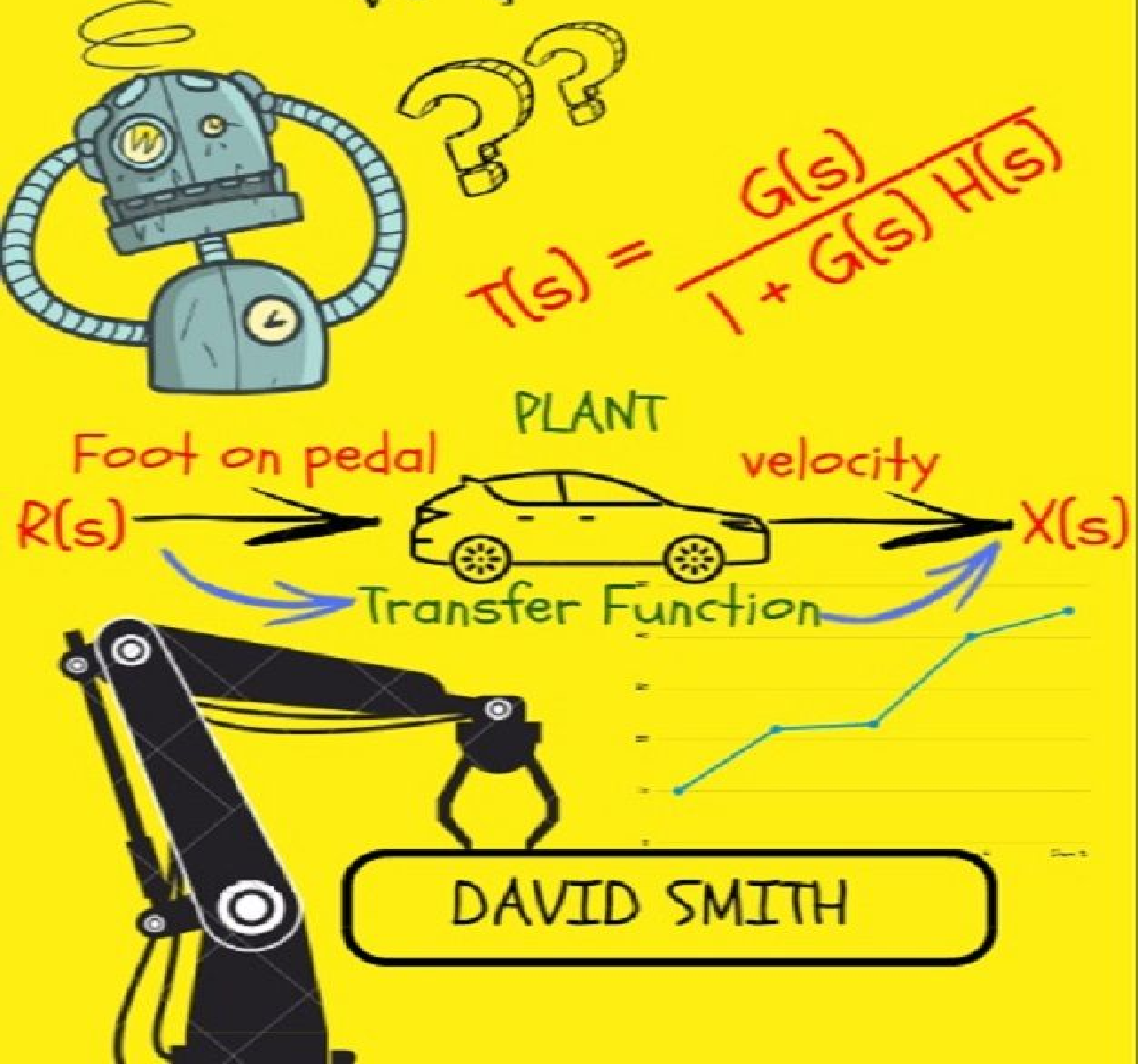
# CONTROL SYSTEMS FOR COMPLETE IDIOTS

$$c(t)_{max} = 1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right) t_p + \phi\right]$$




DAVID SMITH

# Control Systems for Complete Idiots

*by David Smith*

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# PREFACE

In this day and age everything around us is automatic and our desire to automate more stuff is only increasing. Control systems finds its applications in everything you can possibly think of. The concept of Control system plays an important role in the working of, everything from home appliances to guided missiles to self-driving cars. These are just the examples of Control systems we create. Control systems also exist in nature. Within our own body, there are numerous control systems, such as the pancreas, which regulate our blood sugar. In the most abstract sense it is possible to consider every physical object a control system. Hence from an engineering perspective, it is absolutely crucial to be familiar with the analysis and designing methods of such Control systems. Control systems is one of those subjects that go beyond a particular branch of engineering. Control systems find its application in Mechanical, Electrical, Electronics, Civil Engineering and many other branches of engineering. Although this book is written in an Electrical engineering context, we are sure that others can also easily follow the topics and learn a thing or two about Control systems.

In this book we provide a concise introduction into classical Control theory. A basic knowledge of Calculus and some Physics are the only prerequisites required to follow the topics discussed in the book. In this book, We've tried to explain the various fundamental concepts of Control Theory in an intuitive manner with minimum math. Also, We've tried to connect the various topics with real life situations wherever possible. This way even first timers can learn the basics of Control systems with minimum effort. Hopefully the students will enjoy this different approach to Control Systems. The various concepts of the subject are arranged logically and explained in a simple reader-friendly language with MATLAB examples.

This book is not meant to be a replacement for those standard Control systems textbooks, rather this book should be viewed as an introductory text for beginners to come in grips with advanced level topics covered in those books. This book will hopefully serve as inspiration to learn Control systems in greater depths.

Readers are welcome to give constructive suggestions for the improvement of the book and please do leave a review.

# 1. INTRODUCTION

## 1.1 INTRODUCTION

In this day and age everything around us is automatic and our desire to automate more stuff is only increasing. Control systems finds its applications in everything you can possibly think of. The concept of Control system plays an important role in the working of, everything from home appliances to guided missiles to self-driving cars. These are just the examples of Control systems we create. Control systems also exist in nature. Within our own body, there are numerous control systems, such as the pancreas, which regulate our blood sugar. In the most abstract sense it is possible to consider every physical object a control system. Hence from an engineering perspective, it is absolutely crucial to be familiar with the analysis and designing methods of such Control systems.

## 1.2 TERMS

So far we have used the term “Control system” many a times. To understand the meaning of the word Control system, first we will define the word system and then we will define a Control system.

A system is nothing but an arrangement of physical components which act together as a unit to achieve a certain objective. A room with furniture, fans, lighting etc. is an example of a system. To control means to regulate or direct. Hence a Control system is an arrangement of physical components connected in such a manner to direct or regulate itself or another system. For example, if a lamp is switched ON or OFF using a switch, the entire system can be called a Control system. In short, a Control system is in the broadest sense, an interconnection of physical components to provide a desired function, involving some kind of controlling action in it.

For each system, there is an excitation and a response or to simply put it, an input and an output. The Input is the stimulus, excitation or command applied to a Control system to produce a specific response and the Output is the actual response obtained from the Control system. It may or may not be equal to the response we intended to produce from the input provided (That's whole other story). Now consider an example, when you clap your hands, it makes sound, this is a system. So what are the input and output of this system?? Is the input, the motion of your hands?? Or Is it the trigger from your brain or something else. Similarly, what is the output of the system?? The sound?? Or the vibrations generated?? For a system, the inputs and outputs can have many different forms. In fact, it's completely up to us to decide what a systems input and output parameters should be, based on the application and most times, Control systems have more than one input or output.

The purpose of the control system usually identifies or defines the output and input. If the output and input are given, it is possible to identify and define the nature of the system components. Fortunately for us, more often than not, all inputs and outputs are well defined by the system description. Sometimes they are not, like noises or disturbances in the output. Disturbance is any signal that adversely effects the output of a system. Some of these disturbances are internally generated, others from outside the system.

## 1.3 CLASSIFICATION OF CONTROL SYSTEMS

There are a lot of ways to classify Control systems, Natural-Manmade, Single input single output -Single input multiple output etc., but we will focus on some of the important classifications:

### 1.3.1 Time Varying and Time Invariant Systems

Time Invariant Control systems are those in which the system parameters are independent of time. In other words, the system behavior doesn't change with time. For example, consider an electrical network, say your mobile charger, consisting of resistors and capacitors. It produces an output of 5V when connected to a 230V AC supply. This system is a time invariant one, because it is expected to behave exactly the same way, whether it's used at 6 pm or 11 pm or any other time.

On the other hand, Systems whose parameters are functions of time are called Time varying systems. The behavior of such systems not only depends on the input, but also the time at which the input is applied. The complexity of design increases considerably for such type of Control systems.

## 1.3.2 Linear and Non-Linear Systems

A Linear system is one that obeys the Superposition property. Superposition property is a basically a combination of 2 system properties:

**1. Additivity property**: A system is said to be additive if the response of a system when 2 or more inputs are applied together is equal to the sum of responses when the signals are applied individually. Suppose we provide an input ($\mathbf{x_1}$) to a system and we measure its response. Next, we provide a second input ($\mathbf{x_2}$) and measure its response. Then, we provide the sum of the two inputs $\mathbf{x_1 + x_2}$. If the system is linear, then the measured response will be just the sum of the responses, we noted down while providing the two inputs separately.

i.e. if $\mathbf{x_1(t) \rightarrow y_1(t)}$ and $\mathbf{x_2(t) \rightarrow y_2(t)}$,

Then the system is additive if, $\mathbf{x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)}$

**2. Homogeneity or Scaling property**: A system is said to obey Homogeneity property, if the response of a system to a scaled input is the scaled version of the response to the unscaled input. This means that, as we increase the strength of a simple input to a linear system, say we double it, then the output strength will also be doubled. For example, if a person's voice becomes twice as loud, the ear should respond twice as much if it's a linear system.

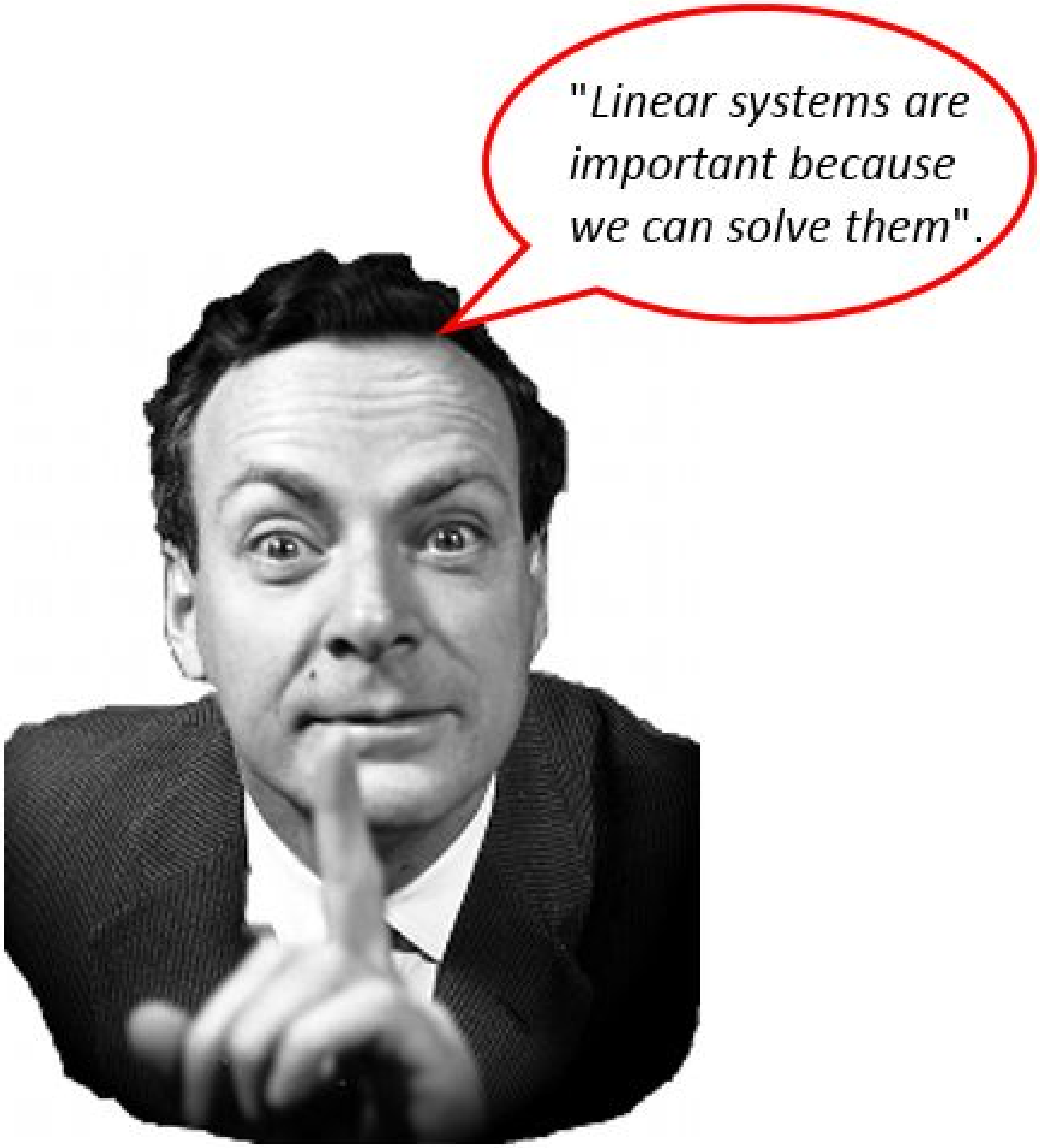
if $\mathbf{x(t) \rightarrow y(t)}$, then the system obeys homogeneity, if $\mathbf{a\ x(t) \rightarrow a\ y(t)}$, where $\mathbf{a}$ is constant.

Combining both these properties we get the superposition property.

$$\mathbf{a\ x_1(t) + b\ x_2(t) \rightarrow a\ y_1(t) + b\ y_2(t)}$$

An interesting observation to be made from this property is that, for linear systems zero input yields zero output (assume **a = 0**, then output is zero).

Both Linear and Time Invariant systems are two very important classes of Control Systems. In practice, it is difficult to find a perfectly linear time invariant system. Most of the practical systems are non-linear to a certain extent. However, more often than not, we model real world systems as Linear Time Invariant systems or LTI systems. This is because, analyzing and finding solutions to non-linear time variant systems are extremely difficult and time consuming. But if the presence of certain non-linearity is negligible and not effecting the system response badly, they can be treated as Linear (for limited range of operation) by making some assumptions and approximations. This makes the math a lot easier and allows us to use more mathematical tools for analysis or in Richard Feynman's words "*Linear systems are important because we can solve them*". The advantages of making this approximation is far greater than any disadvantages that arises from the assumption. We will be dealing with LTI systems from this point on.




## 1.3.3 Continuous time and Discrete time Systems

Mathematical functions are of two basic types, Continuous functions and Discrete functions. Continuous time functions are those functions that are defined for every instant of time. This means that you can draw a continuous function without lifting your pen or pencil from the paper. Discrete time functions on the other hand, are those functions, whose values are defined only for certain instants of time. For example, if you take the temperature reading of your room after every hour and plot it, what you get is a discrete time function. The temperature values are only defined at the hour marks and not for the entire duration of time. The value of temperature at other instants (say at half or quarter hour marks) are simply not defined.

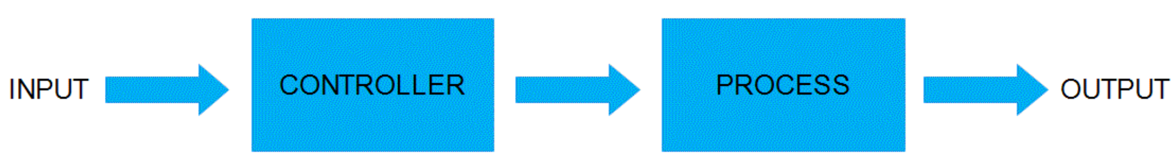
In a continuous-time Control system all the system variables are continuous time functions. In a discrete-time Control system at least one of the system variables is a discrete function. Microprocessor and computer based systems are discrete time systems.

### 1.3.4 Open and Closed Loop Systems

This is another very important classification of control systems and the features of both these types are discussed in detail in the coming sections.

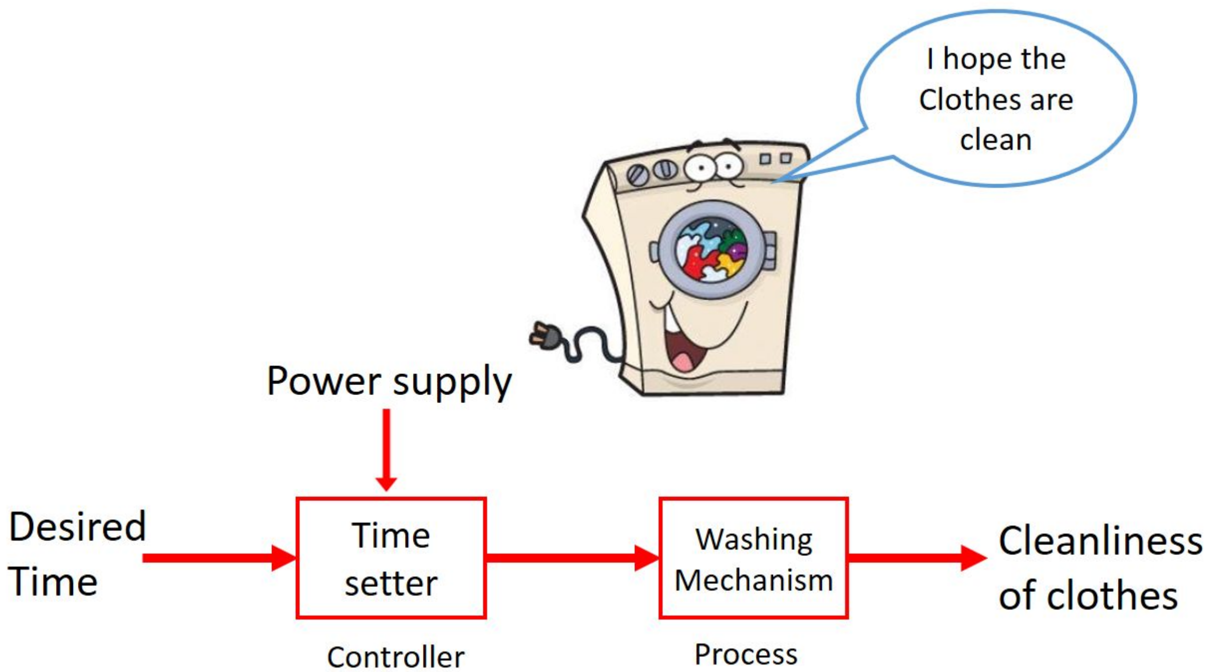
## 1.4 OPEN LOOP SYSTEM

Open loop system is a system in which output or rather the variation in output has no bearing on the controlling action. In other words, in an Open loop system, the output is neither measured nor fed back for comparison with the input. This simply means that, in an Open loop system there is no mechanism to correct the output if it goes out of track. Thus the accuracy of the output in an Open loop system is completely dependent on the accuracy of the input we provide and the calibration of the system. The Open loop control systems have a major drawback; the output of the system is adversely effected the presence of disturbances. This is because the changes in the output due to disturbances are not followed by changes in the input to correct the output. So any necessary changes, need to be made manually and since the nature of disturbances aren't the same always, it is quite difficult to maintain the accuracy in the output.



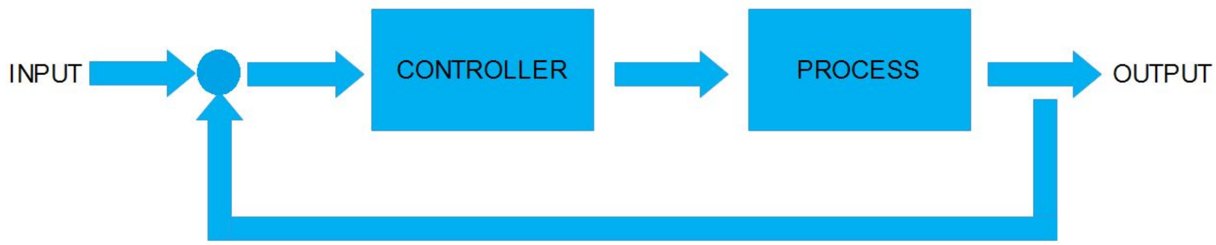
Due to these reasons, the practical applications of Open loop control systems are minimal and used only in places where the input-output relation is quite clear and the disturbances (internal and external) are minimum.

One good example of a practical Open loop system is your washing machine. Soaking, washing, rinsing in the washing machine operate on a time basis. The machine does not measure the output i.e. the cleanliness of the clothes (at least not the ordinary ones).

