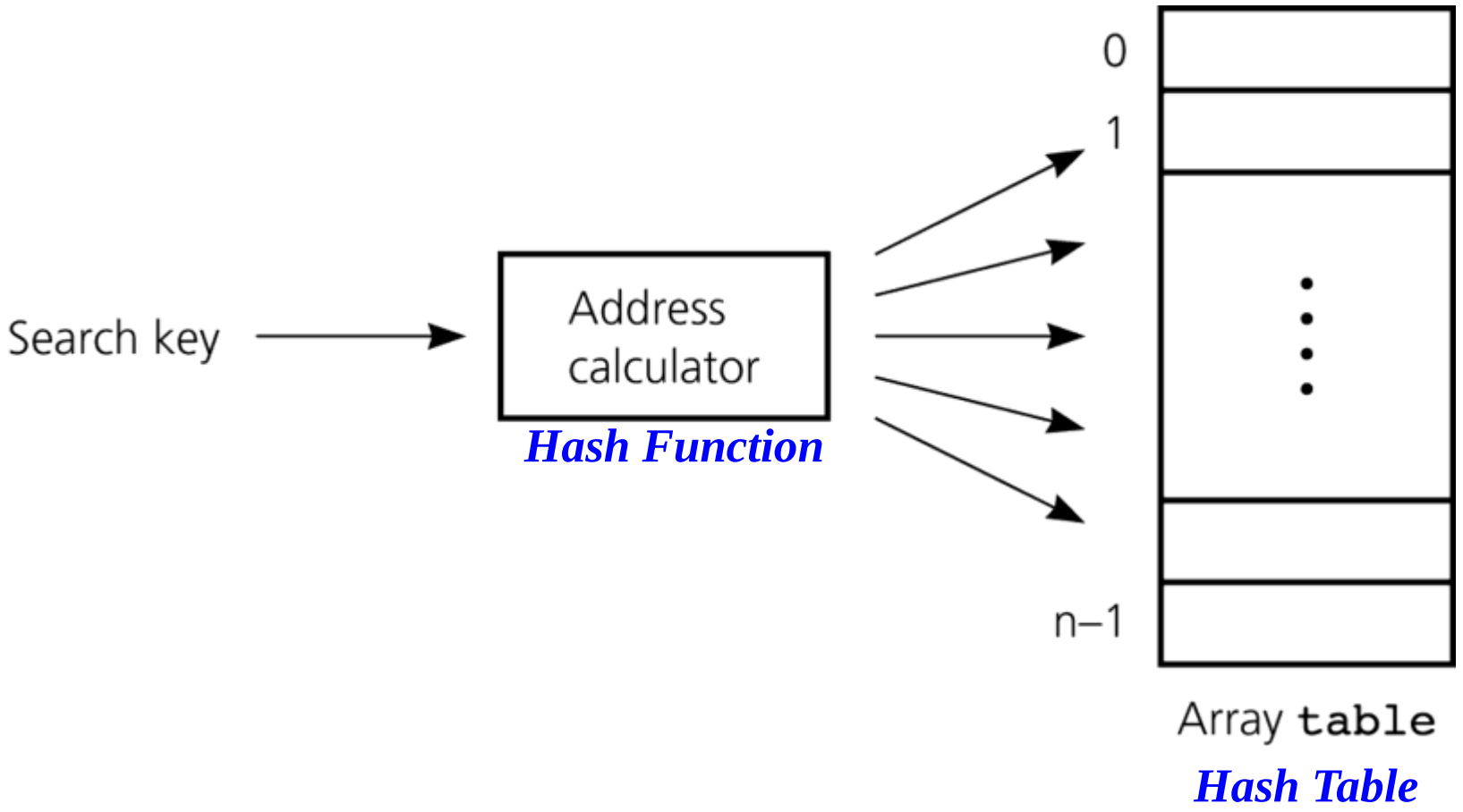
Hashing

- Using balanced search trees (2-3, 2-3-4, red-black, and AVL trees), we implement **table operations in O (logN) time**
  - retrieval, insertion, and deletion
- Can we find a data structure so that we can perform these table operations **even faster** (**e.g., in O(1) time**)?
  - **Hash Tables**

Hash Tables

- In hash tables, we have
  - **An array** (index ranges 0 … n – 1) and
    - Each array location is called a ***bucket***
  - **An address calculator** (***hash function***), which maps a search key into an array index between 0 … n – 1

Hash Function -- Address Calculator



Hashing

- A **hash function** tells us where to place an item in array called a **hash table**.
  - This method is known as **hashing**.

- Hash function **maps a search key into an integer** between 0 and n – 1.
  - We can have different hash functions.
  - Hash function depends on key type (int, string, ...)
    - E.g., **h(x) = x mod n**, where x is an integer

Collisions

- A **perfect hash function** maps each search key into a unique location of the hash table.
  - A perfect hash function is possible if we know all search keys in advance.
  - In practice we do not know all search keys, and thus, a hash function can map more than one key into the same location.

- **Collisions** occur when a hash function maps more than one item into the same array location.
  - We have to resolve the collisions using a certain mechanism.