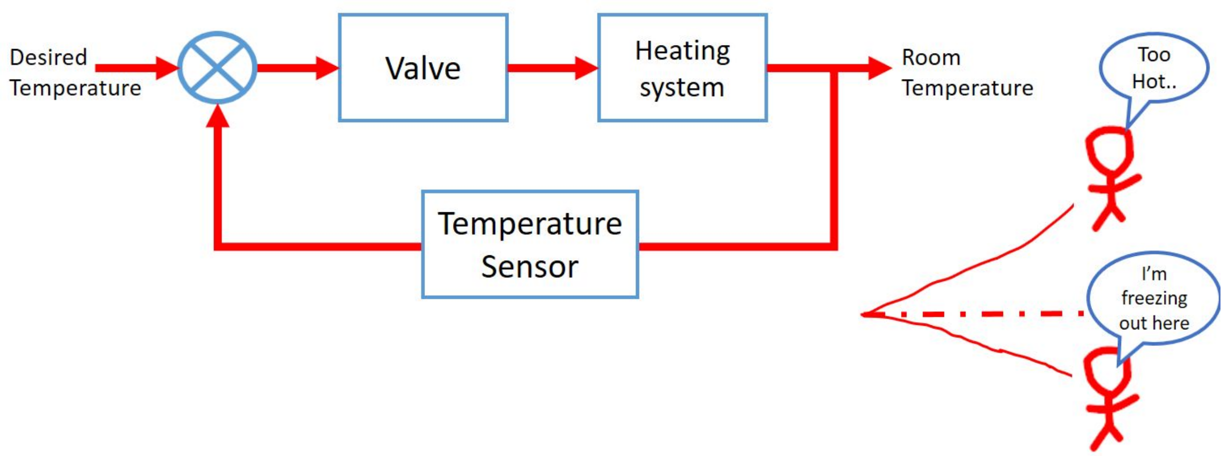



# 1.5 CLOSED LOOP SYSTEM

A Closed loop system is a system in which the variations in the output has a bearing on the control mechanism of the system. In a Closed loop system, an actuating error signal, which is the difference between the input signal and the output signal (in some form, not always directly), is fed to the controller. It is this error signal that drives the system. If the output of the system falls lower than the desired value, the corresponding error signal ensures that the output rises to the desired value and if the output rises above the desired value, its corresponding error signal makes the output decrease to the desired value. This way the output is always scrutinized and the corrections are made automatically. Doesn't Closed loop systems already seem more useful??




A Home heating system is an example of a Closed loop system. When the temperature is above the desired value, an error signal is generated and the valve is actuated to bring the temperature down to desired value. Similarly, it automatically raises the temperature, when it's too cold.




Something to be noted is that, it is not always possible, sometimes not desirable, to feed back the available output signal directly. Depending on the nature of the controller and the plant, it may be required to attenuate it, amplify it, or sometimes even change its nature (to digital etc.). This changed input is called the reference input. While the Closed loop systems have much more desirable properties than its counterpart, they do have some undesirable properties. Firstly, with all the complexity, designing these systems is a challenge in itself. But the critical area where they lose out to the Open loop systems is stability. That's a bit odd, isn't it?? One might have figured, that with the self-correcting nature of these systems, they are far more stable. The problem is that these systems have a tendency to over correct the errors and that may overtime lead to oscillations. The problem of stability is a severe one and must be taken care of in the design stages (which again adds complexity in designing).

## 1.6 OPEN LOOP V/S CLOSED LOOP SYSTEM

| OPEN LOOP SYSTEM | CLOSED LOOP SYSTEM |
|---|---|
| No feed back element | Feed back element is present |
| Low accuracy | Highly accurate |
| Highly sensitive to disturbances | Less sensitive to disturbances |
| Simple to design | Difficult to design |
| Generally stable in nature | Stability depends on design |
| Cheap | Costly |
| Highly affected by nonlinearities | Less affected by nonlinearities |

# 2. LAPLACE TRANSFORM

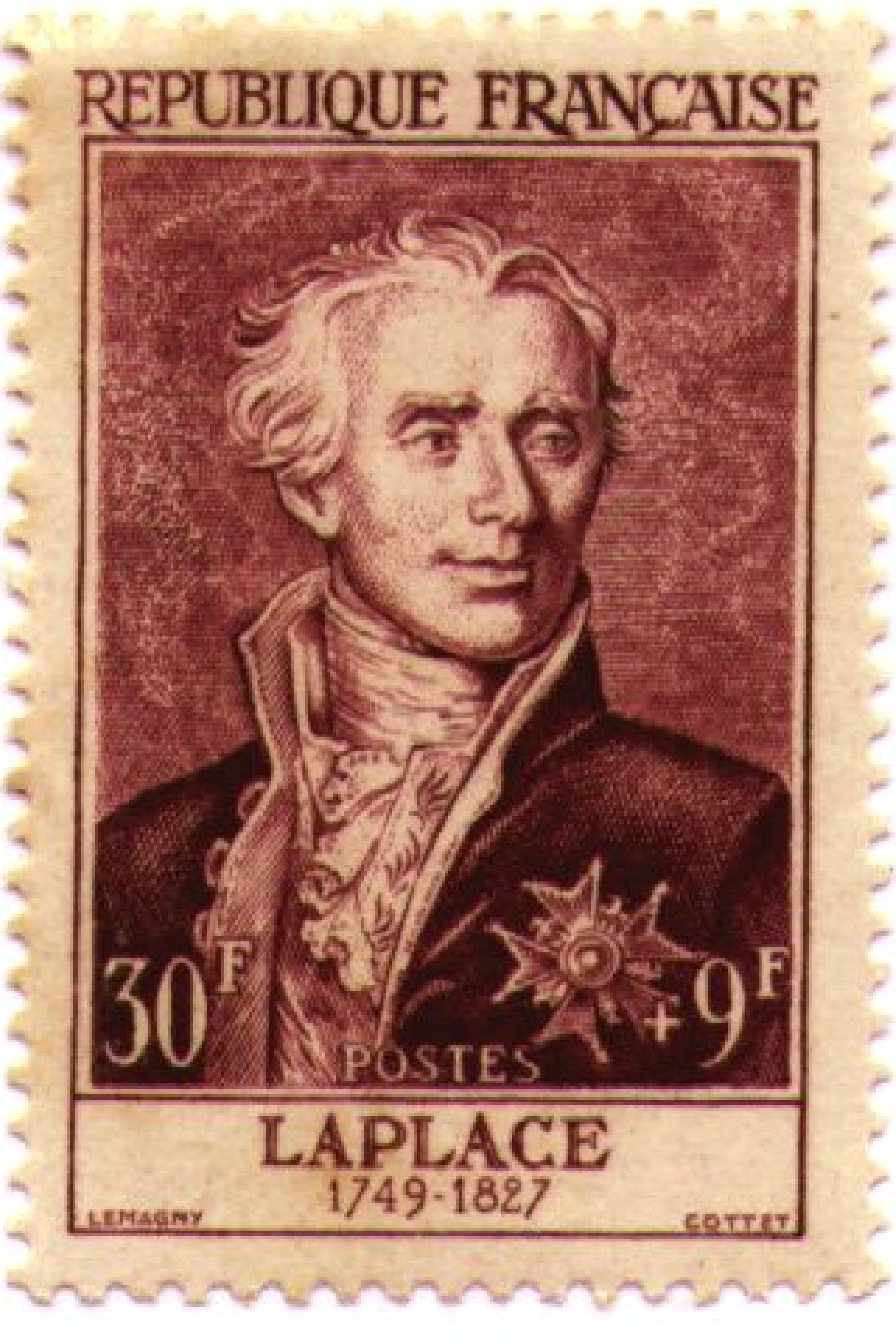
## 2.1 INTRODUCTION

Like any other engineering subject, Control systems has its fair share of math, but at the same time it isn't anywhere as math intensive as say, DSP is or EM theory is. In control systems, we are mainly dealing with Laplace transform and its discrete version, z- transform (for discrete Control systems).

You know, it's always a little scary when we devote a whole chapter just for a Math topic. Laplace transforms (or just transforms) can seem scary at first. However, as we will see, they aren't as bad as they may appear at first. To be honest, only the application part of Laplace transform is relevant to us, as far Control systems is concerned. Just the ability is to use the Laplace operator is more than enough. But just for completeness of the topic and considering how useful a tool it is, we will try and explain the concept and develop an intuitive understanding about the Laplace transform.

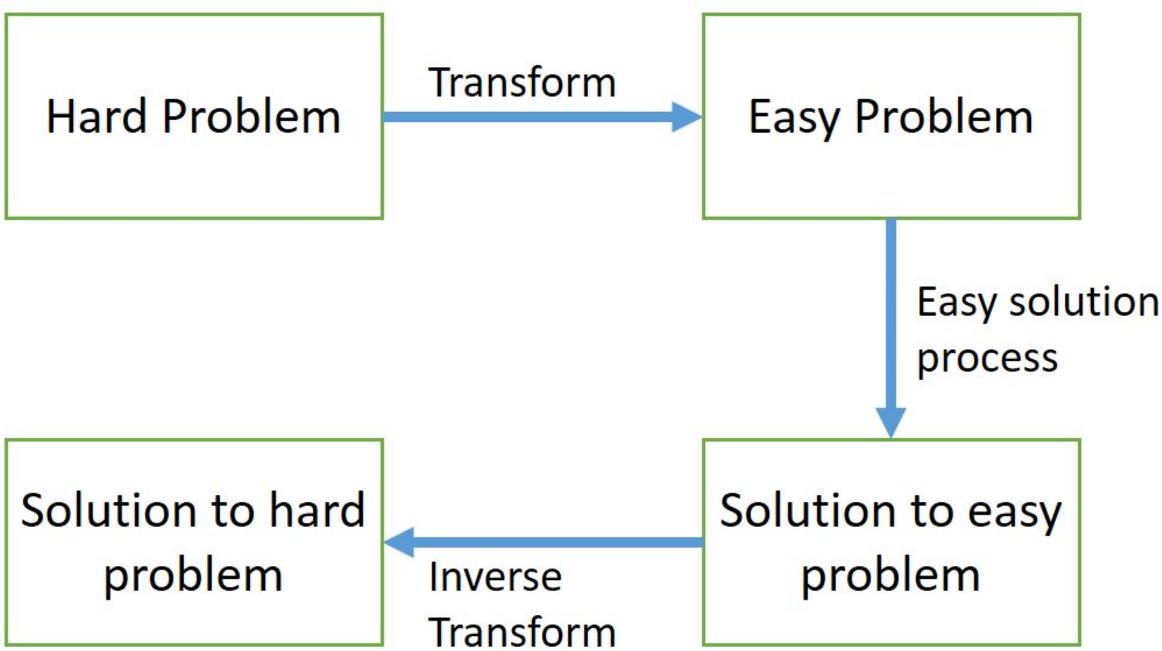
## 2.2 CONCEPT

The Laplace transform is a well-established mathematical technique for solving differential equations. It is named in honor of the great French mathematician, Pierre Simon De Laplace (1749-1827). Like all transforms, the Laplace transform changes one signal into another according to some fixed set of rules or equations. WAIT….did I say signals? That's right, here by signals we mean a continuous function (of time mostly) and also, it is the more appropriate term in an electrical engineering context.


REPUBLIQUE FRANÇAISE
30F
+9F
POSTES
LAPLACE
1749-1827
LEMAGNY
COTTET

Before we dig into Laplace transform, let's look into transforms in general. So what is a transform? Why do we need them?

Let's begin by considering a simple computational problem: compute the value of $\mathbf{x = 3.4^{2.4}}$ . It is not easy to get the exact value using straightforward methods. What we can do to make this problem solvable is to take natural log on both sides: now the equation becomes $\mathbf{\ln(x) = 2.4 \ln (3.4)}$. Now the value of ln(x) can be easily obtained and to obtain the value of x, all we have to do is to take the antilog of the value obtained. What we did was to take the hard problem, convert it into an easier equivalent problem. This is the very idea behind transforms. The concept of transformation can be illustrated with the simple diagram below:




The peculiarity of physical systems (LTI systems) are that they can be modeled by Differential equations. But solving Differential equations isn't the easiest of tasks. What kind of transformation might we use with ODEs? Based on our experience with logarithms, the dream would be a transformation, which allows us to replace the operation of differentiation by some easier operation, perhaps something similar to multiplication. Even if we don't get this exactly, coming close might still be useful. This is exactly what the Laplace transform is used for. The Laplace transform, transforms the differential equations into algebraic equations which are easier to manipulate and solve. Once the solution in the Laplace transform domain is obtained, the inverse Laplace transform is used to obtain the solution to the differential equation.

The Laplace transform of a function **f(t)**, denoted as **F(s)**, is defined as:

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

Yes, I know this equation looks menacing at first glance. But fortunately, most times you don't need to use this equation, you can easily get away with knowing some standard results.

## 2.3 PHYSICAL MEANING

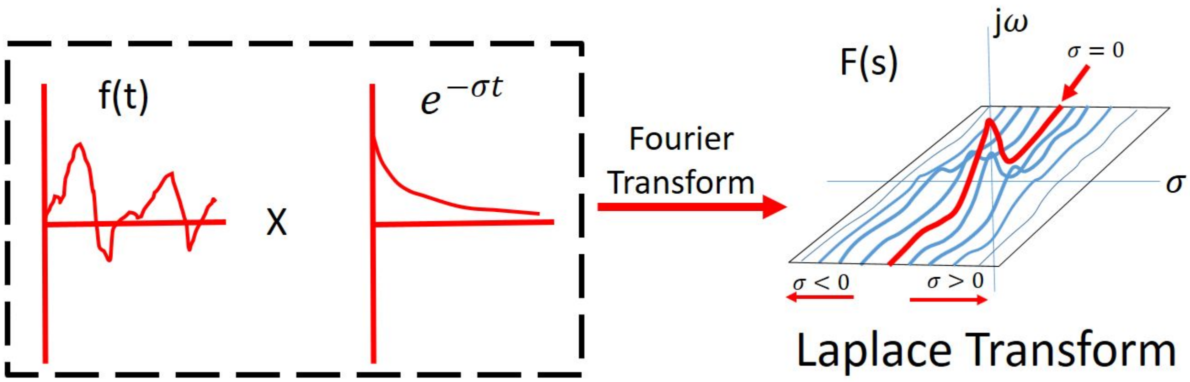
Making sense of the Laplace transform and getting your head around its physical meaning isn't the easiest of tasks. To understand the meaning of Laplace transform you need to have some idea about the Fourier transform (Link). If you look at Fourier transform equation, you can spot a striking similarity with the Laplace transform equation. The two equations are very similar, except that in the Laplace transform equation, term 's' is used in place of 'j $\omega$'. This similarity is because the Laplace transform was developed in order to overcome some limitations of the Fourier transform. From a mathematical standpoint, the Fourier transform is a subset of the Laplace transform. The connection between the two will become more apparent as we write the expansion of ‘s’,

$$s = \sigma + \underbrace{j\omega}_{\text{Term in Fourier Transform}}$$

Where $\sigma$ is a real number.

The Laplace transform is basically the Fourier transform with an additional term ($e^{-\sigma t}$). Although the Fourier transform is an extremely useful tool for analyzing many kinds of systems it has some shortcomings that can be overcome, in many ways, by the Laplace Transform. In particular, the Fourier transform is not very useful for studying the stability of systems because, in studying instabilities, it is often necessary to deal with signals that diverge in time. We know that the Fourier integral does not converge for signals that diverge because such signals are not absolutely integrable.

Because of the exponential weighting, the Laplace transform can converge for signals for which the Fourier transform does not converge. Depending upon the value of σ, which is the real part of s, a signal (whose transform we are taking) is be multiplied by a decaying or expanding exponential. By tacitly choosing the value of σ, thereby multiplying the signal with a decaying exponential, we can ensure that it becomes convergent. The region in the “s” plane where this infinite integral converges is called the region of convergence (ROC).



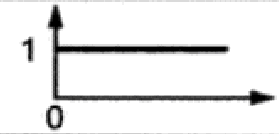
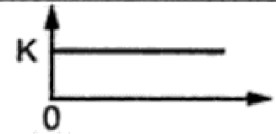
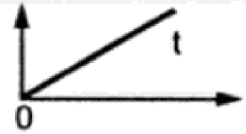
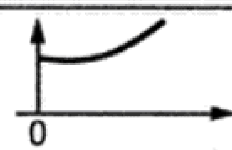
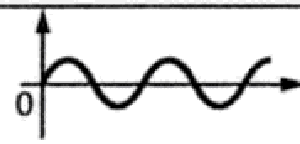
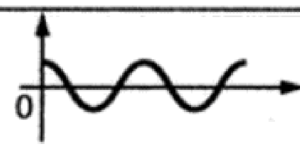
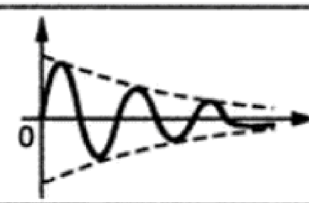
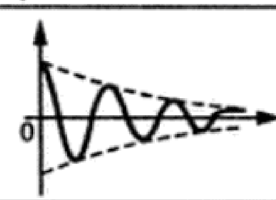

Intuitively, this means that the Laplace transform analyses the signals both in terms of exponentials and sinusoids, just as the Fourier transform analyses signals in terms of sinusoids. The center line in the s-plane (at $\sigma = 0$) corresponds to the Fourier.

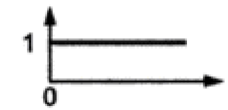
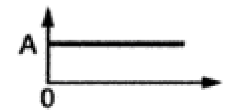
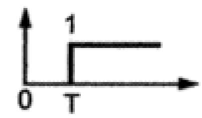
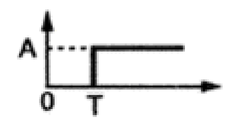
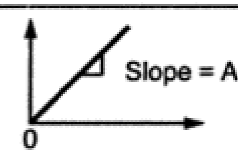
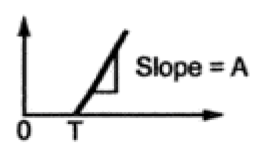
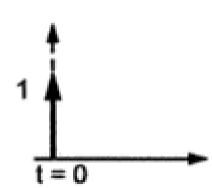
# 2.4 PROPERTIES OF LAPLACE TRANSFORM

Some of the basic properties of Laplace transform,

| Property | Operation in time domain | Operation in s domain |
|---|---|---|
| Linearity | $a_1 x_1(t) + a_2 x_2(t)$ | $a_1 X_1(s) + a_2 X_2(s)$ |
| Differentiation | $\frac{d^n x(t)}{dt^n}$ | $s^n X(s) - s^{n-1} x(0^-) \ldots - x^{n-1}(0^-)$ |
| Integration | $\int_{-\infty}^{t} x(\tau) d\tau$ | $\frac{X(s)}{s} + \frac{x^{n-1}(0^-)}{s}$ |
| Initial value theorem | $x(0) = \lim_{t \to 0} x(t)$ | $x(0) = \lim_{s \to \infty} sX(s)$ |
| Final value theorem | $x(\infty) = \lim_{t \to \infty} x(t)$ | $x(0) = \lim_{s \to 0} sX(s)$ |
| Time scaling | x(at) | $a^{-1} X(\frac{s}{a})$ |

# 2.5 STANDARD LAPLACE TRANSFORM PAIRS

| f(t) | F(s) | Waveform |
|---|---|---|
| 1 | $\frac{1}{s}$ |  |
| Constant K | $\frac{K}{s}$ |  |
| t | $\frac{1}{s^2}$ |  |
| $t^n$ | $\frac{n!}{s^{n+1}}$ | |
| $e^{-at}$ | $\frac{1}{s+a}$ |  |
| $e^{at}$ | $\frac{1}{s-a}$ |  |
| $e^{-at}\, t^n$ | $\frac{n!}{(s+a)^{n+1}}$ | |
| $\sin \omega t$ | $\frac{\omega}{s^2+\omega^2}$ |  |
| $\cos \omega t$ | $\frac{s}{s^2+\omega^2}$ |  |
| $e^{-at} \sin \omega t$ | $\frac{\omega}{(s+a)^2+\omega^2}$ |  |
| $e^{-at} \cos \omega t$ | $\frac{(s+a)}{(s+a)^2+\omega^2}$ |  |

| Function f(t) | Laplace Transform F(s) | Waveforms |
|---|---|---|
| Unit step = u(t) | $\frac{1}{s}$ |  |
| A u(t) | $\frac{A}{s}$ |  |
| Delayed unit step = u(t – T) | $\frac{e^{-Ts}}{s}$ |  |
| A u(t – T) | $\frac{A\,e^{-Ts}}{s}$ |  |
| Unit ramp = r(t) = t u(t) | $\frac{1}{s^2}$ |  |
| A t u(t) | $\frac{A}{s^2}$ |  |
| Delayed unit ramp = r(t – T) = (t – T) u(t – T) | $\frac{e^{-Ts}}{s^2}$ |  |
| A (t – T) u(t – T) | $\frac{A\,e^{-Ts}}{s^2}$ |  |
| Unit impulse = δ(t) | 1 |  |

## 2.6 INVERSE LAPLACE TRANSFORM

Finding the Inverse Laplace transforms of functions isn't terribly difficult. Most times Inverse Laplace transforms of functions can be figured out by inspection. The general method to find the Inverse Laplace transforms of functions is to express them as partial fractions and then make it into a convenient form and figure out which function's Laplace each term is. The table below shows some basic Laplace inverse pairs.

| $f(t)$ | Laplace transformed $F(s)$ |
|---|---|
| $\delta$ pulse $\delta(t)$ | 1 |
| unit step $\sigma(t)$ | $\frac{1}{s}$ |
| $t$ | $\frac{1}{s^2}$ |
| $t^2$ | $\frac{2}{s^3}$ |
| $\frac{t^n}{n!}$ | $\frac{1}{s^{n+1}}$ |
| $e^{-at}$ | $\frac{1}{s+a}$ |
| $te^{-at}$ | $\frac{1}{(s+a)^2}$ |
| $t^2e^{-at}$ | $\frac{2}{(s+a)^3}$ |
| $t^ne^{-at}$ | $\frac{n!}{(s+a)^{n+1}}$ |

| | |
|---|---|
| $\sin \omega_0 t$ | $\frac{\omega_0}{s^2 + \omega_0^2}$ |
| $\cos \omega_0 t$ | $\frac{s}{s^2 + \omega_0^2}$ |
| $e^{-at} \sin \omega_0 t$ | $\frac{\omega_0}{(s+a)^2 + \omega_0^2}$ |
| $e^{-at} \cos \omega_0 t$ | $\frac{s+a}{(s+a)^2 + \omega_0^2}$ |
| $\frac{1}{a} f\left(\frac{t}{a}\right)$ | $F(as)\ (a > 0)$ |
| $e^{at} f(t)$ | $F(s-a)$ |
| $f(t-a)$ for $t > a \geq 0$<br>$0$ for $t < a$ | $e^{-as} F(s)$ |
| $-t\, f(t)$ | $\frac{dF(s)}{ds}$ |
| $(-t)^n f(t)$ | $\frac{d^n F(s)}{ds^n}$ |

# 2.7 SOLVING DIFFERENTIAL EQUATIONS

As mentioned earlier, one of the biggest uses of Laplace transform is in solving differential equations. The procedure is best illustrated with an example.