Hash Functions

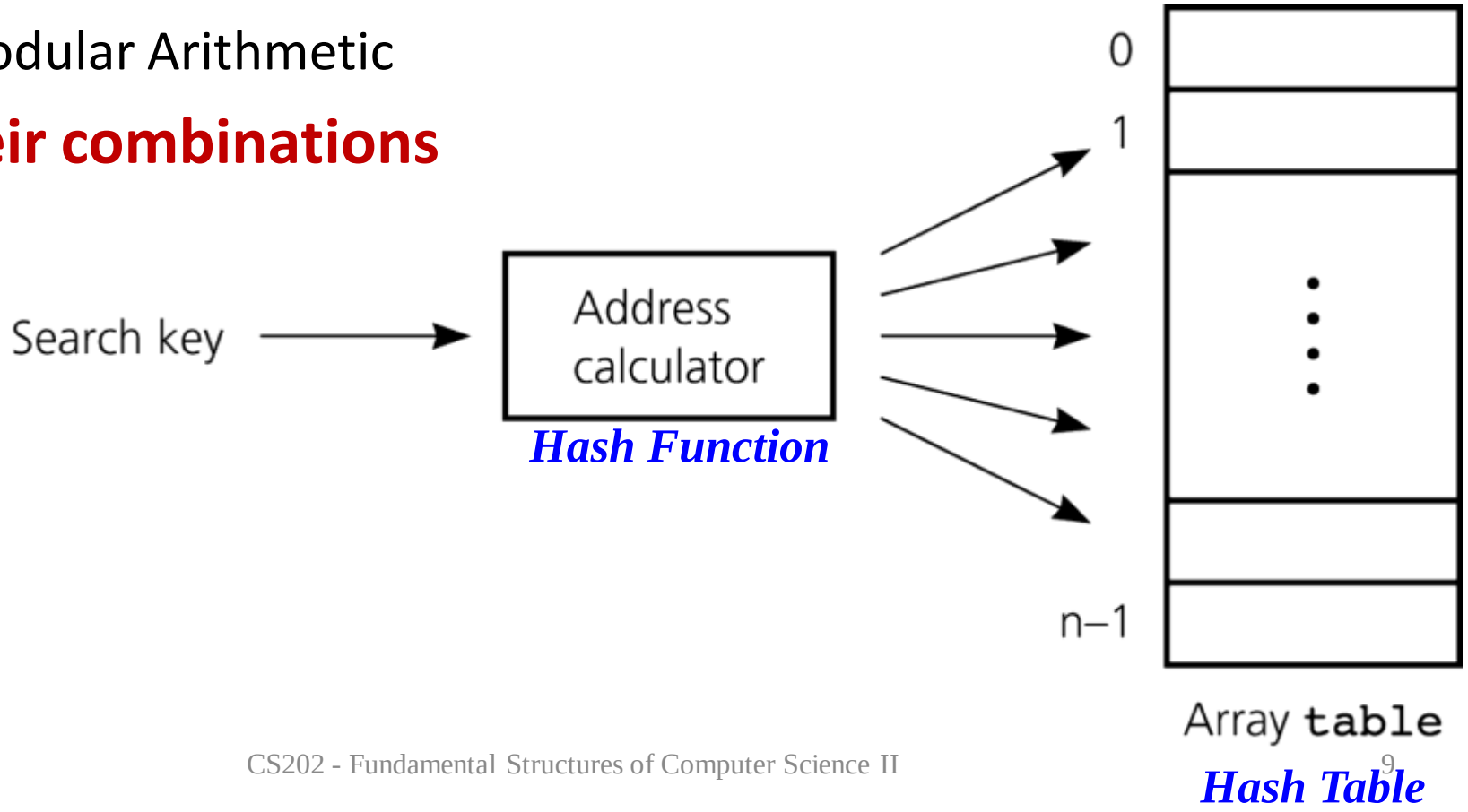
- We can design different hash functions.
- But a **good hash function** should
  - be easy and fast to compute
  - place items uniformly (evenly) throughout the hash table.
- We will consider only **integer hash functions**
  - On a computer, everything is represented with bits.
  - They can be converted into integers.
    - 10010101010010100001 10…. remember?

Everything is an Integer

- If search keys are strings, think of them as integers, and apply a hash function for integers.
- For example, strings can be encoded using ASCII codes of characters.
- Consider the string "NOTE"
  - ASCII code of **N** is 4Eh (01001110), **O** is 4Fh (01001111),
    T is 54h(01010100), **E** is 45h (01000101)
  - Concatenate four binary numbers to get a new binary number
    01001110010011110101010001000101= 4E4F5445h = **1313821765**

How to Design a Hash Function?

- **Three possibilities**
  1. Selecting digits
  2. Folding
  3. Modular Arithmetic
- **Or, their combinations**



Hash Functions -- Selecting Digits

- **Select certain digits** and combine to create the address.
- For example, suppose that we have 11-digit Turkish nationality ID's
  - Define a hash function that selects the 2nd and 5th most significant digits

    h(0**3**34**7**5678) = **37**

    h(0**2**34**5**5678) = **25**
  - Define the table size as 100
- Is this a good hash function?
  - No, since it does not place items uniformly.

Hash Functions -- Folding

- **Folding –** selecting all digits and adding them.

- For example, suppose previous nine-digit numbers
  - Define a hash function that selects all digits and adds them
    h(033475678) = **0 + 3 + 3 + 4 + 7 + 5 + 6 + 7 + 8** = 43
    h(023455678) = **0 + 2 + 3 + 4 + 5 + 5 + 6 + 7 + 8** = 40
  - Define the table size as 82

- We can select a group of digits and add the digits in this group as well.

Hash Functions -- Modular Arithmetic

- **Modular arithmetic –** provides a simple and effective hash function.

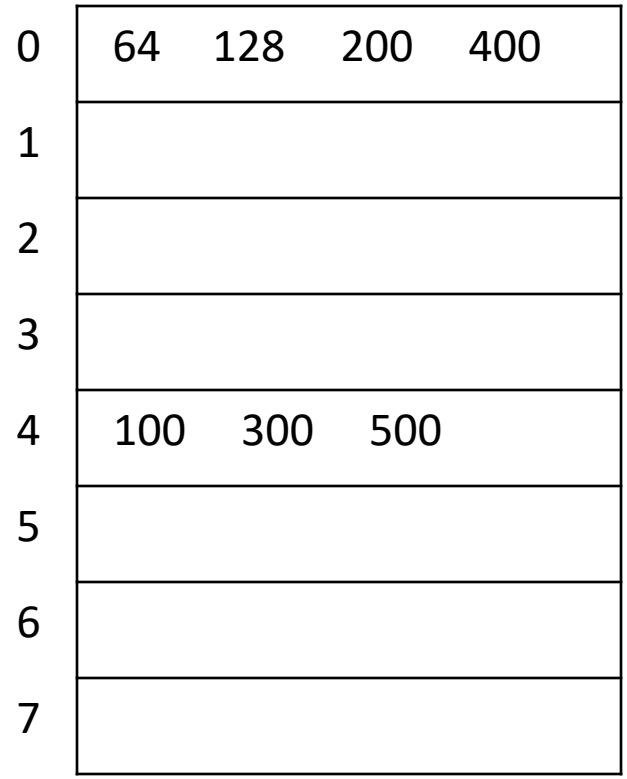
$$h(x) = x \bmod \text{tableSize}$$

- The table size should be a prime number.
  - ***Why? Think about it.***

- We will use modular arithmetic as our hash function in the rest of our discussions.

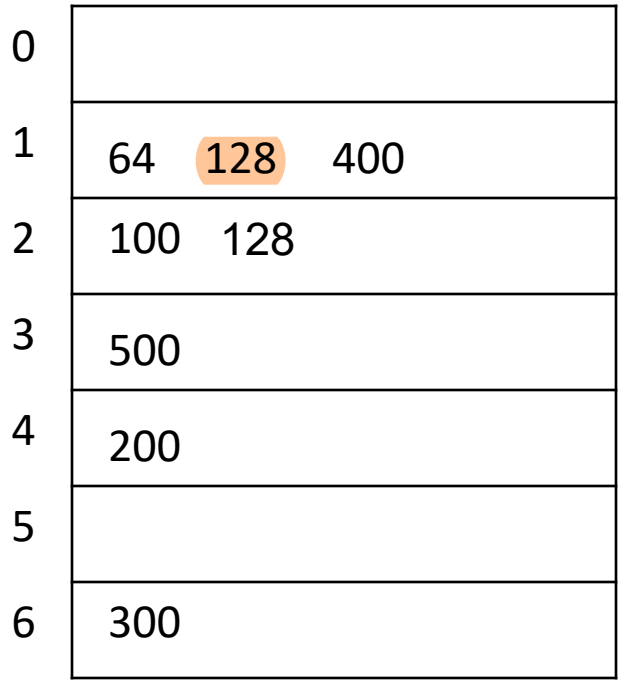
Why Primes?

- Assume you hash the following with x mod 8:
  - 64, 100, 128, 200, 300, 400, 500



Why Primes?

- Now try it with x mod 7
  - 64, 100, 128, 200, 300, 400, 500



Rationale

- If we are adding numbers a_1, a_2, a_3 … a_4 to a table of size m
  - All values will be hashed into multiples of
    gcd(a_1, a_2, a_3 … a_4 ,m)
  - For example, if we are adding 64, 100, 128, 200, 300, 400, 500 to a table of size 8, all values will be hashed to 0 or 4
    gcd(64,100,128,200,300,400,500, **8**) = 4
  - When m is a prime gcd(a_1, a_2, a_3 … a_4 ,m) = 1, all values will be hashed to anywhere
    gcd(64,100,128,200,300,400,500,**7**) = 1
    unless gcd(a_1, a_2, a_3 … a_4) = m, which is rare.