**Example:**

$f''(t) + 3 f'(t) + 2 f(t) = e^{-t}$, with the initial conditions $f(0) = f'(0) = 0$

$f''(t) + 3\,f'(t) + 2\,f(t) = e^{-t}$

$s^2 F(s) + 3s\,F(s) + 2\,F(s) = \frac{1}{s+1}$ ← Taking Laplace transform on both sides

$F(s) = \frac{1}{s+1}\,\frac{1}{s^2+3s+2}$

Decomposing into partial fractions,

$F(s) = \frac{1}{s+2} - \frac{1}{s+1} + \frac{1}{(s+1)^2}$

$f(t) = e^{-2t} - e^{-t} + te^{-t}$ ← Taking Inverse Laplace transform on both sides

More Examples (Link)

# 3. TRANSFER FUNCTION

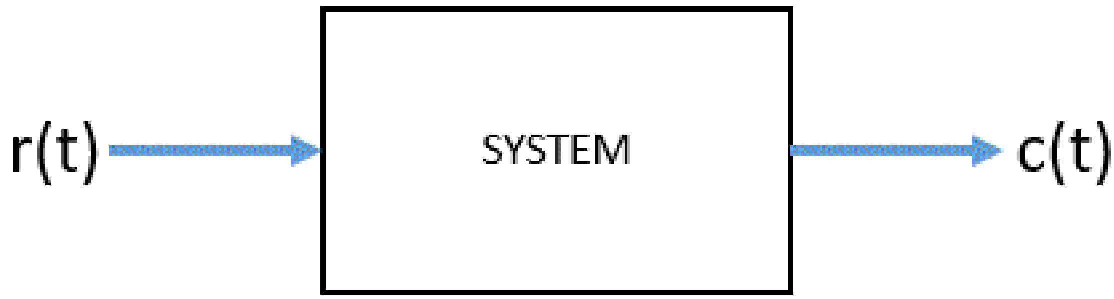
## 3.1 CONCEPT

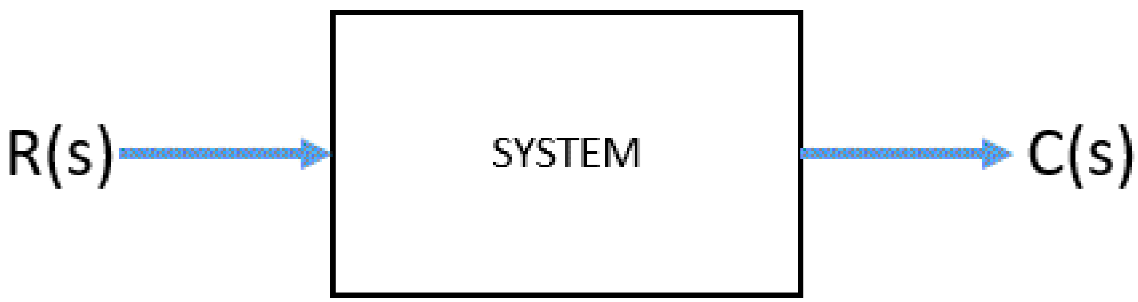
A Transfer function is nothing but a mathematical indication of the relationship existing between the input and the output of a Control system. In other words, the transfer function is a mathematical expression that tells what the system is doing with our input. In designing a system, first the system parameters are designed and their values are chosen as per requirement. Then we provide an input and see the performance of the system. We can roughly say that the output of the system is the product of the input and effect of the system parameters. Therefore, the effect of the system parameters can be expressed as the ratio of the Output to the Input. Due to the characteristics of system parameters, the input gets transferred into output, once applied to the system. This is the concept of Transfer function.

$$\text{Output} = \text{Input} \times \text{Effect of the system parameters}$$

$$\therefore \text{Effect of the system parameters} = \frac{Output}{Input}$$

**Mathematically, the Transfer function is defined as the ratio of Laplace transform of the output of the system to the Laplace transform of the input of the system, assuming zero initial conditions.**



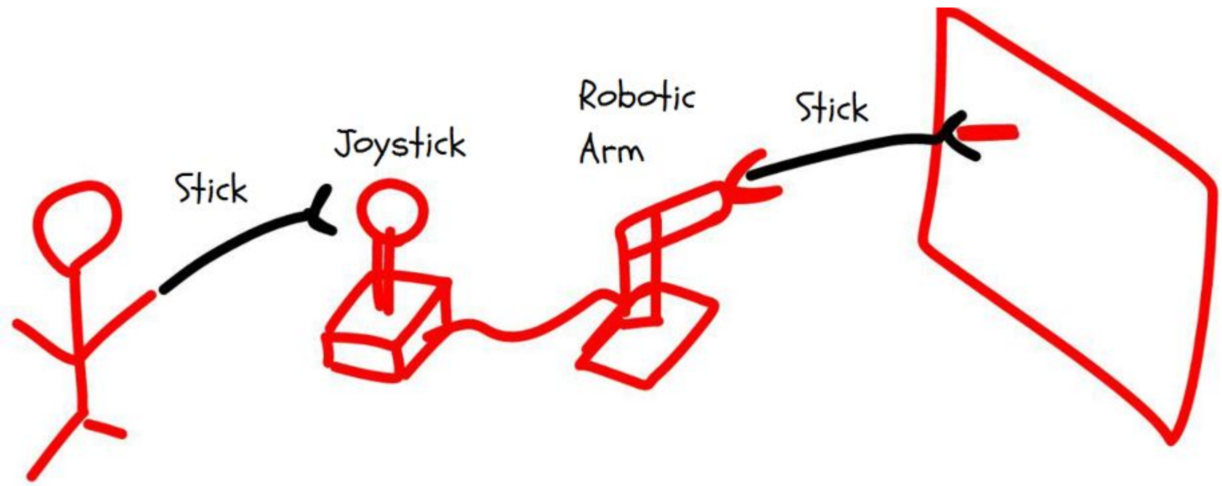

The Transfer function of the system **G(s)** is given by,

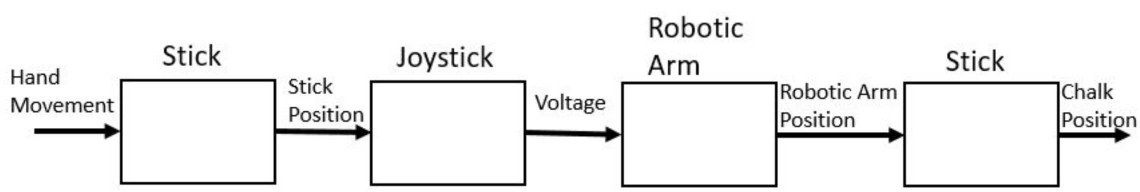
$$G(s) = \frac{\textit{Laplace transform of the Output}}{\textit{Laplace transform of the Input}} = \frac{C(s)}{R(s)}$$

The reason we consider the Transfer function in the Laplace domain will become clear in the next section. For now, just understand that the Transfer function is a function of 's'.

(ex) $G(s) = \dfrac{s}{s(s+1)}$

The following example will make the concept of Transfer function clear: Suppose you are asked to write your name on a black board. But there's a catch, you can’t do this directly, instead you have to use a contraption, as shown below.




Is this even possible??, you must be thinking. I can totally understand your skepticism, for a system with these many transformations, drawing a straight line would be a challenge, let alone something as complex as your name. Disregarding its complexity, this is nothing but another Control system with an input and an output. The difference here is that this Control system is itself made up of 4 other Control systems. So to understand the effect of the Control system as a whole, we need to characterize the behavior of each part separately and combine them. It may seem like a lot of effort, but it’s not. All we have to do is to find the transfer function of each part and combine them in some manner. Since the output of one part is the input to the next part, the Transfer function of the system as a whole is simply the product of individual Transfer functions.

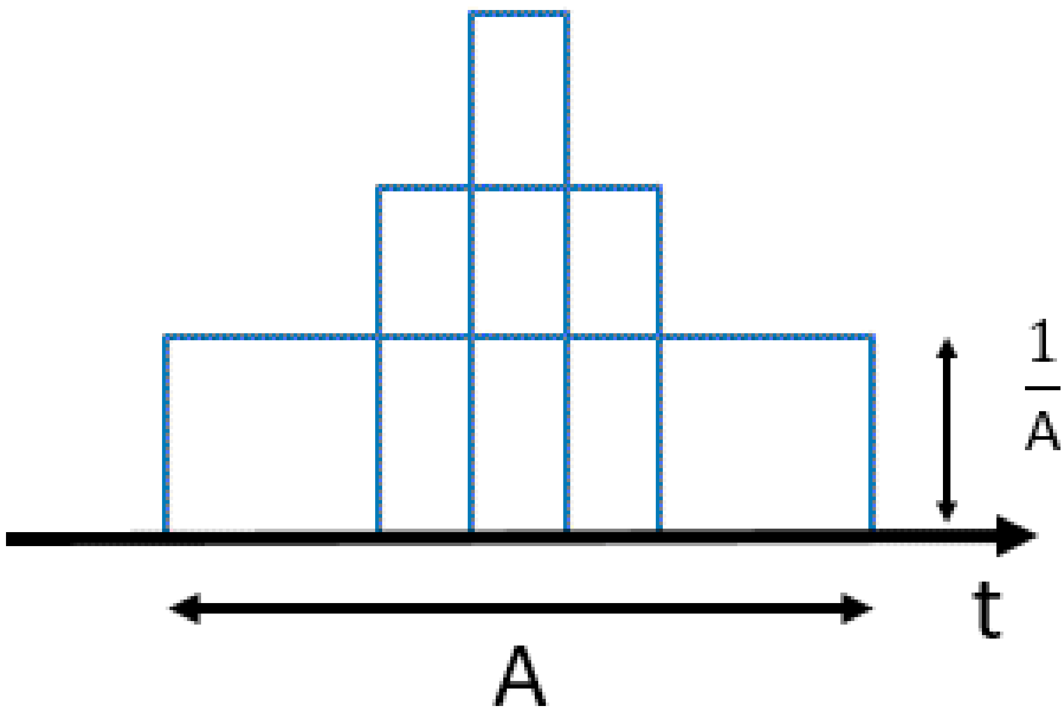
$$G(s) = G_1(s) . G_2(s) . G_3(s) . G_4(s)$$

## 3.2 UNIT IMPULSE FUNCTION

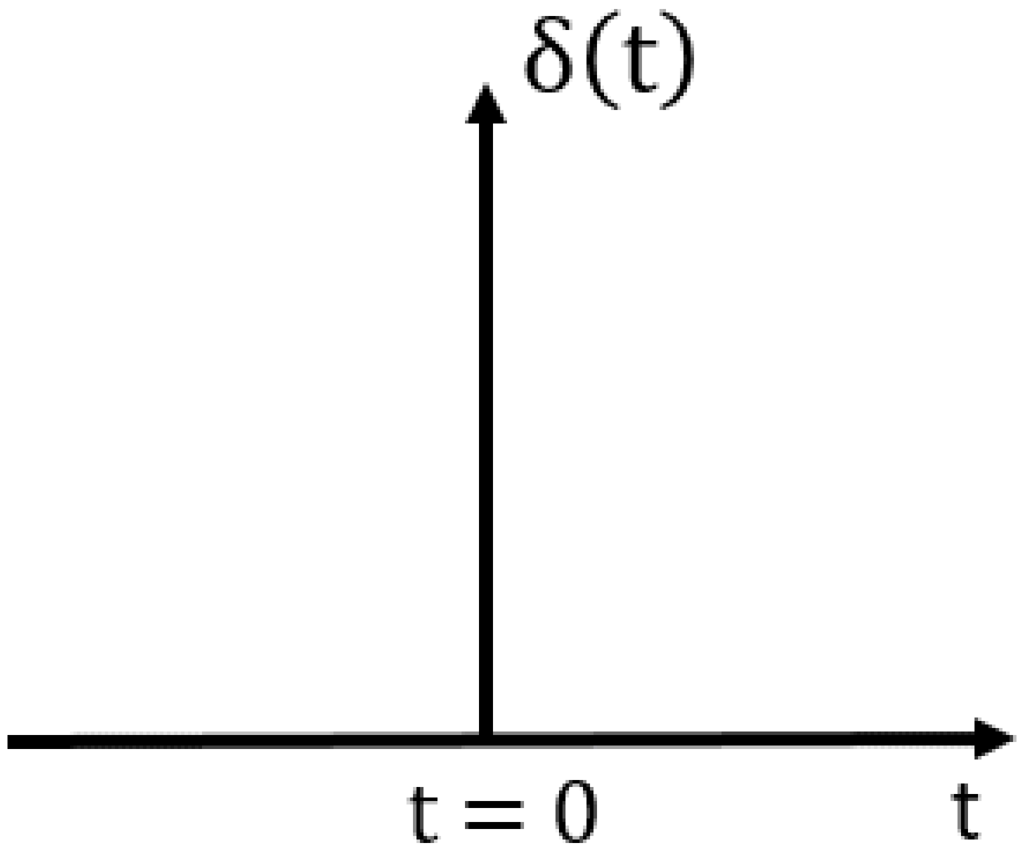
The Unit Impulse function is a very important function in Control systems and Signal Processing. Mathematically, it is defined as:

$$\begin{aligned} \delta(t) &= 0, & t &\neq 0 \\ &= \infty, & t &= 0 \end{aligned}$$

Think of the Impulse signal like a short pulse, like the output when a switch is turned on and off as fast as you can. But, if the value of the Unit Impulse function is $\infty$ at $t = 0$, then why the name Unit Impulse function?? The name comes from the fact that the Unit impulse function has a unit area at $t = 0$. Consider a narrow rectangular pulse of width $A$ and height $1/A$, so that the area under the pulse = 1. Now if we go on reducing the width $A$ and maintain the area as unity, then the height $1/A$ will go on increasing. Ultimately when A $\rightarrow$ 0, $1/A \rightarrow \infty$ and it results in a pulse of infinite magnitude. It is very clear from this, that the Unit impulse function has infinite magnitude at $t = 0$.

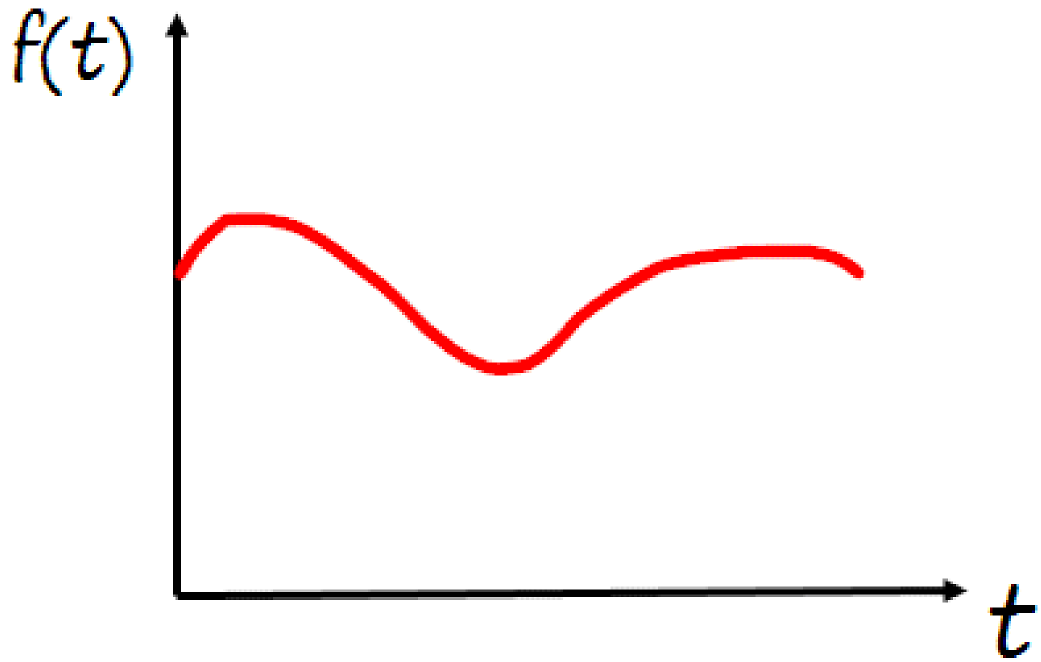


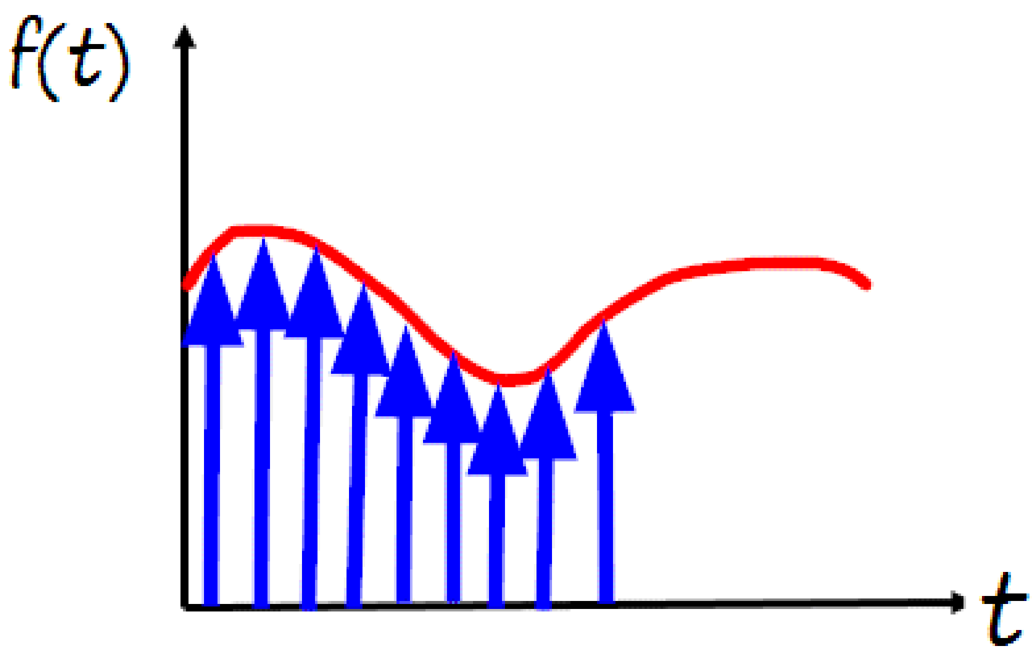
1
A
t
A




The height of the arrow is used to depict the area of a scaled impulse. The unit Impulse function is also known as the delta function or the delta-dirac function.

The obvious question that may come to your mind at this point is, “What is so special about the Unit Impulse function?”. The answer to that lies in the figures shown below.