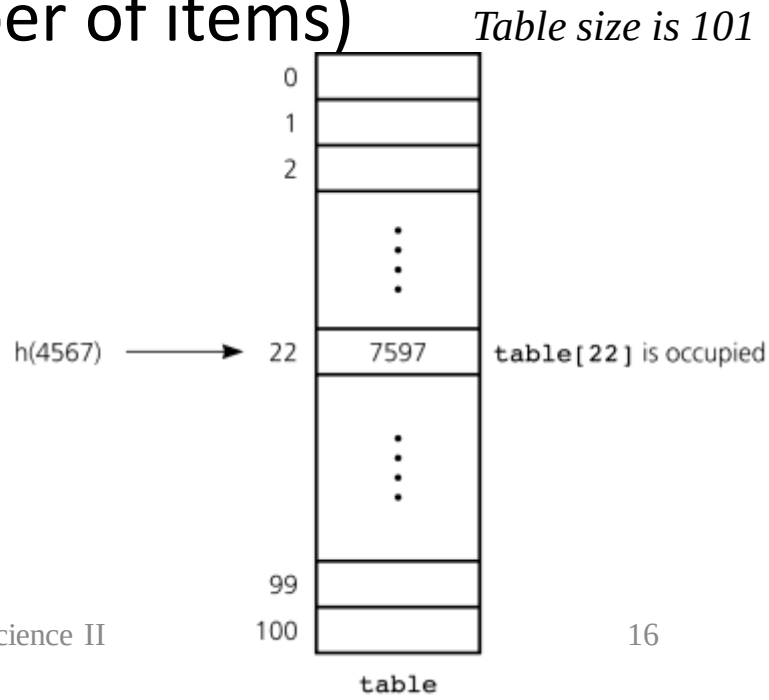
Collision Resolution

- **Collision resolution –** two general approaches
  - **Open Addressing**

    Each entry holds one item
  - **Chaining**

    Each entry can hold more than one item

    (**Buckets** – hold certain number of items)



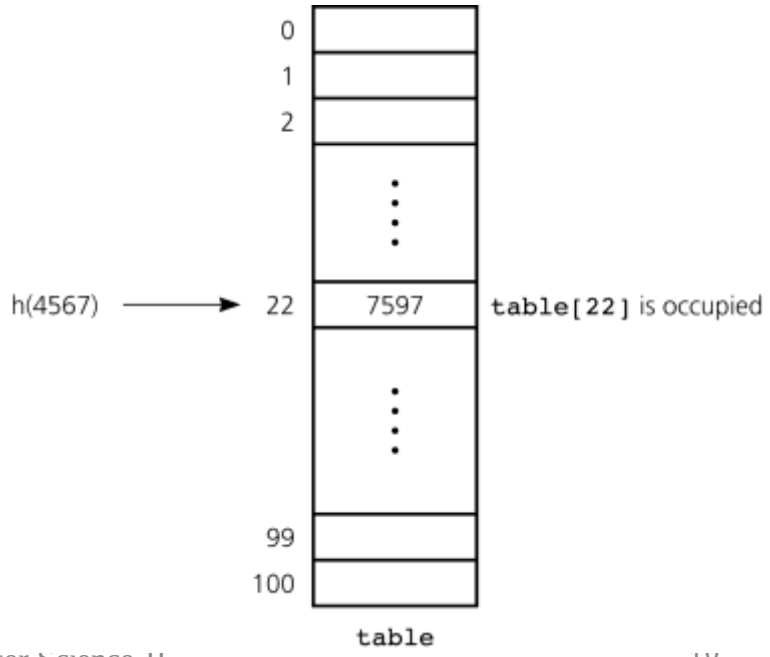
Open Addressing

- **Open addressing** – probes for some other empty location when a collision occurs.

- **Probe sequence:** sequence of examined locations. Different open-addressing schemes:
  - Linear Probing
  - Quadratic Probing
  - Double Hashing

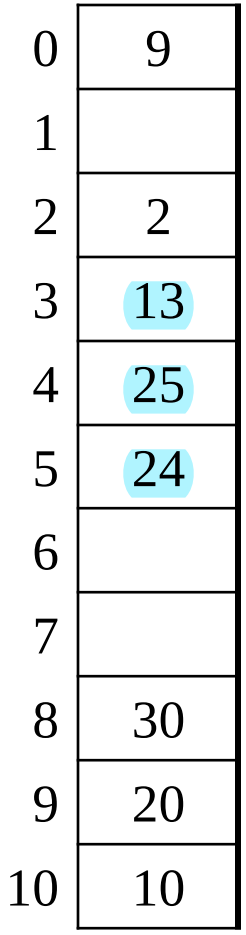
Open Addressing -- Linear Probing

- **linear probing**: search table sequentially starting from the original hash location.
  - Check next location, if location is occupied.
  - Wrap around from last to first table location



Linear Probing -- Example

- Example:
  - Table Size is 11 (0..10)
  - Hash Function: **h(x) = x mod 11**
  - Insert keys: 20, 30, 2, 13, 25, 24, 10, 9
    - 20 mod 11 = 9
    - 30 mod 11 = 8
    - 2 mod 11 = 2
    - 13 mod 11 = 2 ➔ 2+1=3
    - 25 mod 11 = 3 ➔ 3+1=4
    - 24 mod 11 = 2 ➔ 2+1, 2+2, 2+3=5
    - 10 mod 11 = 10
    - 9 mod 11 = 9 ➔ 9+1, 9+2 mod 11 =0



Linear Probing -- Clustering Problem

- One of the problems with linear probing is that table items tend to **cluster** together in the hash table.
  - i.e. table contains groups of consecutively occupied locations.

- This phenomenon is called **primary clustering**.
  - Clusters can get close to one another, and merge into a larger cluster.
  - Thus, the one part of the table might be quite dense, even though another part has relatively few items.
  - Primary clustering causes long probe searches, and therefore, decreases the overall efficiency.

Open Addressing -- Quadratic Probing

- **Quadratic probing:** almost eliminates clustering problem

- Approach:
  - Start from the original hash location `i`
  - If location is occupied, check locations `i+1²`, `i+2²`, `i+3²`, `i+4²` `...`
  - Wrap around table, if necessary.