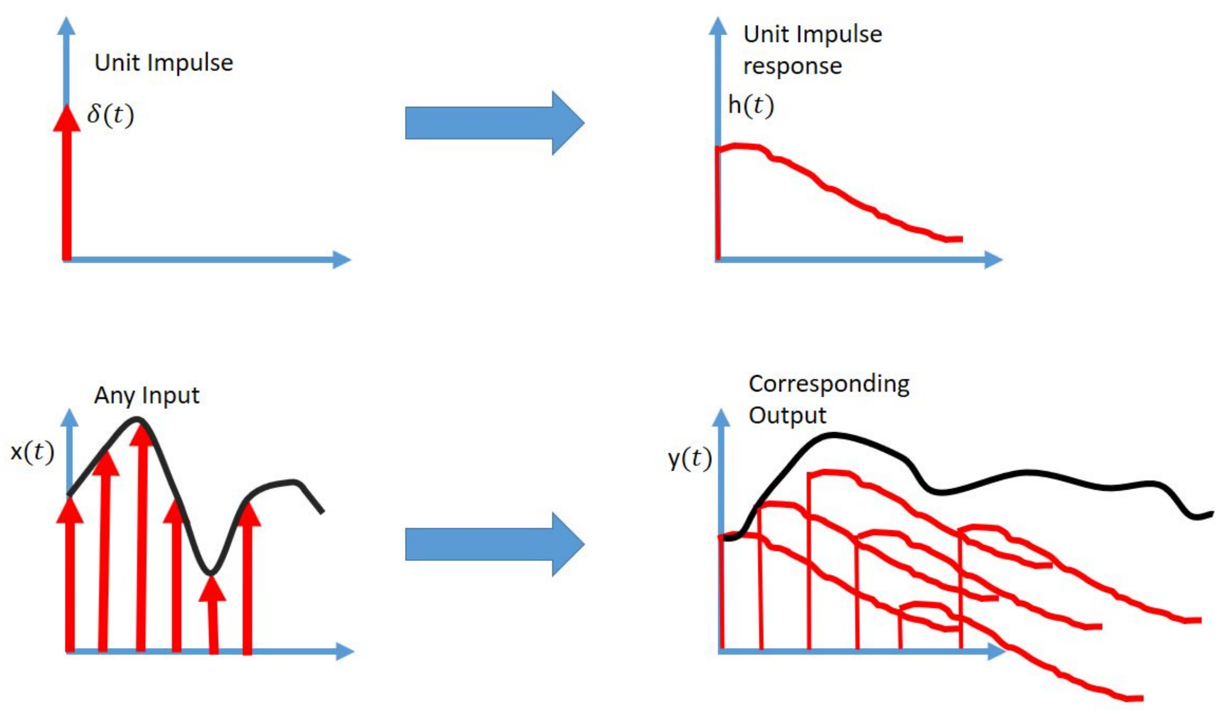



Figured it out yet? In the second figure shown above, we have substituted the signal in the first figure, by infinite number of Impulses of appropriate magnitudes at each instant of time. In this manner, any signal can be constructed out of scaled and shifted Unit Impulse signals. This is called the Sifting property of Unit Impulse function. This may not seem like a big deal, but trust me it is.

# 3.3 IMPULSE RESPONSE

When working with a control system, we would want to test out its response to a signal or a range of signals. How do we do that? Surely, we can't go around testing the response of every signal one after the other. It's too cumbersome and most times uneconomical. There's got to be an easier way. Actually there is, in the last section, we saw how a signal can be expressed as the sum of appropriately scaled and shifted Unit Impulses. We can put that property to good use. By figuring out how the system responds to a Unit Impulse signal, we can predict the system response to any input signal. The response of a system to a Unit Impulse signal is called the Unit Impulse Response (denoted by h(t)).




Because these are LTI systems, the Impulse response is scaled and shifted by the same factor as the input Impulse signal. So the system response to any input can be obtained by summing up the scaled and shifted impulse responses (as shown in above figure).

But since summing up infinite impulse responses isn't possible, mathematicians came up with what is known as the Convolution integral (denoted by * operator).

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) dt$$

Woah!! That seems awfully hard. Fortunately for us, the convolution integral corresponds to simple multiplication in the Laplace domain (told ya Laplace would come in handy).

$$Y(s) = X(s)\, H(s)$$

Where H(s) is the Unit Impulse response in the Laplace domain, which by the way is the Transfer function of the system.

## 3.4 TERMINOLOGIES RELATED TO TRANSFER FUNCTION

The Transfer function of a system can expressed as the ratio of two functions of 's', G(s) = $\frac{P(s)}{Q(s)}$. In most cases, it is more convenient to represent the rational transfer function in the factorized form.

$$G(s) = \frac{K\,(s-b_0)(s-b_1)....(s-b_m)}{(s-a_0)(s-a_1)....(s-a_n)}$$

$$K \rightarrow \mathit{Gain\ factor}$$

## 3.4.1 DC Gain

The term K in the transfer function (in factorized form) is called the Gain factor or the DC gain. It is simply the value of the transfer function at zero frequency i.e. s=0.

$$\text{DC Gain} = K = G(s)\ \Big|_{s=0}$$

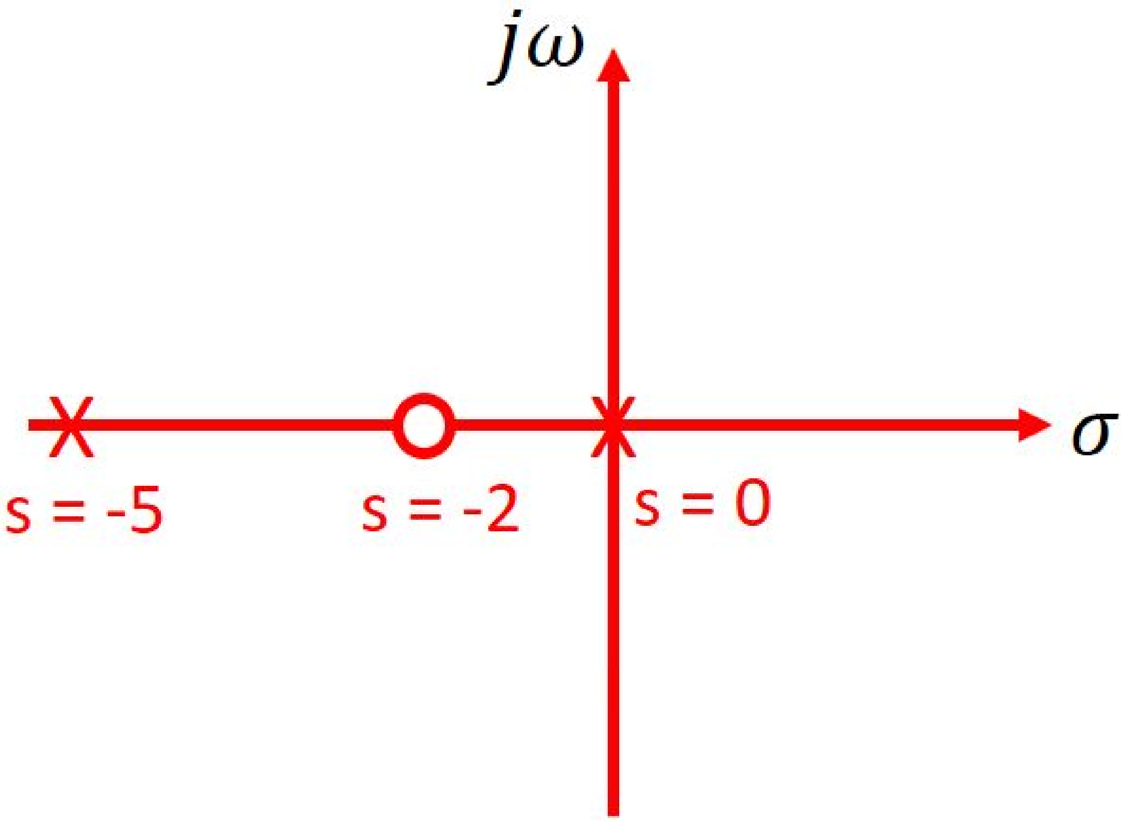
## 3.4.2 Poles & Zeros

The values of 's' for which the magnitude of the transfer function becomes infinity are called the Poles of the system. These are nothing but the roots of the equation obtained after equating the denominator of the transfer function to zero.

And the values of 's' for which the magnitude of the transfer function becomes zero are called the Zeros of the system. Poles and Zeros may be real or complex-conjugates or combination of both.

(ex) $G(s) = \dfrac{(s + 2)}{s(s + 5)}$

For this transfer function, there are two poles at s= 0 and s= -5 and a zero at s= -2. These values can be plotted on the complex s-plane, using an X for poles and O for zeros.




### 3.4.3 Order of the System

The order of the system is the highest power of 's' present in the denominator polynomial of the transfer function. The system in our last example is a second order system.

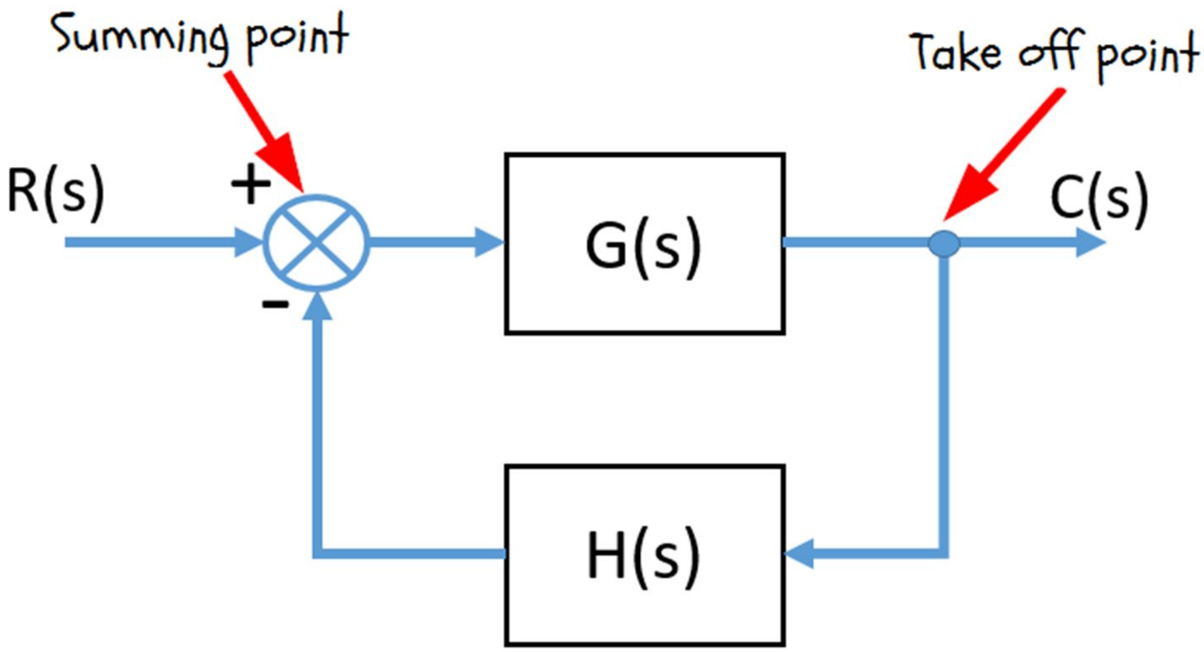
# 4. REPRESENTATION

## 4.1 BLOCK DIAGRAM

If a given system is a complicated one (like our example from the last chapter), it is very difficult to analyze the system as a whole. For such Control systems, we can find the transfer function of each and every element of the system. And by showing connection between the elements, the complete system can be split into different blocks and analyzed conveniently.

In block diagram representation, the interconnection of system components to form a system can be conveniently shown by the blocks arranged in proper sequence. It explains the cause and effect relationship existing between the input and output of the system, through the blocks. Each block in a block diagram is called a Functional block, it means the block explains the mathematical operation on the input by the element to produce the corresponding output. The transfer function of the element is mentioned inside the block.

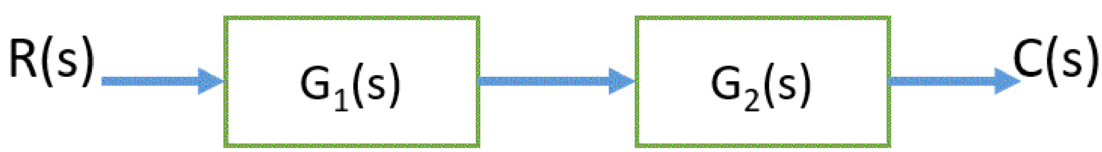
There are four basic components of a block diagram. First are the blocks themselves describing the relation between input and output through transfer function. Then there are summing points, where the output of two or more blocks are added algebraically. The third component of a block diagram is called a take-off point, which represent the application of total output from a point as the input to some other block. Finally block diagrams contain arrows indicating the direction of flow of signals.

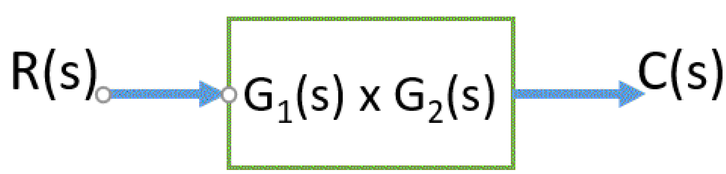



Utilizing some basic rules, it is possible to reduce a complex block diagram to a simple form. We shall discuss those rules and later then try out an example:

**1. Combining Cascade blocks**

Blocks connected in cascade can be replaced by a single block with transfer function equal to the product of the respective transfer functions.

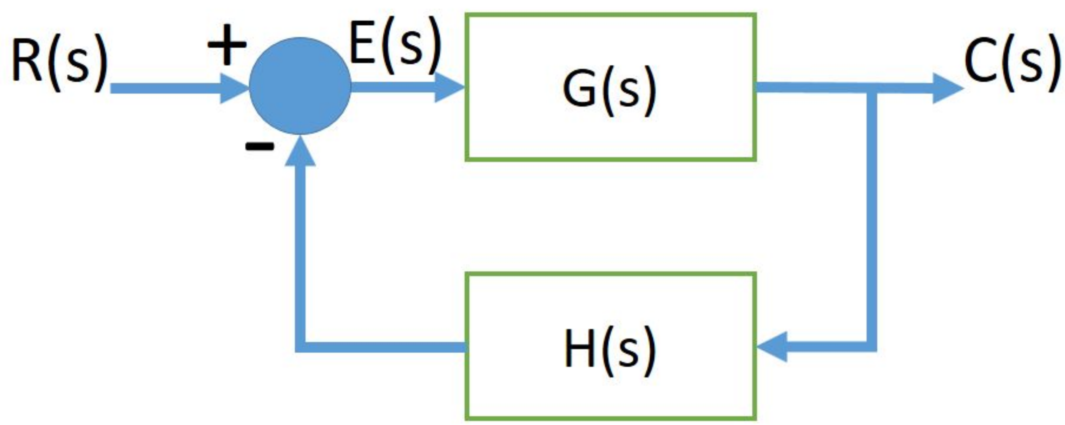


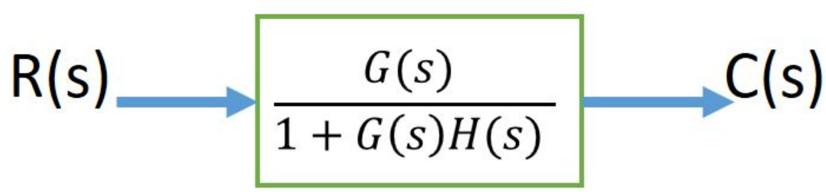



## 2. Eliminating a Feedback loop

The transfer function of a simple feedback loop with, where G(s) is the forward path transfer function and H(s) is the feedback path transfer function, is given by

$$\frac{G(s)}{1+G(s)H(s)}$$




This can be easily proved as follows:

$$C(s) = G(s)E(s)$$

$$E(s) = R(s) \blacksquare H(s)\, C(s)$$

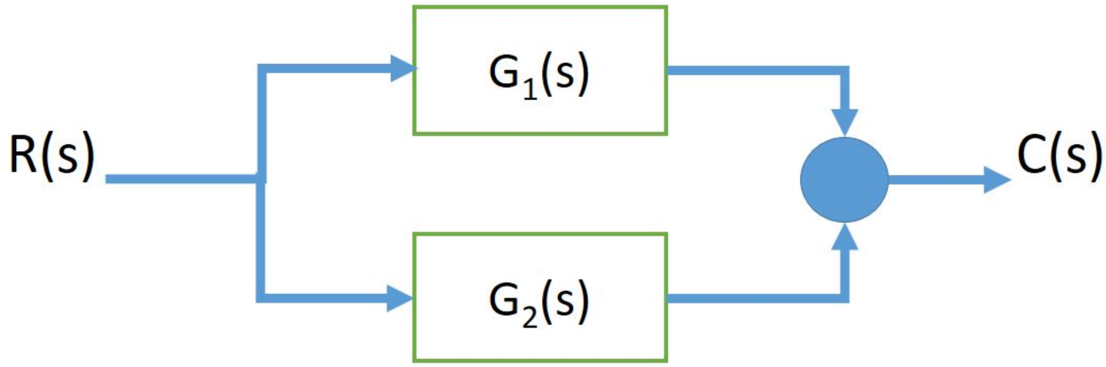
$$C(s)$$

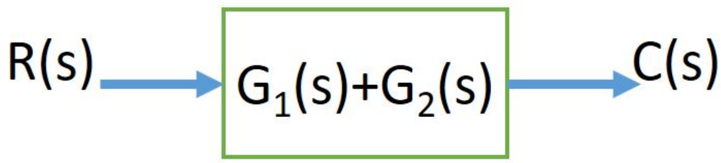
$$= G(s)\, [R(s) \blacksquare H(s)\, C(s)]$$

$$\therefore \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

**3. Parallel Blocks**

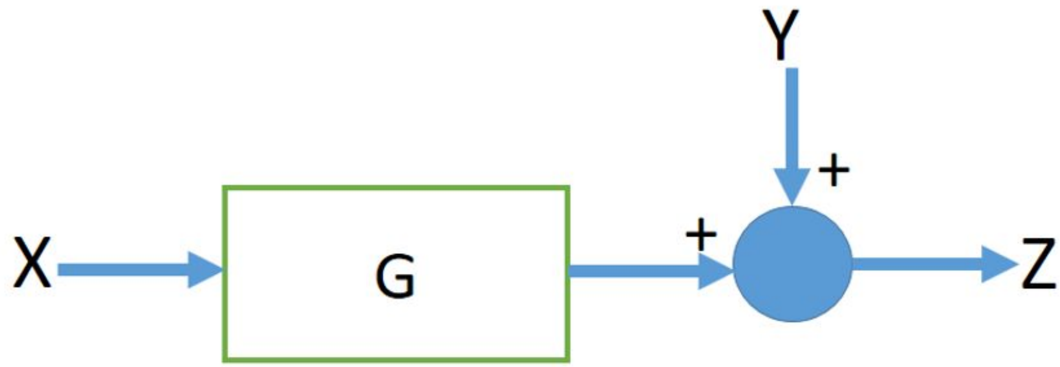
Blocks are said to be in parallel if they have a common input and the overall output is the sum of the individual outputs. The overall transfer function is simply the sum of the transfer functions of the individual blocks

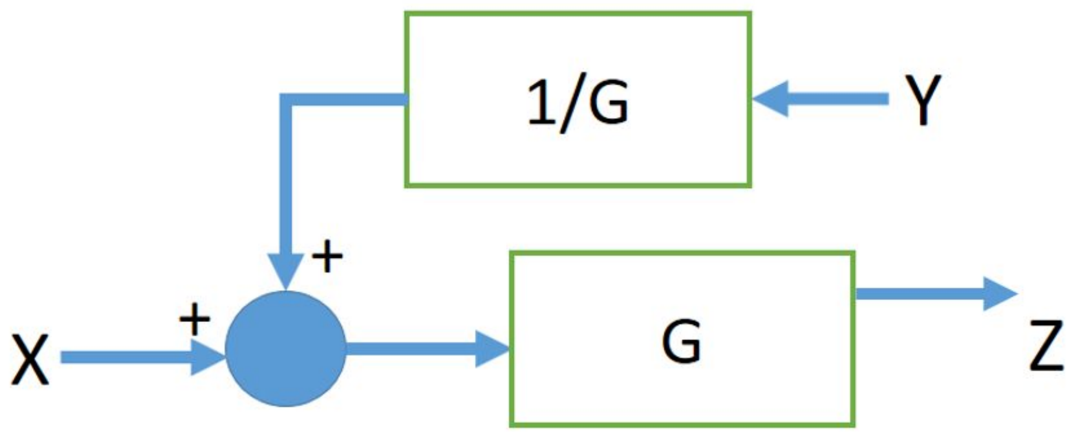


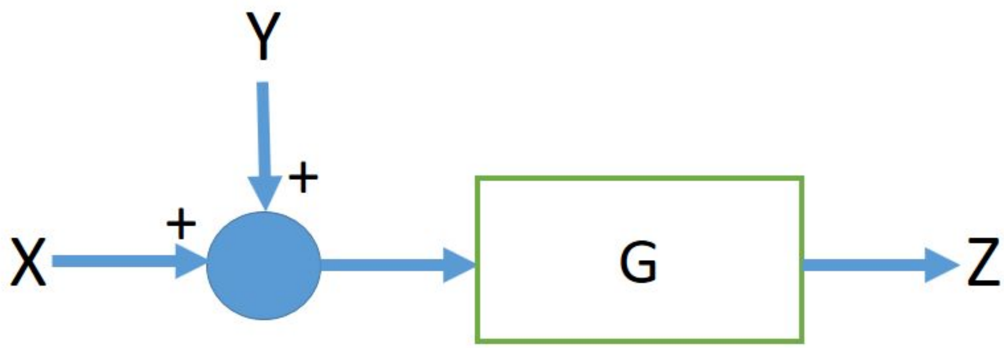

4. **Moving a summing point ahead of a block**

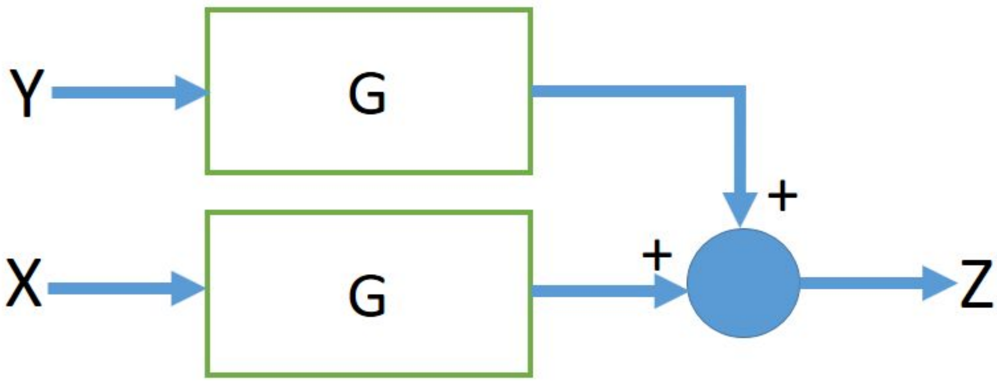
It is sometimes necessary to move a summing point ahead of a block to simplify the block diagram. This can be done provided the transfer function of the blocks are modified accordingly.



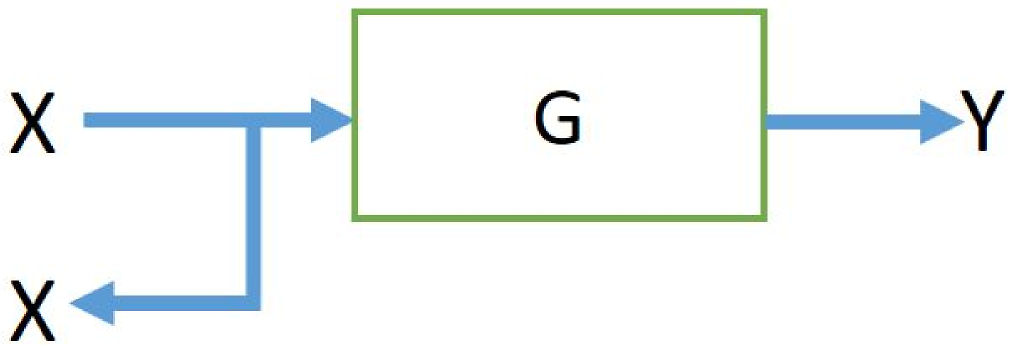



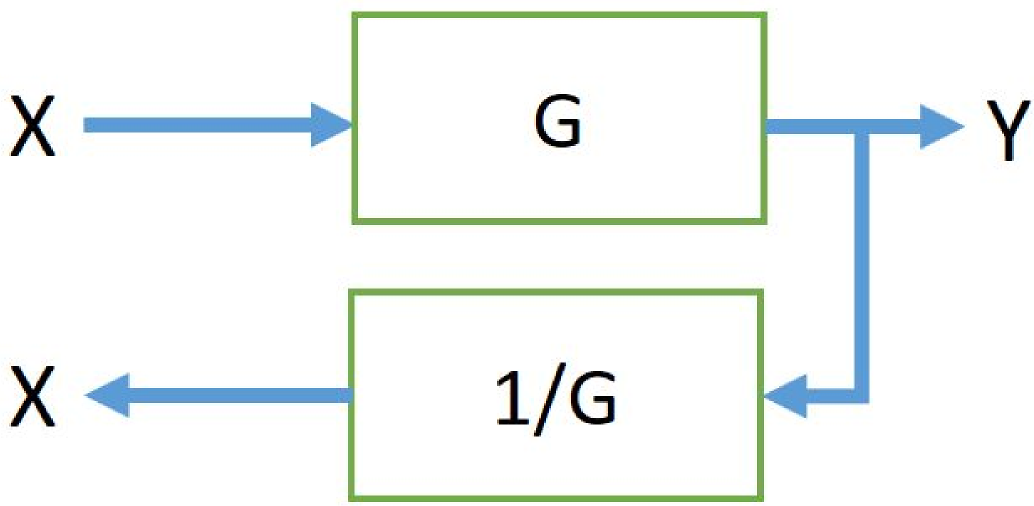

**5. Moving a summing point behind a block**



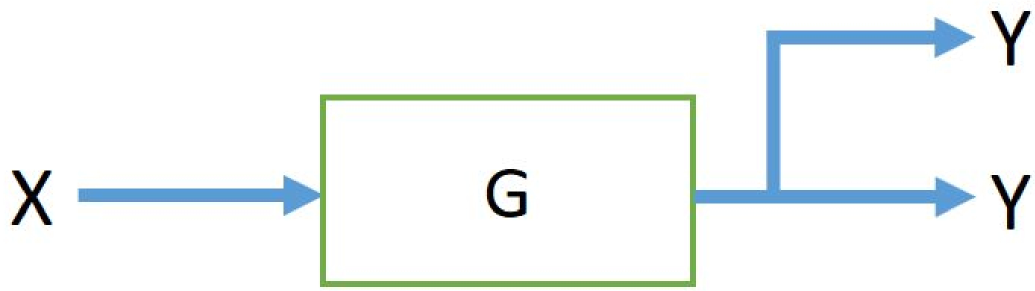



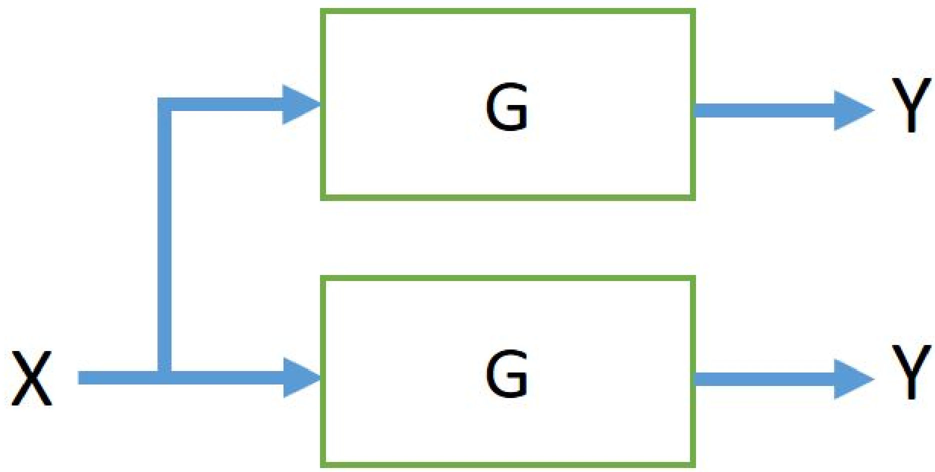

**6. Moving a pick-off point behind a block**



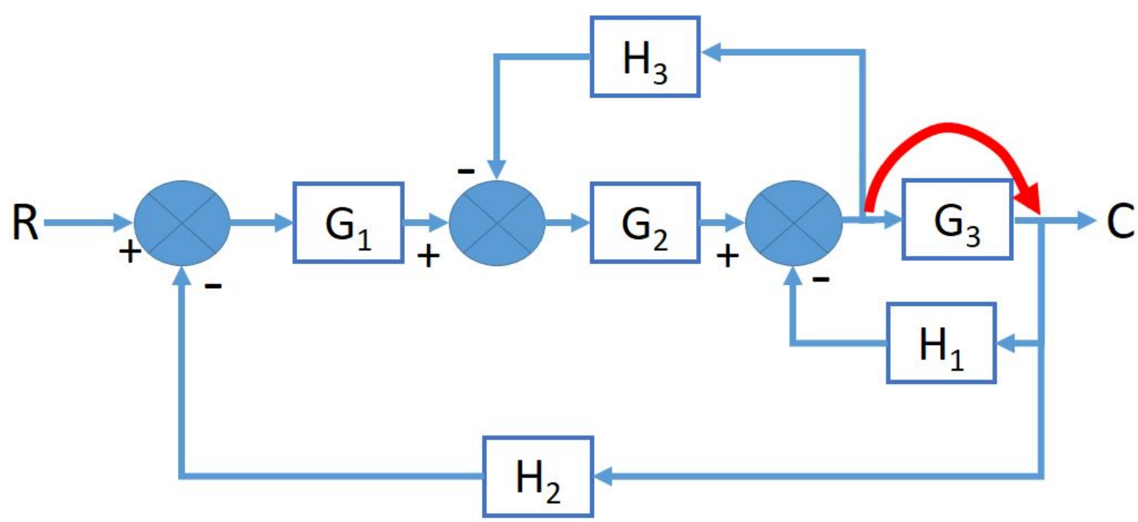

**7. Moving a pick-off point ahead of a block**

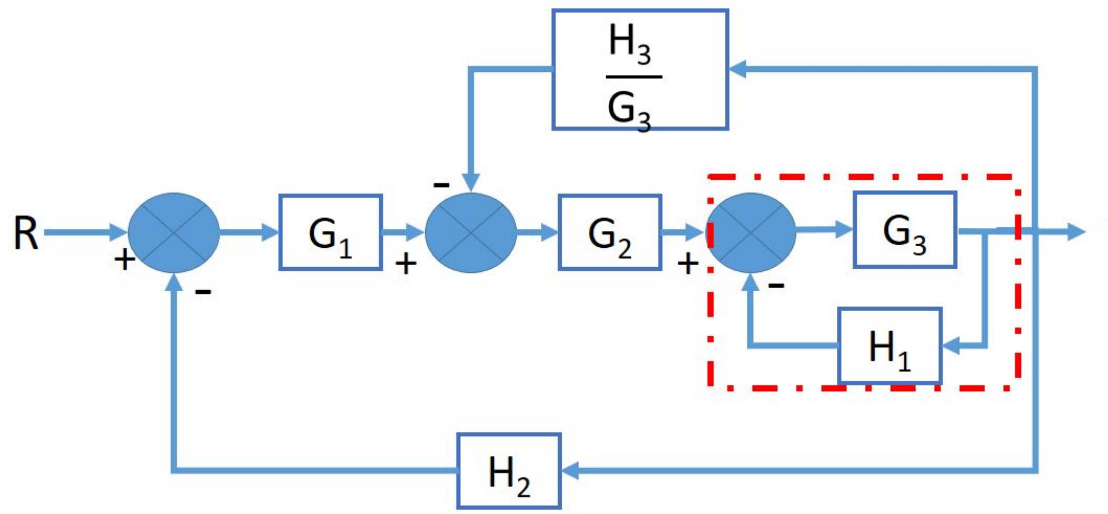


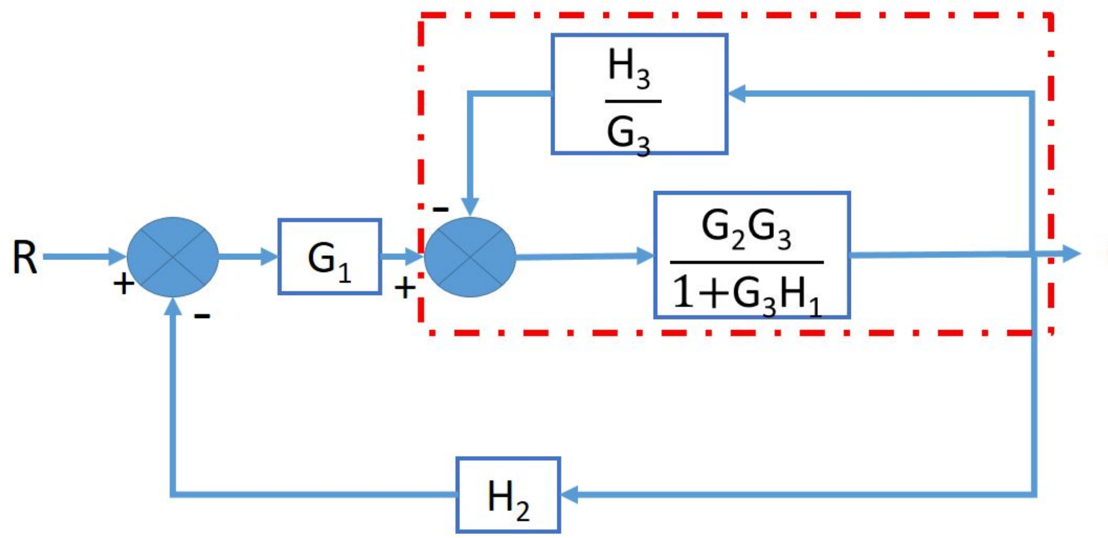



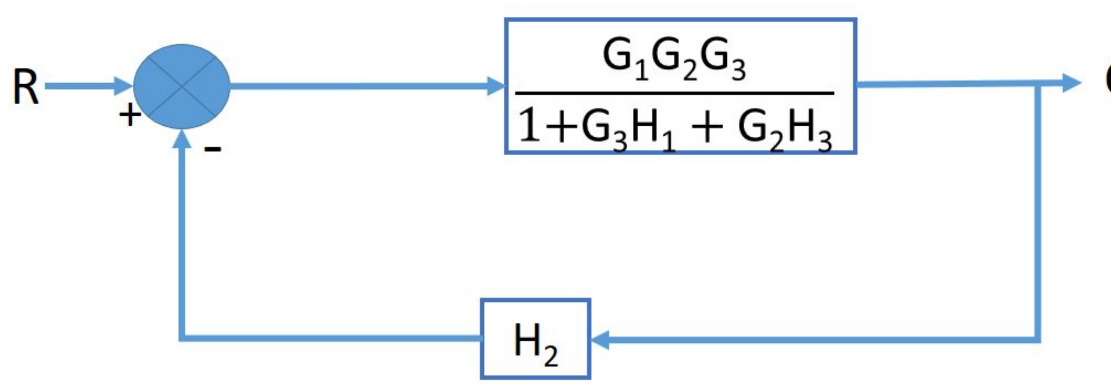

**Example:**

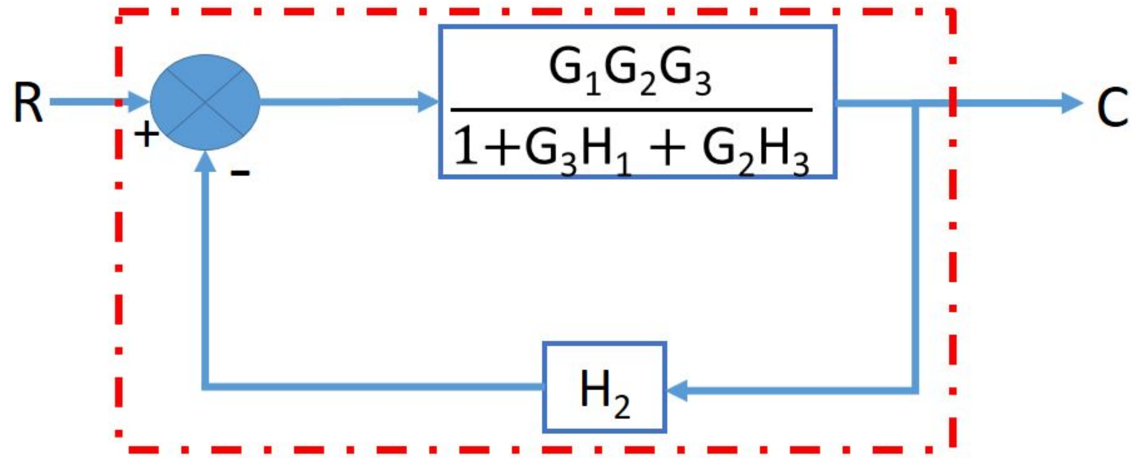
Let's solve this example using the block diagram reduction technique. Keep in mind that it is not necessary to reduce the block in this exact same manner. Nonetheless the final answer will the same.

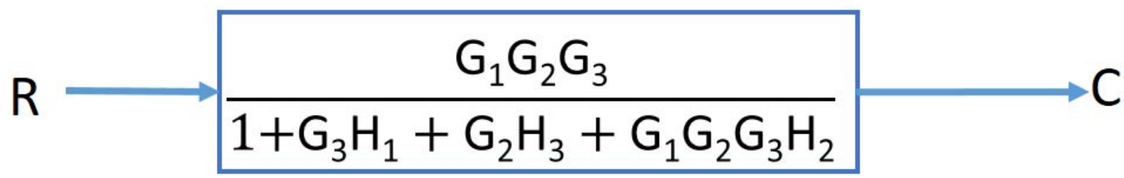



H3
G3
R
G1
G2
G3
C
H1
H2
+
-


H3
G3
R
G1
G2G3
1+G3H1
C
H2
+
-


R
G1G2G3
1+G3H1 + G2H3
C
H2
+
-

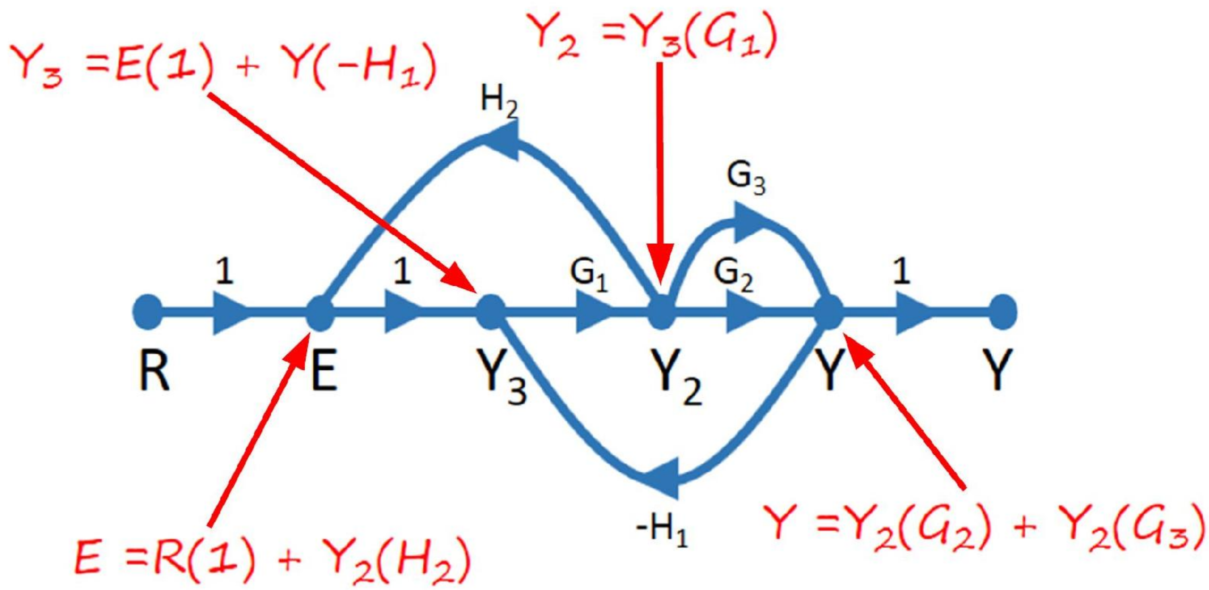



# 4.2 SIGNAL FLOW GRAPHS

Signal flow graphs are an alternate way of representing systems, particularly when the set of equations describing the system are available. Unlike block diagrams, which consist of blocks, arrows, summing junctions, and pickoff points, a signal-flow graph consists only of branches, which represent systems, and nodes, which represent signals. In the signal flow graph, all the variables, dependent and independent are represented by small circles called Nodes. The relationships between various nodes are represented by joining the nodes as per the equations. The lines joining the nodes are called branches. Each branch is associated with a transfer function and an arrow. The transfer function represents the relationship between adjoining variables and the arrow indicates the direction of flow of signals.