Quadratic Probing -- Example

- Example:
  - Table Size is 11 (0..10)
  - Hash Function: **h(x) = x mod 11**
  - Insert keys: 20, 30, 2, 13, 25, 24, 10, 9
    - 20 mod 11 = 9
    - 30 mod 11 = 8
    - 2 mod 11 = 2
    - 13 mod 11 = 2 ➔ $2+1^2=3$
    - 25 mod 11 = 3 ➔ $3+1^2=4$
    - 24 mod 11 = 2 ➔ $2+1^2$, $2+2^2=6$
    - 10 mod 11 = 10
    - 9 mod 11 = 9 ➔ $9+1^2$, $9+2^2$ mod 11, $9+3^2$ mod 11 =7

0	
1	
2	2
3	13
4	25
5	
6	**24**
7	**9**
8	30
9	20
10	10

Open Addressing -- Double Hashing

- **Double hashing** also reduces clustering.
- **Idea**: increment using a **second hash function h_2**. Should satisfy:

$$h_2(\text{key}) \neq 0$$

$$h_2 \neq h_1$$

- Probes following locations until it finds an unoccupied place

$$h_1(\text{key})$$

$$h_1(\text{key}) + h_2(\text{key})$$

$$h_1(\text{key}) + 2*h_2(\text{key}),$$

...

Double Hashing -- Example

- Example:
  - Table Size is 11 (0..10)
  - Hash Function:

    $h_1(x) = x \bmod 11$

    $h_2(x) = 7 - (x \bmod 7)$
  - Insert keys: 58, 14, 91
    - 58 mod 11 = 3
    - 14 mod 11 = 3 ➔ 3+7=10
    - 91 mod 11 = 3 ➔ 3+7, 3+2*7 mod 11=6

to wrap around --> x mod(table size)

0	
1	
2	
3	58
4	
5	
6	91
7	
8	
9	
10	14

Open Addressing -- Retrieval & Deletion

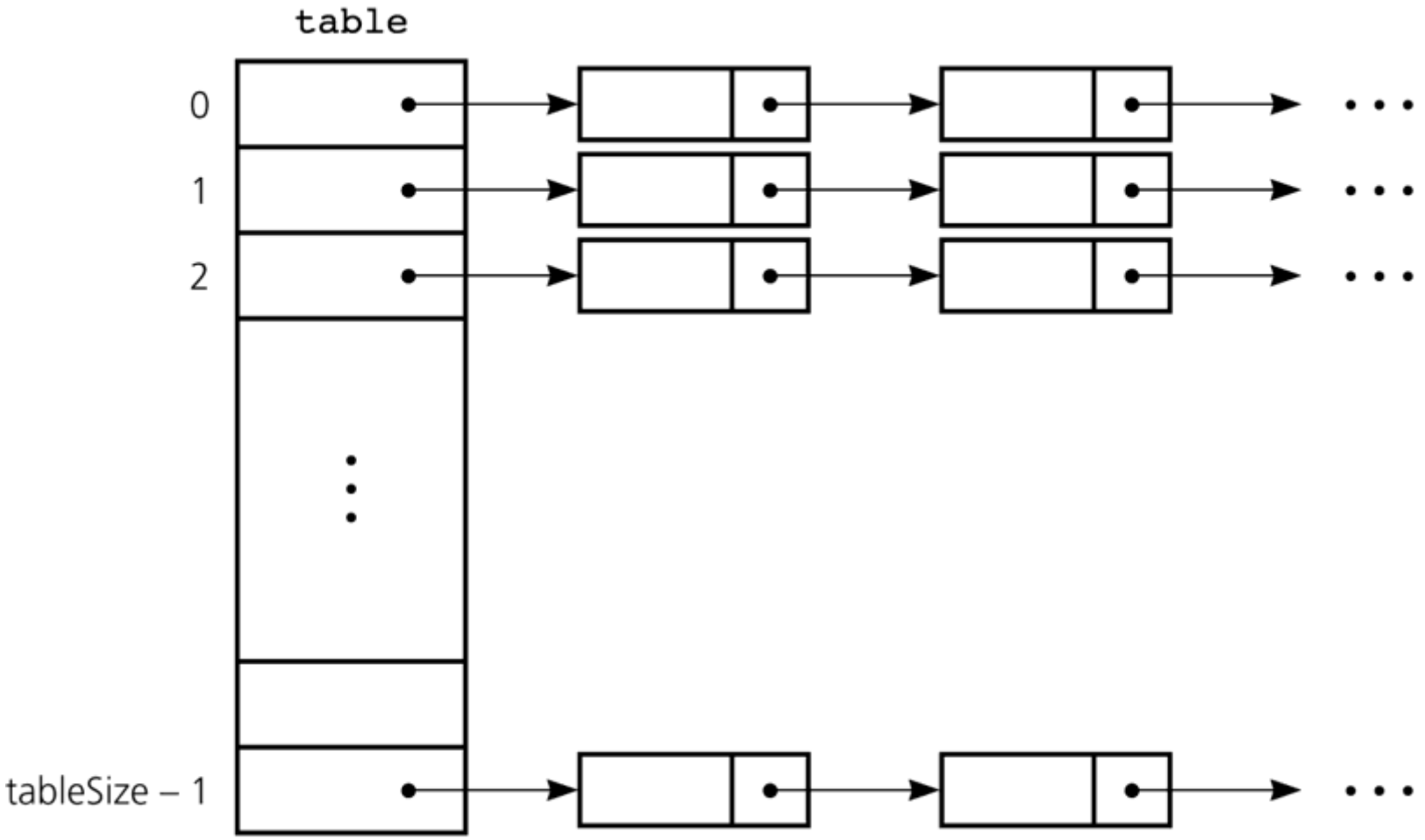
- **Retrieving an item** with a given key:
  - (same as insertion): probe the locations until we find the desired item or we reach to an empty location.