Example:

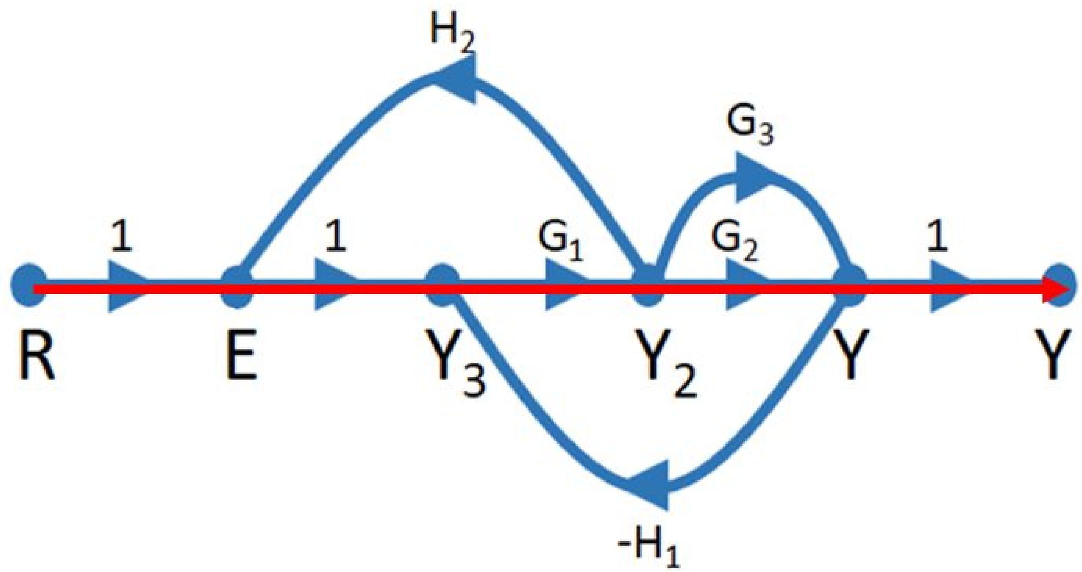



# 4.3 MASON'S RULE

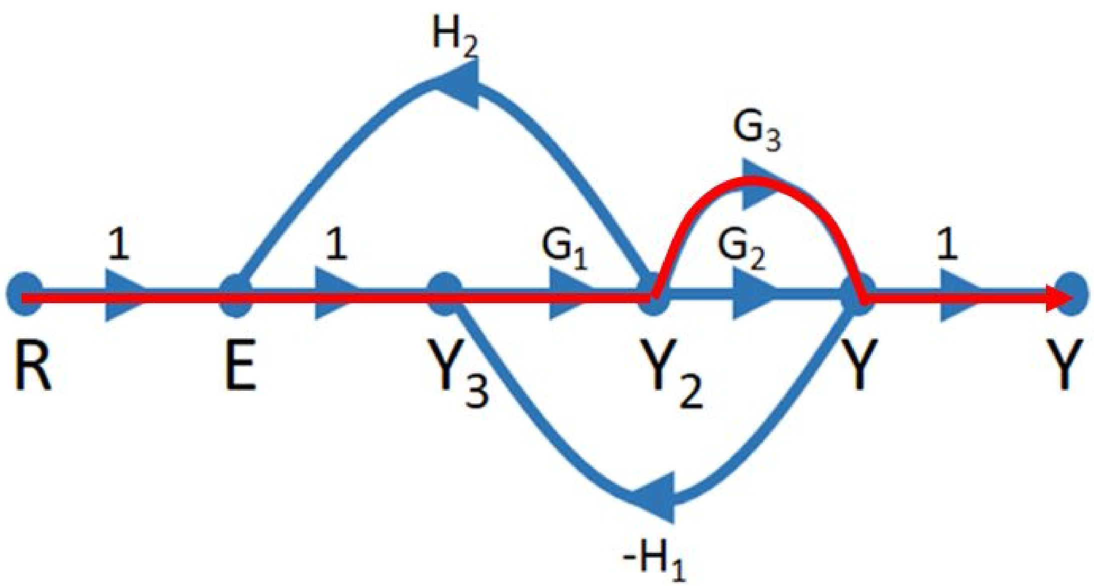
If the block diagram is an extremely complicated one, it may take a while to find the transfer function using block reduction technique and most times it's not worth the effort. Instead the transfer function of a block diagram can be obtained directly by using the Mason's Gain formula. Although the Mason's Gain formula was first developed as a method to solve signal flow graphs, it can be used with Block diagrams as well.

There are a few terms that you need to be familiar with before using this method.

1. **Forward Path:** is a path from the input node to the output node. In our example from above there are 2 forward paths.




Forward path gain = $G_1\,G_2$

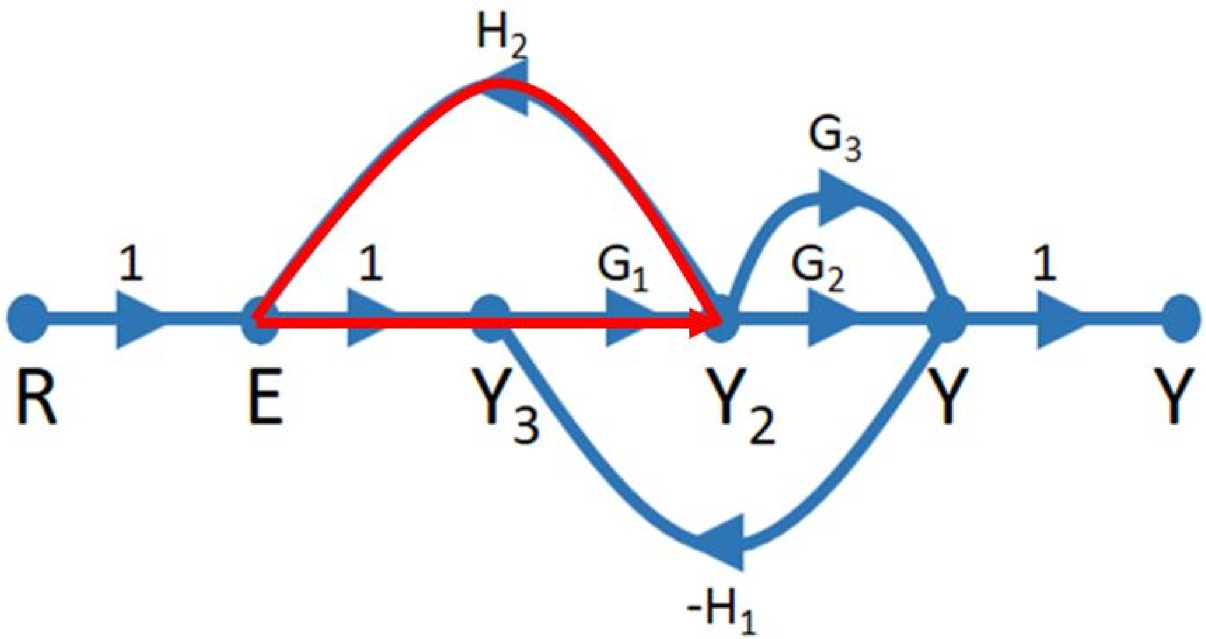



Forward path gain = $G_1\,G_3$

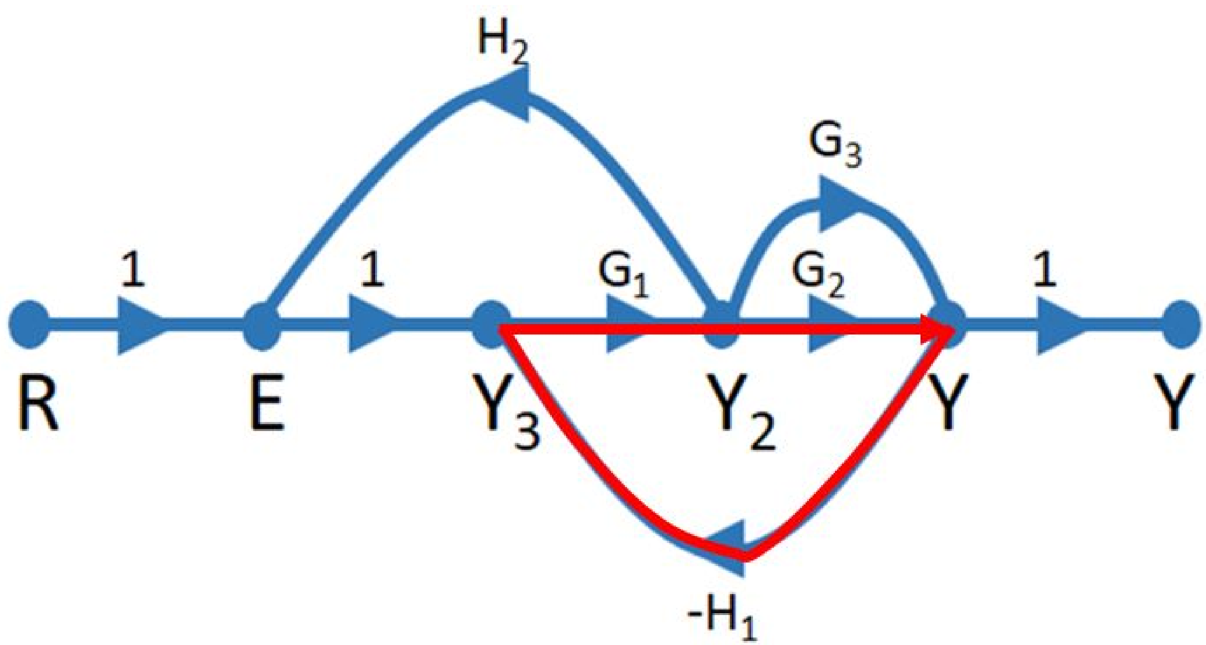
The forward path gain can be obtained by simply multiplying all the transfer functions along the path.

2. **Loop:** A loop is a path which originates from a node and terminates at the same node, without passing through any node

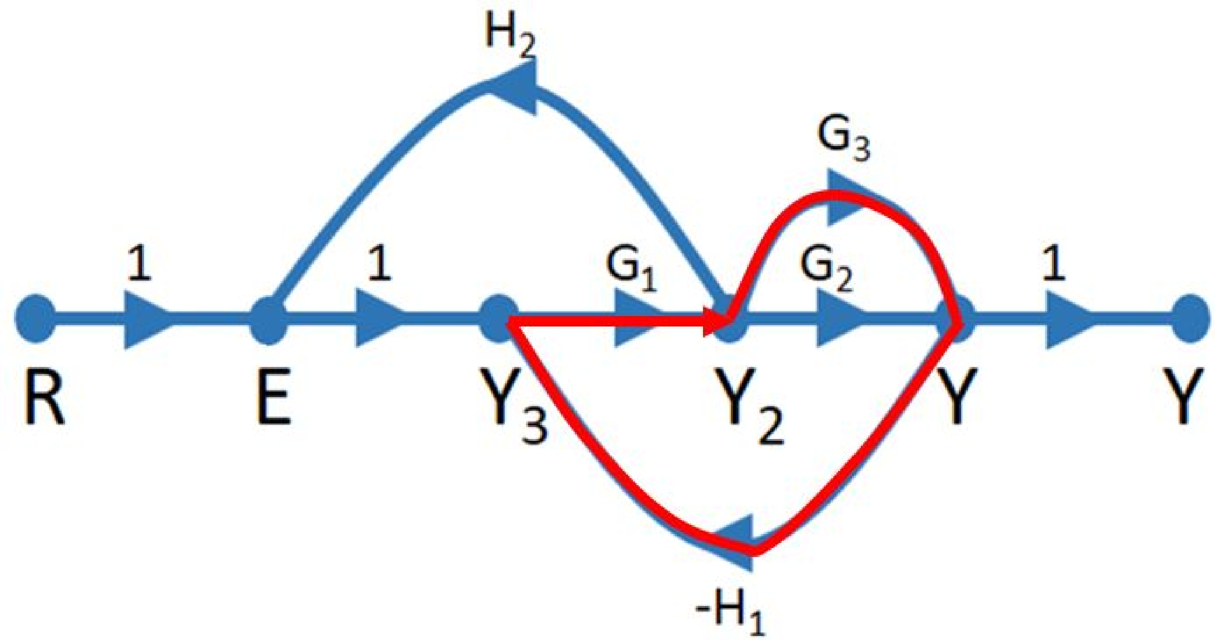
twice. There are three loops in our example.




Loop gain = $G_1 H_2$




Loop gain = $-G_1 G_2 H_1$




Loop gain = $-G_1 G_3 H_1$

Loops that contain only one node are called self-loops. The loop gains can be obtained by multiplying the transfer functions of all the branches in the loop.

3. **Non touching Loops:** Loops that do not share a node are called non touching loops. There are no non touching loops in our example.

OK! Now that you are familiar with these basic concepts, let's try and use the Mason gain formula to obtain the transfer function of the system in our example.

The first step in using the Mason's gain formula is to identify all the possible forward paths and loops in the system. Then find the forward path gains and the loop gains for all the identified forward paths and loops. Denote the forward path gains as $P_1$, $P_{2,}$ $P_3$ etc. and the loop gains as $L_1$, $L_{2,}$ $L_3$ etc.

The next step is to identify the non-touching loops (if there are any) and then calculate the gain product of all possible combinations of two non-touching loops. In our example there are no 2 non-touching loops.

The Mason’s Gain formula is given by,

$$\text{Overall TF} = \frac{\sum P_k \Delta_K}{\Delta}$$

K = no. of forward paths

$P_K$ = Gain of the $k^{th}$ forward path

$\Delta$ = 1 – Σ (all individual loop gains) + Σ (Gain product of all possible combinations of 2 non touching loops) - Σ (Gain product of all possible combinations of 3 non touching loops) +……

$\Delta_K$ = Value of $\Delta$ eliminating any loops that touch the $k^{th}$ forward path

In our example,

$P_1 = G_1\, G_2$

$P_2 = G_1\, G_3$

$L_1 = G_1\, H_2$

$L_2 = -G_1\, G_2\, H_1$

$L_3 = -G_1\, G_3\, H_1$

$$\Delta = 1 - (L_1 + L_2 + L_3) + 0 = 1 - G_1H_2 + G_1G_2H_1 + G_1\, G_3\, H_1$$

$\Delta_1 = 1 \quad \because$ all the 3 loops touches the forward path 1

$\Delta_2 = 1 \quad \because$ all the 3 loops touches the forward path 2

$$\therefore \text{TF} = \frac{P_1\, \Delta_1 + P_2\, \Delta_2}{\Delta} = \frac{G_1\, G_2 + G_1\, G_3}{1 - G_1H_2 + G_1G_2H_1 + G_1\, G_3\, H_1}$$

As we saw with our example, the Mason's Gain formula is a far easier method to find the overall transfer functions of system. All there is to do to identify the forward paths and loops correctly.

# 5. TIME RESPONSE ANALYSIS

## 5.1 INTRODUCTION

Time response analysis of a Control system means to analyze the variation of the output of a system with respect to time. The output behavior with respect to time should be within specified limits to have satisfactory performance of the system. The complete base of stability analysis (chapter 6) lies in the time response analysis. In practical systems, abrupt changes don't usually occur and the output takes a definite time to reach its final value. This time taken varies from system to system and depends on many factors.

The total output response can be analyzed in two separate parts. First is the part of the output during the time it takes to reach its final value. The second is the final value attained by the output, which will be close as possible to the desired value, if things go as planned.

This can be further explained by considering a practical example. Suppose we want to travel from city A to city B. It will take finite time to reach city B. This time depends on whether we travel by bus or train or plane. Similarly, whether we reach city B or not depends on number of factors like weather, condition of road etc. So in short we can classify the output as,

1. Where to reach?
2. How to reach?

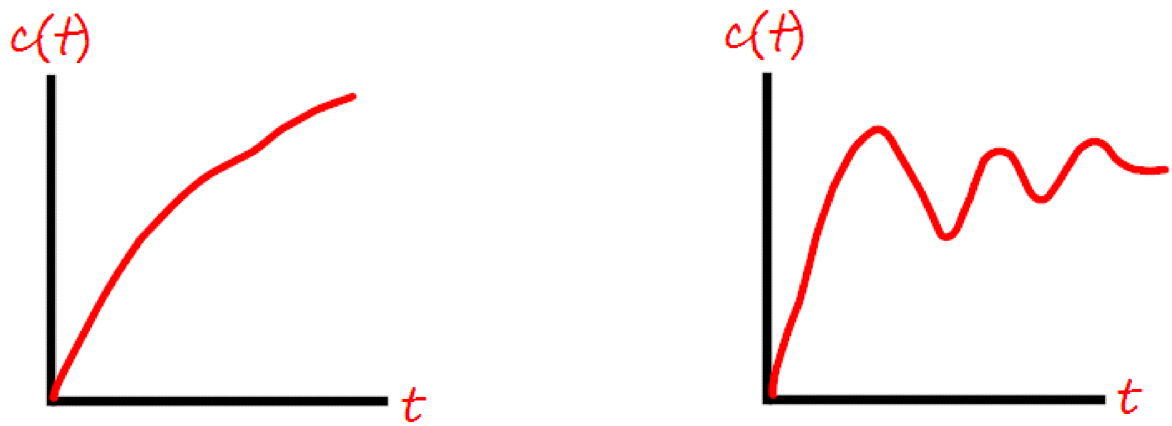
So the effectiveness and accuracy of the system depends on the final value reached by the system, which should be close to desired value as possible and also that final value should be reached in smoothest manner possible. The final value achieved by the output is called the Steady state value, while the output variations within the time taken to steady state is called the transient response of the system.

The transient response of a system is that part of the response that dies to zero after some time as the system reaches its final value. It is denoted as $c_t(t)$. Systems in which the transient response dies out after some time are called stable systems.

i.e. for stable systems,

$$\lim_{t \to \infty} c_t(t) = 0$$

Typically, the transient response is exponential or oscillatory in nature.




The steady state response is that part of the time response that remains after the transients have died down. It is denoted as $C_{ss}$. The steady state response indicates the accuracy of the system and the difference between the desired output and the actual output is known as the steady state error ($e_{ss}$).

Hence the total time response of the system can be written as,

$$c(t) = C_{ss} + c_t(t)$$

## 5.2 STANDARD TEST INPUTS

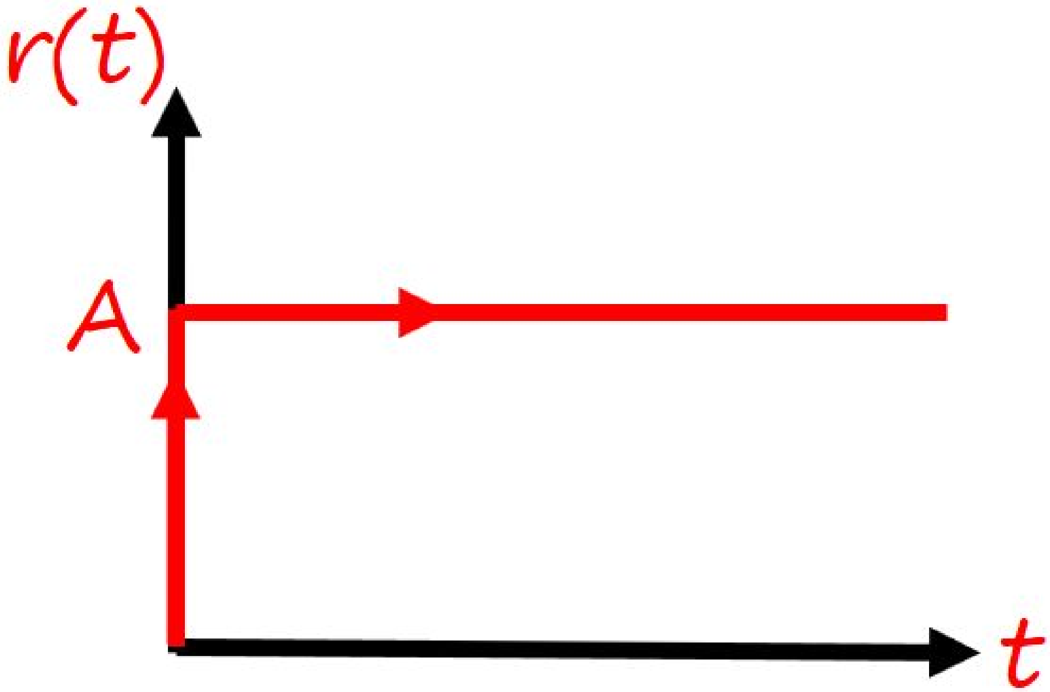
Usually the time response analysis of Control systems is done with the help of Standard test inputs. The evaluation of the system is done on the basis of the system response to these standard test inputs. Once the system behaves satisfactorily to these test input, its time response to the actual inputs is also assumed to be satisfactory. In practice, many signals (that are functions of time) can be used as test inputs, but the commonly used standard test inputs are:

1. **Step input**

Step function is mathematically defined as,

$$\begin{aligned} r(t) &= A, \qquad t \geq 0 \\ &= 0, \qquad t < 0 \end{aligned}$$

Step Input is like the sudden application of an input at a specified time. If A = 1, then it is called the Unit step function.



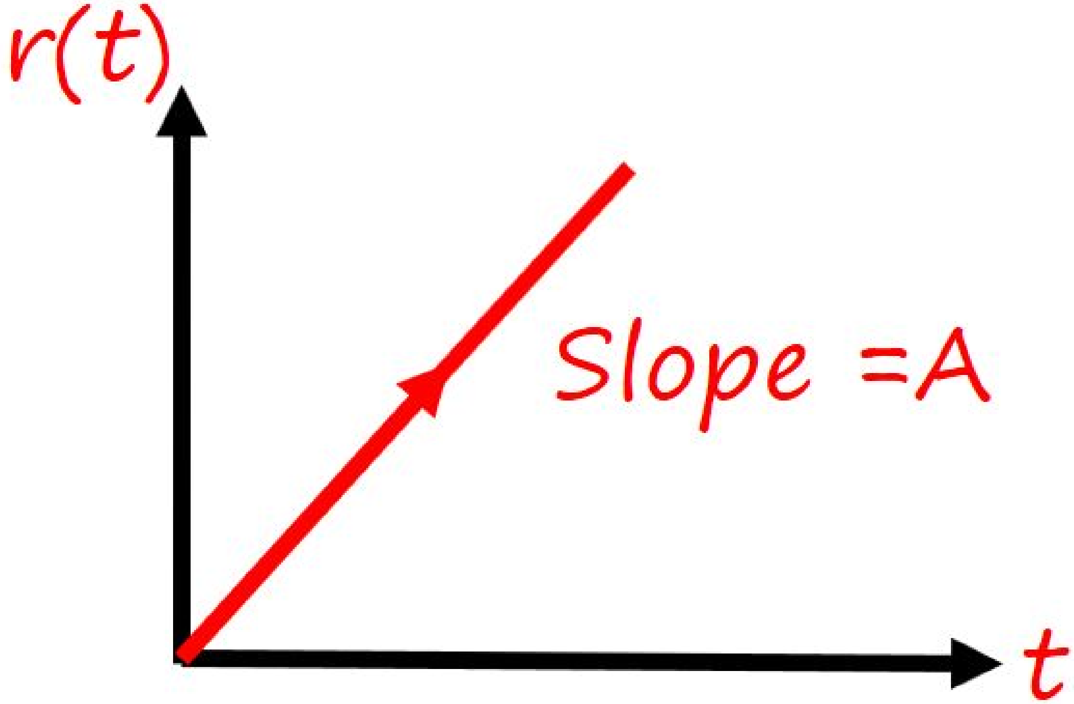

The Laplace transform of step function is A/s.

2. **Ramp Input**

Ramp function is mathematically defined as,

$$
\begin{aligned}
r(t) &= At, \qquad && t \geq 0 \\
&= 0, && t < 0
\end{aligned}
$$

Ramp input is nothing but the gradual application of an input. If A = 1, then it is called the Unit ramp function.



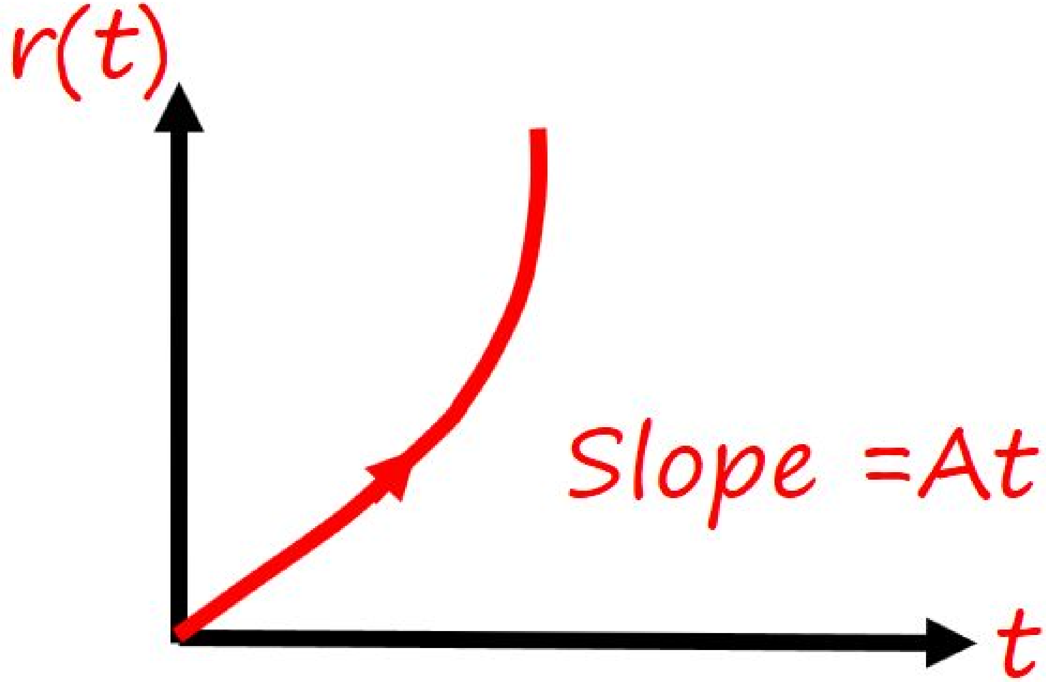

The Laplace transform of the ramp function is $A/s^2$.

**3. Parabolic Input**

Parabolic function is mathematically defined as,

$$r(t) = \frac{At^2}{2}, \qquad t \geq 0$$
$$= 0, \qquad t < 0$$

If A = 1, then it is called the Unit Parabolic function.

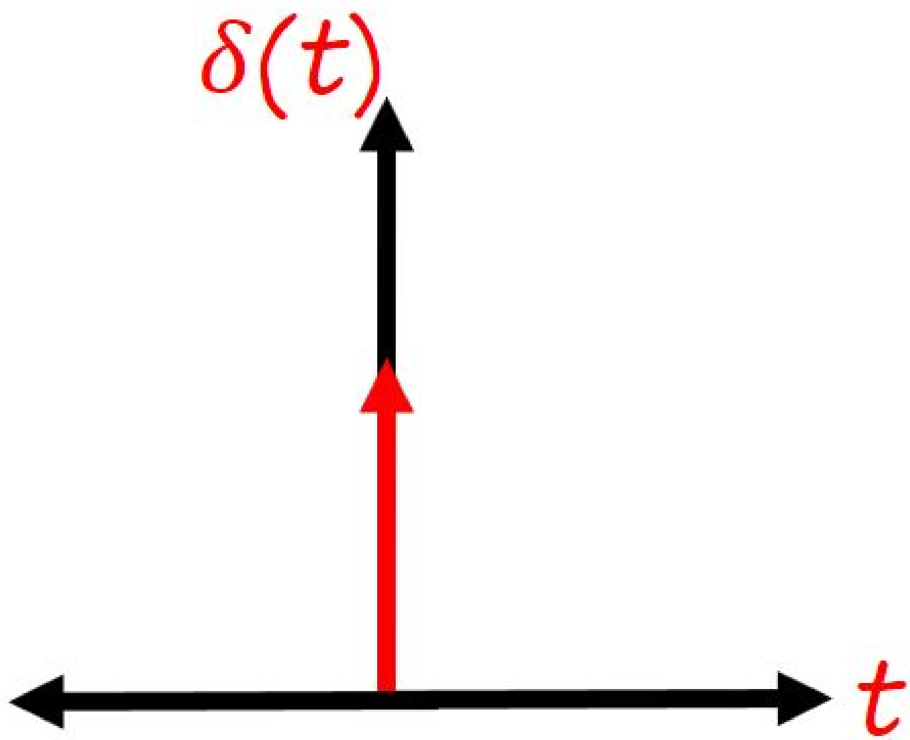



The Laplace transform of the parabolic function is $A/s^3$.

**4. Impulse Input**

We have already discussed the impulse function in detail in chapter 3. Impulse function is mathematically defined as,

$$\begin{aligned} \delta(t) &= 0, & t &\neq 0 \\ &= \infty, & t &= 0 \end{aligned}$$

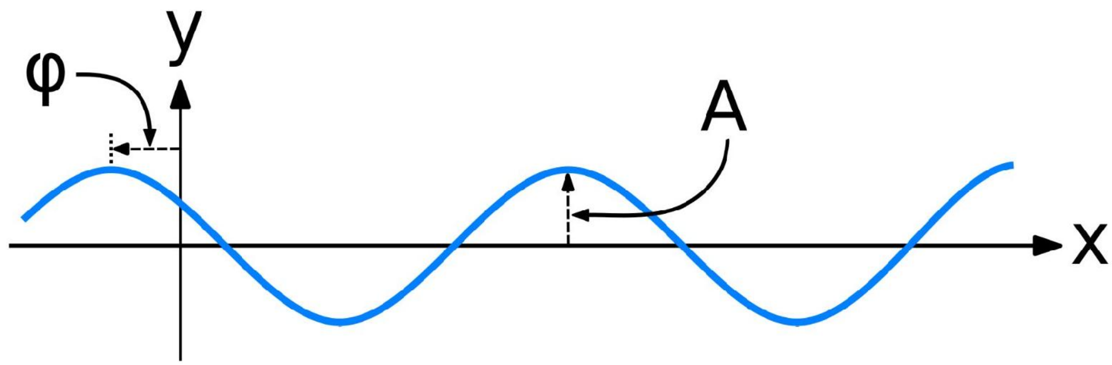



The Laplace transform of the parabolic function is 1.

5. **Sinusoidal input**

Sine waves and Cosines waves are collectively known as sinusoids or sinusoidal signals. Mathematically Sinusoids are mathematically represented as,

$$x(t) = A\cos(\omega t + \Phi)$$

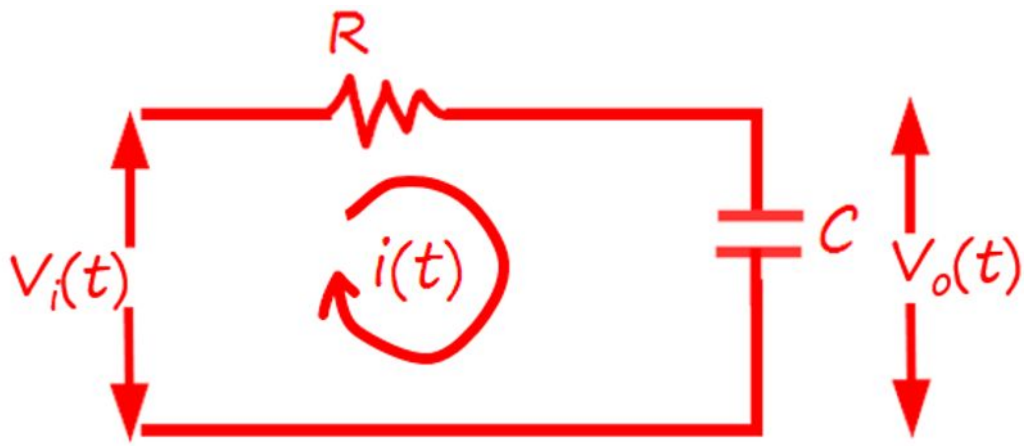



where **A** is the Amplitude (maximum height of the signal), **ω** is the angular frequency and **ϕ** is the phase. Sine waves and Cosines waves are basically the same, except that they start at different times (ie they are 90 degrees of phase).

Usually, the Impulse signal is used for Transient response analysis and the Steady state analysis is carried out using all the above mentioned test signals.

# 5.3 FIRST ORDER SYSTEM

First order systems are those systems which can be described by first order differential equations. In First order systems the highest power of ‘s’ in the denominator of the closed loop transfer function is 1. I'm sure you've run into a fair share of first-order systems before, most popular one being the R-C filter.




$$V_i(s) = I(s)\,R + \frac{1}{sC}\,I(s) \quad \ldots\ldots..(1)$$

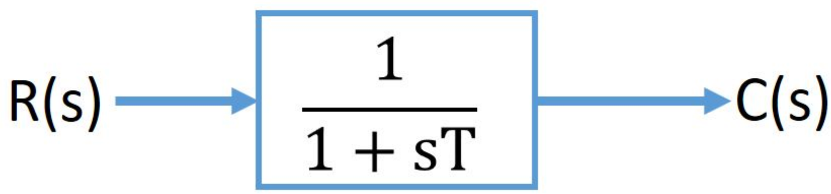
$$V_O(s) = \frac{1}{sC}\,I(s) \quad \ldots\ldots..(2)$$

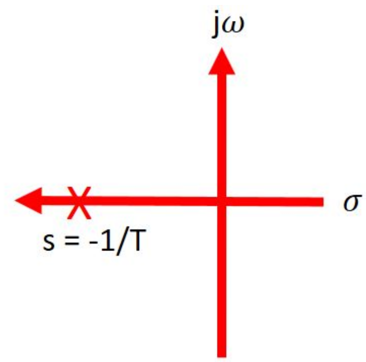
$$TF = \frac{V_O(s)}{V_i(s)} = \frac{1}{1+sRC} \quad , RC= \tau \rightarrow time\ constant$$

Just like our RC circuit, the general form of the transfer function of First order systems is,

$$TF = \frac{C(s)}{R(s)} = \frac{1}{(1+sT)}$$

Where T is the time constant of the system.




Let’s now calculate the Unit Impulse response of a first order system.

$$r(t) = u(t) \Longrightarrow R(s) = \frac{1}{s}$$

$$\therefore C(s) = \frac{1}{s(1+sT)}$$

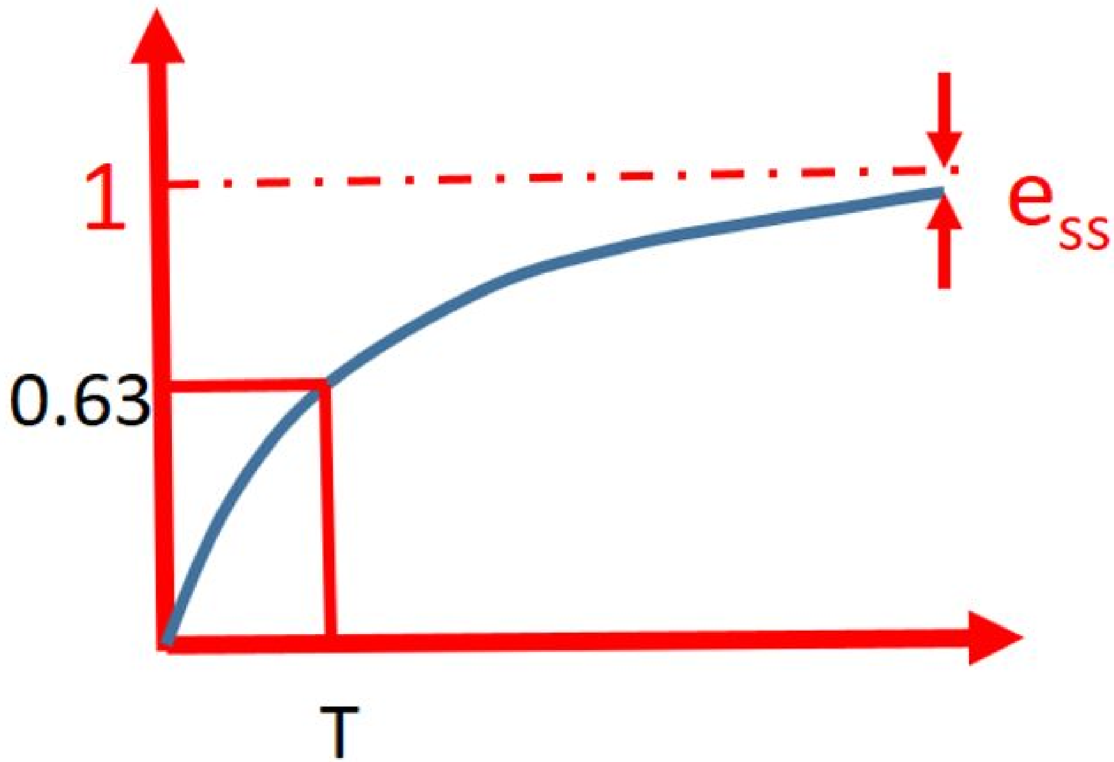
$$= \frac{1}{s} - \frac{1}{(s+\frac{1}{T})}$$

Taking the Laplace inverse,

$$c(t) = 1 - e^{\frac{-t}{T}}$$

Steady state    Transient state

The interesting thing to note here is that the input pole at the origin is responsible for the steady state response and the system pole at s = -T is responsible for the transient response. So the transient response is totally dependent on the parameter T (Time constant). The time constant can be defined as the time taken by the step response to reach 63% of its final value.

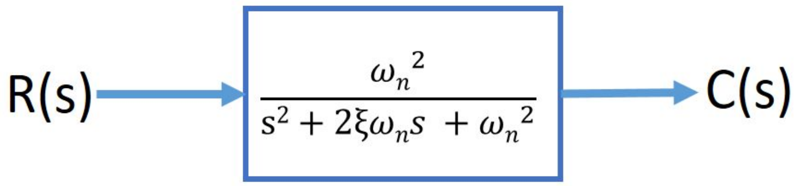


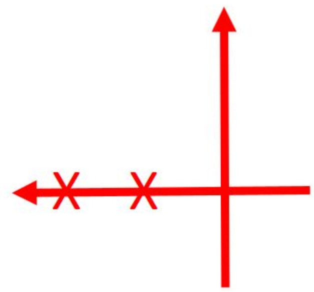

## 5.4 SECOND ORDER SYSTEM

Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described. Whereas varying a first-order system's parameter simply changes the speed of the response, changes in the parameters of a second-order system can change the form of the response.

The transfer function of a 2nd order system is generally of the form,

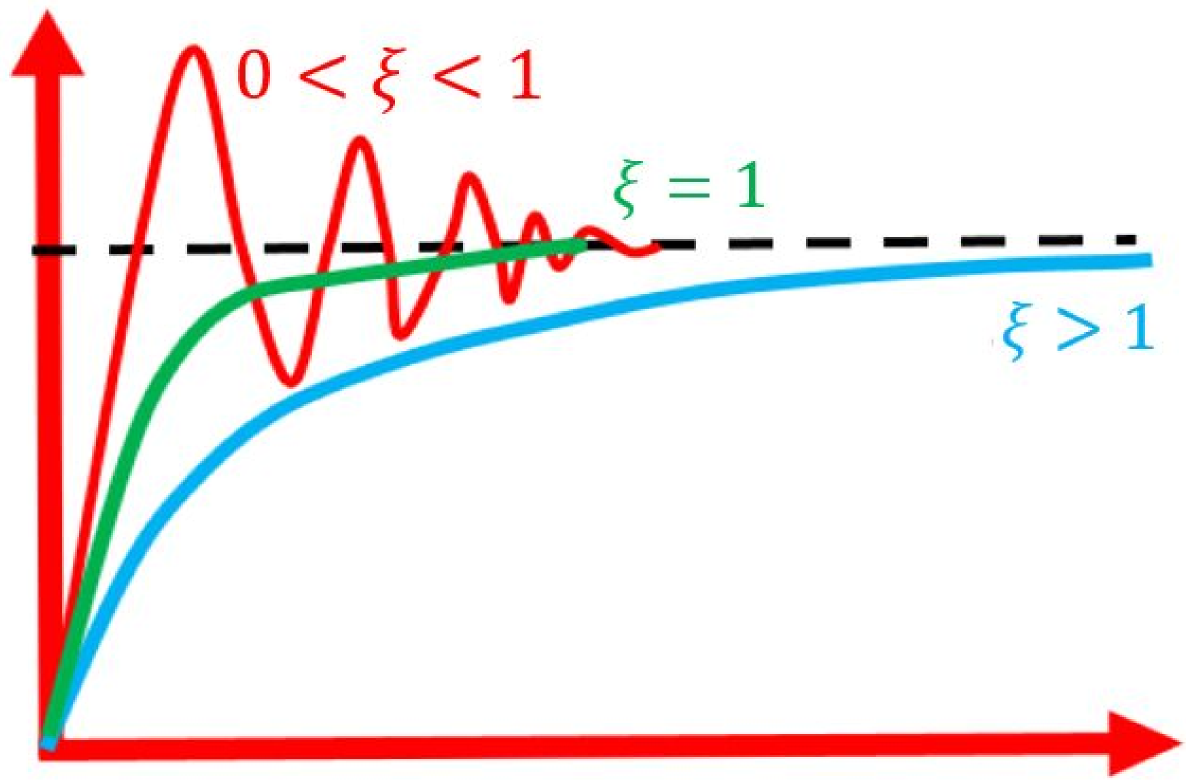
$$TF = \frac{C(s)}{R(s)} = \frac{\omega_n^{\,2}}{s^2 + 2\xi\omega_n s + \omega_n^{\,2}}$$



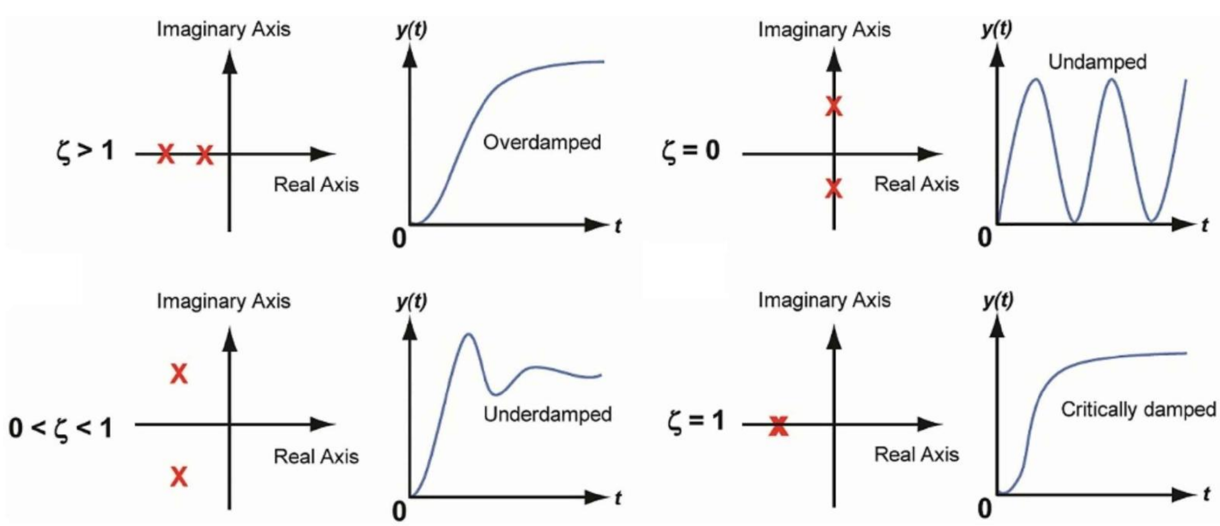



The behavior of the second-order system can then be described in terms of two parameters: the damping ratio ( $\xi$ ) and the natural frequency ( $\omega_n$). Every practical second order system takes a finite time to reach its steady state and during this period the system output oscillates. But practical systems have a tendency to oppose this oscillatory nature of the system, this is called Damping. A factor called the Damping ratio is used to quantify the extent of damping, a system offers. In some systems it may be so low that the oscillations sustain for a longer time. These are called Underdamped systems. While in some other systems, the damping factor maybe so high that the system output will not oscillate at all. Instead the output follows an exponential path (like first order systems). These are called Overdamped systems.