- **Deletions** in open addressing cause complications
  - We CANNOT simply delete an item from the hash table because this new empty (a deleted) location causes to stop prematurely (incorrectly) indicating a failure during a retrieval.
  - Solution: We have to have three kinds of locations in a hash table: ***Occupied, Empty, Deleted***.
  - A deleted location will be treated as an occupied location during retrieval.

Separate Chaining

- Another way to resolve collisions is to change the structure of the hash table.
  - In open-addressing, each location holds only one item.
- **Idea 1**: each location is itself an array called bucket
  - Store items that are hashed into same location in this array.
  - Problem: What will be the size of the bucket?
- **Idea 2:** each location is itself a linked list. Known as **separate-chaining**.
  - Each entry (of the hash table) is a pointer to a linked list (the chain) of the items that the hash function has mapped into that location.

Separate Chaining



Each location of the hash table contains a pointer to a linked list

Hashing Analysis

Hashing -- Analysis

- An analysis of the average-case efficiency of hashing involves the **load factor** α:

  α = (current number of items) / tableSize

- α measures how full a hash table is.
  - Hash table should not be too loaded if we want to get better performance from hashing.
- In average case analyses, we assume that the hash function uniformly distributes keys in the hash table.
- Unsuccessful searches generally require more time than successful searches.

Separate Chaining -- Analysis

- **Separate Chaining –** approximate average number of comparisons (probes) that a search requires :

$$1+\frac{\alpha}{2} \qquad \textit{for a successful search}$$

$$\alpha \qquad \textit{for an unsuccessful search}$$

- It is the most efficient collision resolution scheme.
- But it requires more storage (needs storage for pointers).
- It easily performs the deletion operation. Deletion is more difficult in open-addressing.

Linear Probing -- Analysis

- **Linear Probing –** approximate average number of comparisons (probes) that a search requires :

$$\frac{1}{2}\left[1+\frac{1}{1-\alpha}\right] \qquad \textit{for a successful search}$$

$$\frac{1}{2}\left[1+\frac{1}{(1-\alpha)^2}\right] \qquad \textit{for an unsuccessful search}$$

- Insert and search cost depend on length of cluster.
  - Average length of cluster = α = N / M.
  - Worst case: all keys hash to the same cluster.

Average probes for insert/search miss

$$\frac{1}{2}\left(1+\frac{1}{(1-\alpha)^2}\right) = \frac{(1+\alpha+2\alpha^2+3\alpha^3+4\alpha^4+\ldots)}{2}$$

Average probes for search hit

$$\frac{1}{2}\left(1+\frac{1}{(1-\alpha)}\right) = 1+(\alpha+\alpha^2+\alpha^3+\alpha^4+\ldots)/2$$

Linear Probing -- Analysis

- **Linear Probing –** approximate average number of comparisons (probes) that a search requires :

$$\frac{1}{2}\left[1+\frac{1}{1-\alpha}\right] \qquad \textit{for a successful search}$$

$$\frac{1}{2}\left[1+\frac{1}{(1-\alpha)^2}\right] \qquad \textit{for an unsuccessful search}$$

- As load factor increases, number of collisions increases, causing increased search times.
- To maintain efficiency, it is important to prevent the hash table from filling up.

Linear Probing -- Analysis

Example: Find the average number of probes for a successful search and an unsuccessful search for this hash table? Use the following hash function: **h(x) = x mod 11**

Successful Search: ***Try 20, 30, 2, 13, 25, 24, 10, 9***

20: 9 **30:** 8 **2:** 2 **13:** 2,3

25: 3, 4 **24:** 2, 3, 4, 5 **10:** 10 **9:** 9, 10, 0

Avg. no of probes = (1+1+1+2+2+4+1+3)/8 = 1.9

Unsuccessful Search: ***Try 0, 1, 35, 3, 4, 5, 6, 7, 8, 31, 32***

look one more empty bucket

0: 0, 1 1: 1 35: 2, 3, 4, 5, 6

3: 3, 4, 5, 6 4: 4, 5, 6 5: 5, 6

6: 6 7: 7 8: 8, 9, 10, 0, 1

if it is empty we are done item is not in the table

31: 9, 10, 0, 1 32: 10, 0, 1

Avg. no of probes =(2+1+5+4+3+2+1+1+5+4+3)/11 = 2.8

0	9
1	
2	2
3	13
4	25
5	24
6	
7	
8	30
9	20
10	10

Quadratic Probing & Double Hashing -- Analysis

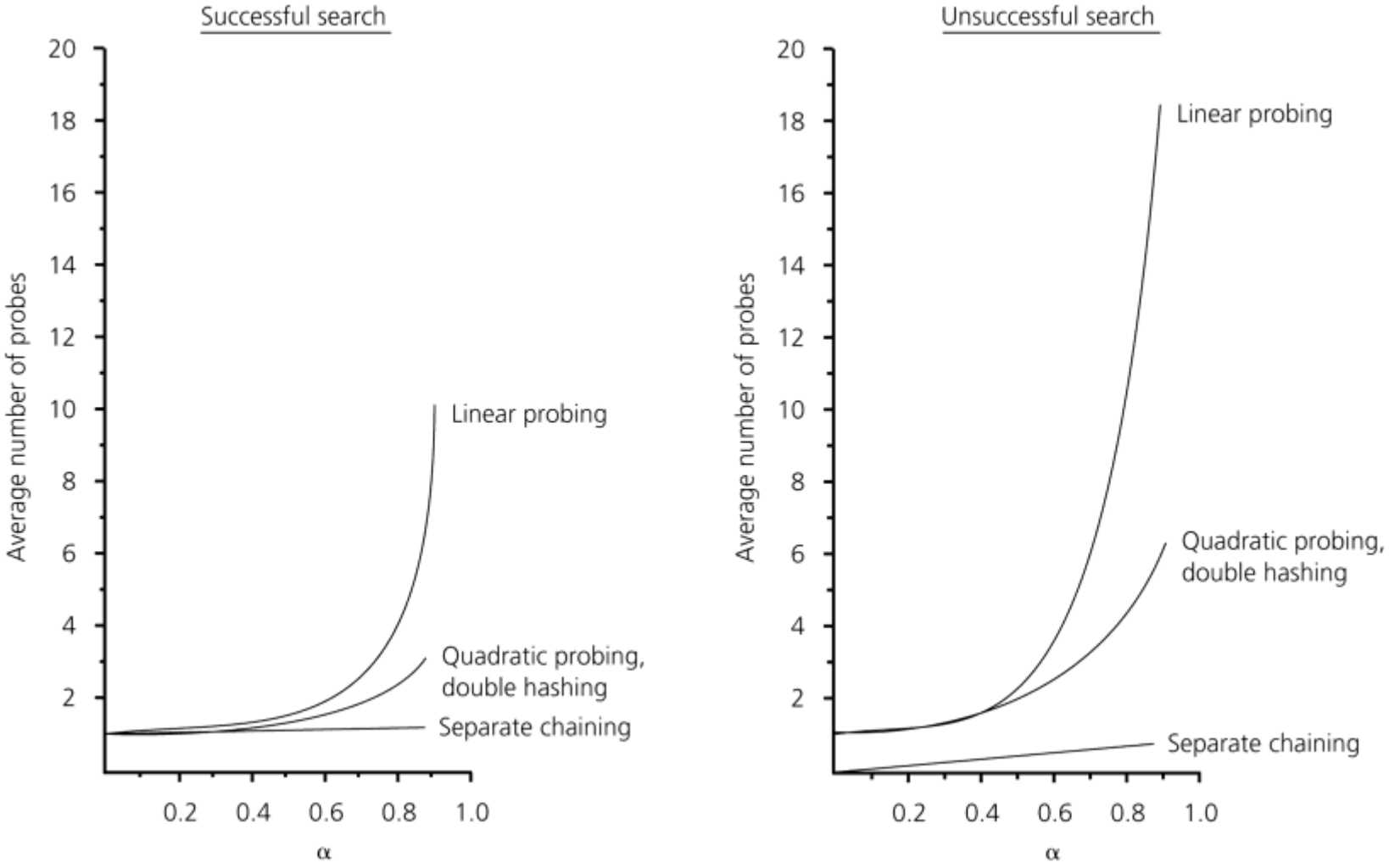
- The approximate average number of comparisons (probes) that a search requires is given as follows:

$$\left[\frac{1}{\alpha}(\log_e \frac{1}{1-\alpha})\right] = \frac{-\log_e(1-\alpha)}{\alpha}$$ *for a successful search*

$$\frac{1}{1-\alpha}$$ *for an unsuccessful search*

- On average, both methods require fewer comparisons than linear probing.

The relative efficiency of four collision-resolution methods



What Constitutes a Good Hash Function

- A hash function should be **easy** and **fast** to compute.
- A hash function should **scatter the data evenly** throughout the hash table.
  - How well does the hash function scatter random data?
  - How well does the hash function scatter non-random data?
- Two general principles :
  1. The hash function should use entire key in the calculation.
  2. If a hash function uses modulo arithmetic, the table size should be prime.

Example: Hash Functions for Strings

Hash Function 1

- Add up the ASCII values of all characters of the key.

```
int hash(const string &key, int tableSize)
{
        int hasVal = 0;

        for (int i = 0; i < key.length(); i++)
                hashVal += key[i];
        return hashVal % tableSize;
}
```

- Simple to implement and fast.
- However, if the table size is large, the function does not distribute the keys well.

last ascii value = 127

  - e.g. Table size =10000, key length <= 8, the hash function can assume values only between 0 and 1016

Hash Function 2

- Examine only the first 3 characters of the key.

```
int hash (const string &key, int tableSize)
{
      return (key[0]+27 * key[1] + 729*key[2]) % tableSize;

}
```

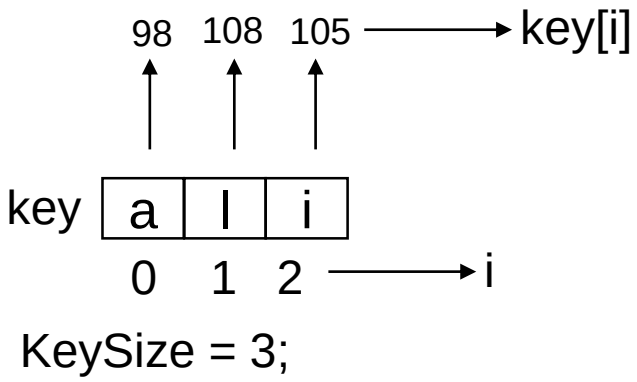
- In theory, **26 * 26 * 26 = 17576** different words can be generated. However, English is not random, only **2851** different combinations are possible.
- Thus, this function although easily computable, is also not appropriate if the hash table is reasonably large.

Hash Function 3

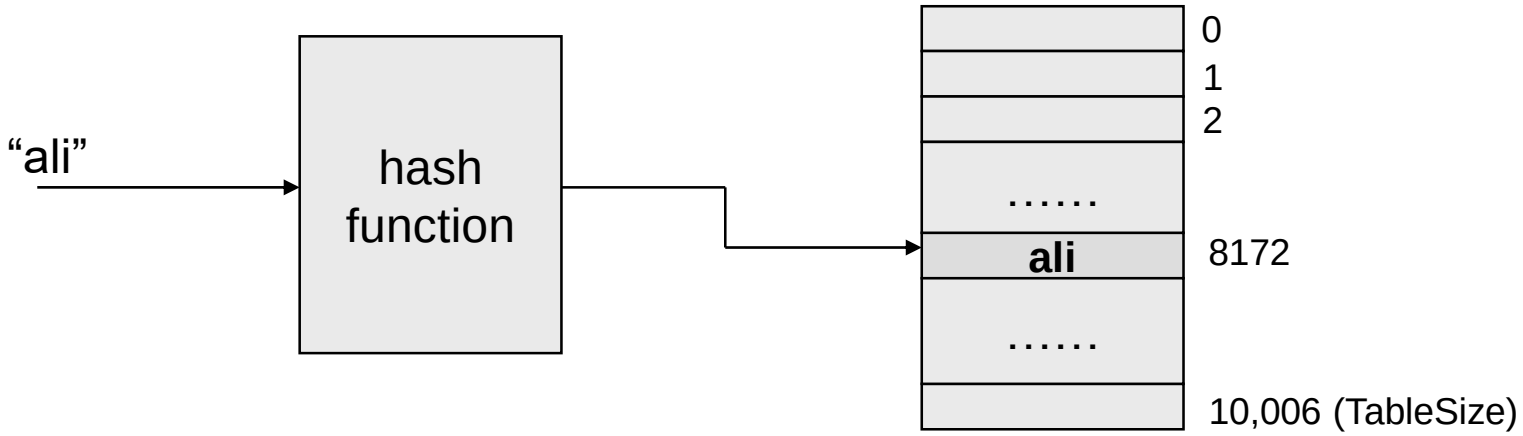
$$hash(key) = \sum_{i=0}^{KeySize-1} Key[KeySize - i - 1] \cdot 37^i$$

```
int hash (const string &key, int tableSize)
{
   int hashVal = 0;

   for (int i = 0; i < key.length(); i++)
       hashVal = 37 * hashVal + key[i];

   hashVal %=tableSize;
   if (hashVal < 0)   /* in case overflows occurs */
       hashVal += tableSize;

   return hashVal;
};
```

Hash function for strings:



hash("ali") = (105 * 1 + 108*37 + 98*37^2) % 10,007 = 8172



Hash Table versus Search Trees

- In most of the operations, the hash table performs better than search trees.

- However, traversing the data in the hash table in a sorted order is very difficult.
  - For similar operations, the hash table will not be good choice (e.g., finding all the items in a certain range).

Performance

- With either chaining or open addressing:
  - **Search** - O(1) expected, O(n) worst case
  - **Insert** - O(1) expected, O(n) worst case
  - **Delete** - O(1) expected, O(n) worst case
  - **Min**, **Max** and **Predecessor**, **Successor** - O(n+m) expected and worst case
- Pragmatically, a hash table is often the best data structure to maintain a dictionary/table. However, the worst-case time is unpredictable.
- The best worst-case bounds come from balanced binary trees.

Other applications of hash tables

- To implement Table ADT, Dictionary ADT
- Compilers
- Spelling checkers
- Games
- Substring Pattern Matching
- Searching
- Document comparison

Applications of Hashing

- Compilers use hash tables to implement the *symbol table* (a data structure to keep track of declared variables).
- Game programs use hash tables to keep track of positions it has encountered (*transposition table*)
- Online spelling checkers.

Substring Pattern Matching

- Input: A text string t and a pattern string p.
- Problem: Does t contain the pattern p as a substring, and if so where?
- e.g: Is *Bilkent* in the news?
- **Brute Force:** search for the presence of pattern string p in text t overlays the pattern string at every position in the text. → O(mn) (m: size of pattern, n: size of text)
- **Via Hashing:** compute a given hash function on both the pattern string p and the m-character substring starting from the ith position of t. → O(n)

Slide by Steven Skiena

Hashing, Hashing, and Hashing

- Udi Manber says that the three most important algorithms at Yahoo are hashing, hashing, and hashing.
- Hashing has a variety of clever applications beyond just speeding up search, by giving you a short but distinctive representation of a larger document.

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Document Comparison

- *Is this new document different from the rest in a large database? – Hash the new document, and compare it to* the hash codes of database.

- *How can I convince you that a file isn't changed? – Check* if the cryptographic hash code of the file you give me today is the same as that of the original. Any changes to the file will change the hash code.

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