In systems which offer no damping at all, the output response continues to oscillate, without ever reaching a steady state value. The frequency of oscillations in such a case is called the natural frequency.




Depending on the value of $\xi$, the location of the closed loop poles vary as shown in the figure below.



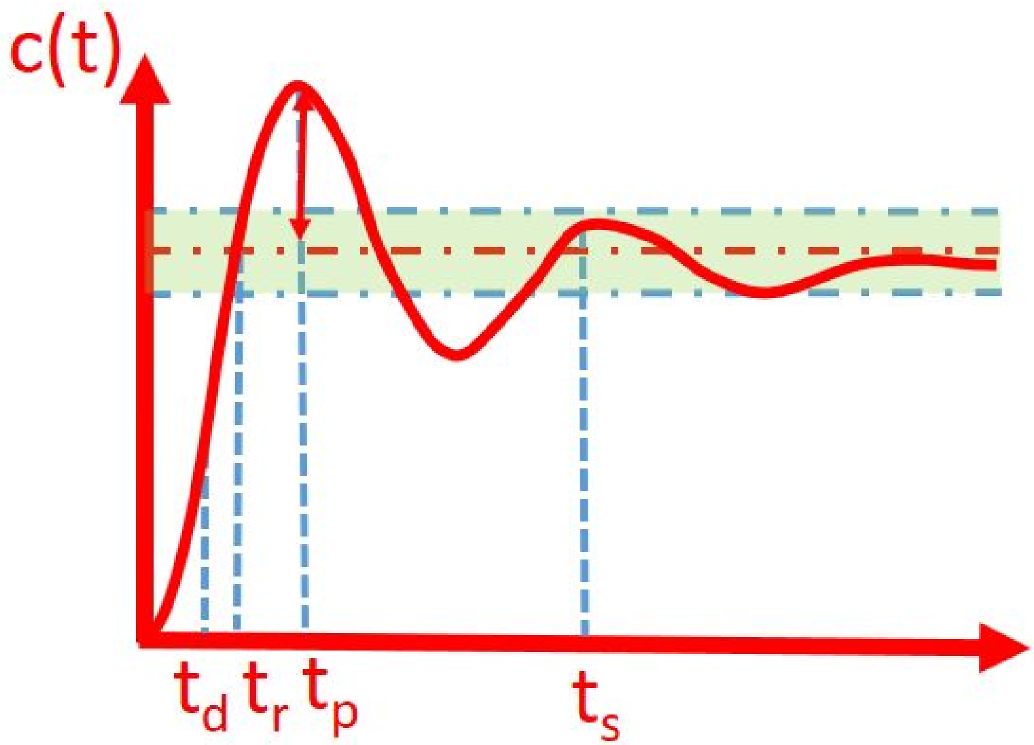

The under-damped case is the most important one from the application point of view since almost all real physical dynamic control systems go through oscillations before settling at the desired steady-state value. The step response of the considered second-order closed-loop system is,

$$y(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right)$$

Derivation (Link)

## 5.5 TRANSIENT RESPONSE SPECIFICATIONS

The general under-damped step response is as shown below




The transient response characteristics of a second order system to a unit step input is specified in terms of the following time domain specifications.

1. **Delay time($t_d$):** It is the time taken for the response to reach 50% of the final value for the first time.

2. **Rise time($t_r$):** It is the time taken for the response to rise from 0% to 100% for the first time.

$$\text{Rise time } t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}}$$

3. **Peak time($t_p$):** It is the time taken by the response to reach its first peak value.

$$\text{Peak time } t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

4. **Peak Overshoot:** It is defined as the ratio of the peak value to the final value, where the peak value is measured from the final value.

$$\text{Peak overshoot } \%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

$$= e^{\frac{\xi\pi}{\omega_n\sqrt{1-\xi^2}}} \text{ x } 100$$

5. **Settling time($t_s$):** It is defined as the time taken by the response to reach and stay within a specified error (usually 2% or 5%)

$$\text{Settling time } t_s = \frac{5}{\xi\omega_n} \text{ (for 2\% tolerance limit)}$$

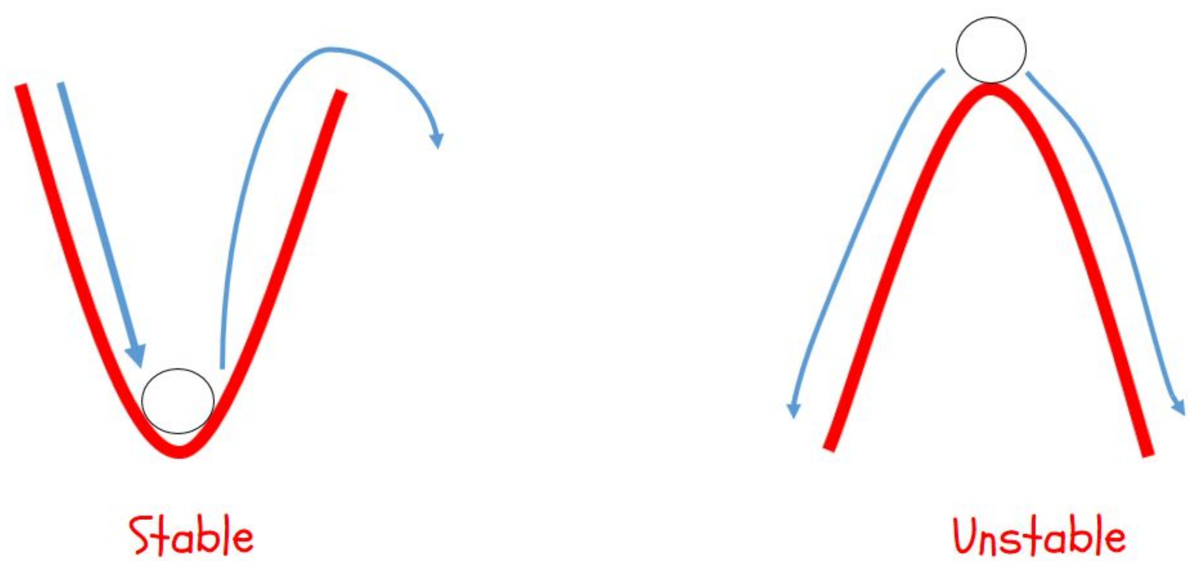
$$= \frac{3}{\xi\omega_n} \text{ (for 5\% tolerance limit)}$$

# 6. STABILITY

## 6.1 INTRODUCTION

What is stability?? The dictionary defines Stability as the ability of a substance to remain unchanged over time under stated or reasonably expected conditions of storage and use. This is actually a very good general definition. A system is said to be stable, if it does not exhibit large changes in its output for a small change in its input.

Before we dig into the stability of Control systems, let’s consider 2 practical cases to have a clear idea on the concept of stability.

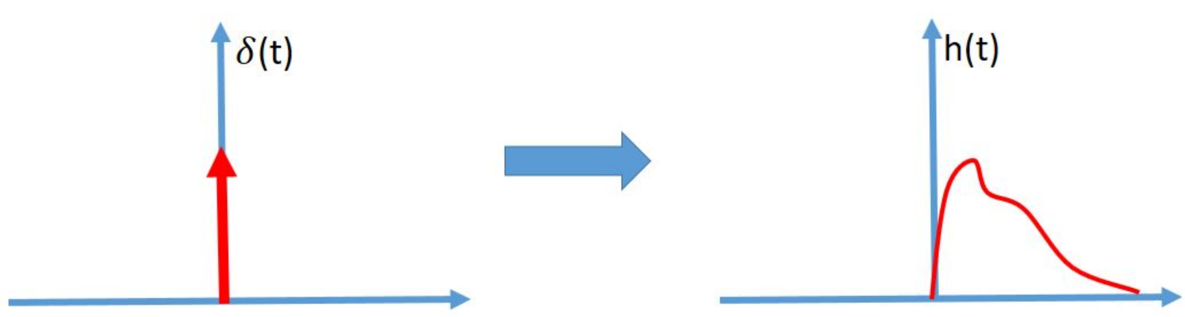



In the first example, a ball is placed in a u-shaped container. If we apply a small force on the ball, it moves back and forth slightly, but ultimately returns to its original position. Such a system is called a Stable system. Do keep in mind that if the force is larger than a certain threshold value, then the ball will not return to its original position. In the second example, the ball is placed on top on a hill, with slope on either side. In this case, the smallest disturbance will make the ball roll down the hill. Such a system is called an Unstable system.

# 6.2 STABILITY OF CONTROL SYSTEMS

As we have seen in the previous chapter, every system has to pass through a definite transient period. Will the system reach the desired steady state value after passing through the transients successfully?? The answer to this question is the basis of stability analysis.

Knowing that an unstable closed-loop system is generally of no practical value, we seek methods to help us analyze and design stable systems. To determine the stability of a Control system, we use a concept called BIBO stability. BIBO stability stands for bounded input, bounded output stability. BIBO stability is the system property that any bounded input yields a bounded output. This is to say that as long as we input a signal with absolute value less than some constant, we are guaranteed to have an output with absolute value less than some other constant. In order to understand this concept, we must first look more closely into exactly what we mean by bounded. A bounded signal is any signal which in which the absolute value of the signal is never greater than some value. Since this value is arbitrary, what we mean is that at no point can the signal tend to infinity, including the end behavior. The Unit impulse is the bounded signal of choice to determine the stability of a system.



Mathematically, a system is said to be BIBO stable if,

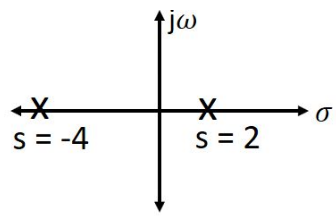
$$\int_{-\infty}^{\infty} |h(t)|dt < \infty$$

Where h(t) is the Unit impulse response.

Although stability can be determined this way. It is much easier to infer the stability of the system from the pole zero plot of the transfer function. The stability of a linear closed loop system depends on the location of the closed loop poles in the s-plane. For stable systems, all poles must be to the left of the imaginary axis.

Let's look at an example to prove this:

$$TF = \frac{C(s)}{R(s)} = \frac{10}{(s-2)(s+4)}$$




Unit step Response,

$$C(s) = \frac{10}{s\,(s-2)(s+4)}$$

R(s) = 1/s for Unit step signal

$$C(s) = \frac{-1.25}{s} + \frac{0.833}{s-2} + \frac{0.416}{s+4}$$

Taking Laplace Inverse to obtain response in time domain

$$c(t) = -1.25 + 0.833\, e^{2t} + 0.416\, e^{-4t}$$

$$\text{As } t \to \infty,\ c_{ss}(t) = \infty$$

In the above example, the pole at s=2 corresponds to the term $e^{2t}$ in the step response. As $t \to \infty$, this term itself tends to $\infty$, not allowing the response to settle down to a steady state value, making the system unstable. If any of the poles of the system lies on the imaginary axis, then the system is said to be marginally stable, meaning the response will keep on oscillating without either settling down or tending to infinity.

## 6.3 ROUTH-HURWITZ CRITERION

The Routh–Hurwitz stability criterion is a mathematical method of determining the location of poles of the system (with respect to imaginary axis) without actually solving the equation. In order to find the close loop poles of a system, we equate the denominator of the closed-loop transfer function to zero. The equation so obtained is called the Characteristic equation of the system. In the example we considered in the previous section, the closed loop transfer function was given in a convenient form, so we had no problem in determining the closed loop poles directly. The general form of the Characteristic equation is,

Characteristic Equation:

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots\ldots\ldots\ldots\ a^n = 0$$

There are two parts to this criterion: The necessary condition and the sufficient condition.

**Necessary Condition**

1. All the coefficients should be real and positive.
2. All the coefficients should be non-zero.

If any of these conditions fail, then the system is Unstable. But do keep in mind that these conditions do not guarantee the stability of the system, for that we need to check the sufficient conditions

**Sufficient Condition**

To check these conditions, we need to construct a Routh array.

The first row will consist of all the even terms of the characteristic equation. The second row will consist of all the odd terms of the characteristic equation.

Characteristic Equation:

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots\ldots\ldots\ldots\ a^n = 0$$

$$\begin{array}{l|llll|}
s^n & a_0 & a_2 & a_4 & \ldots\ldots\ldots\ldots \\
s^{n-1} & a_1 & a_3 & a_6 & \ldots\ldots\ldots\ldots \\
s^{n-2} & b_0 & b_1 & b_2 & \ldots\ldots\ldots\ldots \\
s^{n-3} & c_0 & c_1 & c_2 & \ldots\ldots\ldots\ldots
\end{array}$$

The elements of third row can be calculated as:

$$b_0 = \frac{\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}}{a_1} = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_1 = \frac{\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1} = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

The elements of fourth row can be calculated by multiplying the terms of the second and third row in a similar way. The process is continued till coefficient for $s^0$ is obtained

The necessary and sufficient condition for stability is: All the terms in the first column of the Routh array must have same sign. If any sign change exists, then the system is unstable and the number of sign changes is equal to the number of roots lying in the right half of s-plane.

Looks tough right?? Not really, it looks sort of messy and confusing because we have stated the criterion as a set of rules. But you are one example away from getting grip of this whole thing.

$$S^3 + 4s^2 + s + 16 = 0$$

$$a_0 = 1,\ a_1 = 4,\ a_2 = 1,\ a_3 = 16$$

$$\begin{array}{c|cc|} s^3 & 1 & 1 \\ s^2 & 4 & 16 \\ s^1 & \frac{4(1)-16(1)}{4} = -3 & 0 \\ s^0 & 16 & \end{array}$$

2 sign changes in first row: $4 \rightarrow -3$ , $-3 \rightarrow 16$

System is Unstable, no of poles in RHP = 2

## 6.4 SPECIAL CASES OF ROUTH HURWITZ CRITERION

1. **First element of a row is zero and rest of the row contains at least one non-zero element:**

In such cases, a small positive number $\varepsilon$ is substituted in place of zero, then the array is completed using $\varepsilon$. Then examine the sign change by taking $\lim_{\varepsilon \to 0}$.

2. **All the elements of a row in the array are zero:**

This is a slightly more trickier situation than the previous one. To eliminate this difficulty, an equation is formed by using the coefficients of the row just above the concerned row. Such an equation is called the auxiliary equation.

$$\begin{array}{c|cc|} s^3 & 1 & 5 \\ s^2 & 0 & 0 \\ s^1 & & \\ s^0 & & \end{array} \longleftarrow \text{Zero Row}$$

Auxiliary eqn:

$A(s) = s^3 + 5s$

Then the derivative of the Auxiliary equation with respect to s is taken and the coefficients of that equation are used to replace the row of zeros. Now complete the array in terms of these new terms.

$$\begin{array}{c|cc|} s^3 & 1 & 5 \\ s^2 & 0(3) & 0(5) \\ s^1 & \frac{10}{3} & 0 \\ s^0 & 5 & \end{array}$$

$A(s) = s^3 + 5s$

$A'(s) = 3s^2 + 5$

After completing the Routh array, all we need to do is to check for the sign change in the first row (below the replaced row). If there are any sign changes, that means that the system is unstable. If there is no sign change, the system may be stable. To confirm stability, you need to solve the Auxiliary equation and make sure that the roots doesn't lie on the imaginary axis.

# 6.5 AUXILIARY EQUATION

The auxiliary equation is actually a part of the characteristic equation, which means that the roots of the auxiliary equation are also the roots of the characteristic equation. The other interesting thing is that the roots of the auxiliary equation are the most dominant roots of the characteristic equation. This allows us to determine the stability of the system directly from the roots of the Auxiliary equation. The other roots of the characteristic are always in the left half plane and do not play any significant role in stability analysis. This is why we check for sign changes only below the Auxiliary equation.

**Properties of Auxiliary Equation**

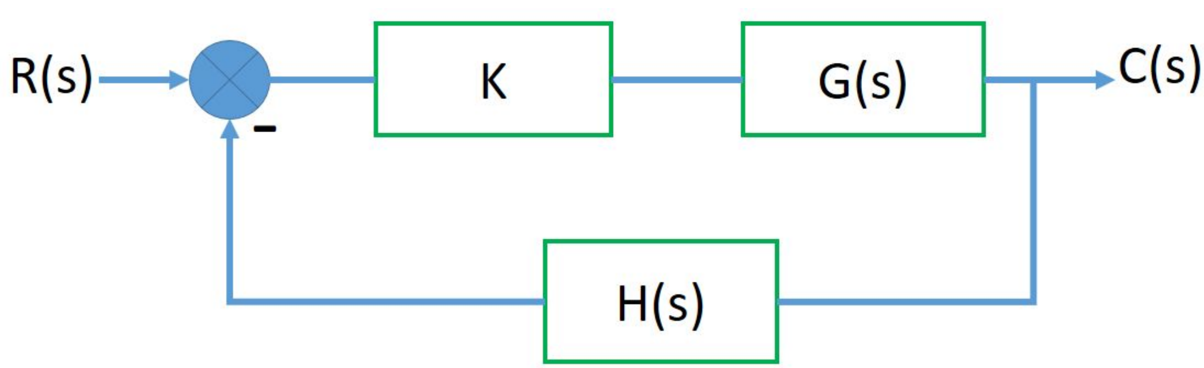
1. The roots of the Auxiliary equation are always symmetrical about the y-axis.
2. No. of roots on the imaginary axis = Order of the AE – 2 x (No. of sign changes below AE).

# 7. ROOT LOCUS

## 7.1 CONCEPT OF ROOT LOCUS

In the previous chapters, we have seen how the location of the poles influence the stability and transient characteristics of the system. Most times one or more parameters of the system are unknown and we are unsure what the optimum values for these parameters are. So it is advantageous to know how the closed loop poles move in the system if some parameters of the system are varied. The knowledge of such movement of the closed loop poles with small changes in the system parameters greatly helps in the design of control systems.

Although the Root locus can be plotted with respect to any parameter, generally the system gain K is chosen as the variable parameter. Consider the system shown below. In this system, the gain of the forward path is K G(s) and that of the feedback path is H(s).




The transfer function of this system is,

$$TF = \frac{K\,G(s)}{1+K\,G(s)\,H(s)}$$

And the corresponding characteristic equation is,

$$1 + K\,G(s)\,H(s) = 0$$

It is very clear that the roots of the above equation are dependent on the values of K. Now if the gain K is varied we will get separate set of roots for the characteristic equation. The root locus is nothing but a plot showing this variation of the roots as the system gain K is varied from 0 to $\infty$.

## 7.2 ANGLE AND MAGNITUDE CONDITION

Every point on the Root locus diagram satisfies two conditions called the Angle condition and the Magnitude condition respectively. Both the conditions can be easily obtained from the characteristic equation as follows:

$$1 + K\,G(s)\,H(s) = 0$$
$$K\,G(s)\,H(s) = -1 + j0$$

$$\boxed{|K\,G(s)\,H(s)| = 1} \longleftarrow \text{Magnitude Condition}$$

$$\boxed{\angle K\,G(s)\,H(s) = (2q + 1)\,180^0} \longleftarrow \text{Angle Condition}$$

Where q = 0,1,2,3…

## 8.3 CONSTRUCTION OF ROOT LOCUS

To assist in the construction of root locus plots, the "Root Locus Rules" for plotting the loci are summarized here. These rules are based upon the interpretation of the angle condition and the analysis of the characteristic equation.

1. **Rule 1 Symmetry**

As all roots are either real or complex conjugate pairs so that the root locus is symmetrical to the real axis.

2. **Rule 2 Number of branches**

The number of branches of the root locus is equal to the number of poles P of the open-loop transfer function.

3. **Rule 3 Locus start and end points**

The root locus starts at finite and infinite open loop poles and it ends at finite and infinite open loop zeroes.

4. **Rule 4 Real Axis Segments of Root Locus**

A point on the real axis lies on the Root locus, if the sum of number of poles and zeros, on the real axis, to the right hand side of this point is odd.

5. **Rule 5 Asymptotes**

No. of branches of root locus approaching / terminating at infinity = no. of asymptotes = (P – Z),
where P = no. of finite open loop poles; Z = no of finite open loop zeroes.

The angle with which asymptotes approaches to infinity are called as angle of asymptotes (θ).

$$\text{Angle of Asymptotes}(\theta) = \left(\frac{(2q+1)180}{No.of\ finite\ poles - No.of\ finite\ zeroes}\right)$$

Where, q = 0, 1, 2, ..... (P –Z – 1)

6. **Rule 6 Centroid**

The meeting point of asymptotes is called as the centroid.

$$\text{Centroid } (\sigma) = \left(\frac{\sum finite\ poles - \sum finite\ zeroes}{No.of\ finite\ poles - No.of\ finite\ zeroes}\right)$$

7. **Rule 7 Break-in and Break-away points**

Break points exists if a branch of the root locus is on the real axis between two poles or zeros. Since the root locus can only start at a pole and end at a zero, the plot breaks from the real axis to intersect with another pole or zero. If the break point lies between two successive poles, then it is called as the Breakaway Point and if the break point lies between two successive zeros, then it is called as the Breakaway Point.

To find these points, write k in terms of ‘s’ using the Characteristic equation. Then find the derivative of k with respect to s. The roots of equation $\frac{dk}{ds}$ = 0, give the locations of break points. To differentiate between break in & break away points, find $\frac{d^2k}{ds^2}$ and substitute the values of k obtained. If $\frac{d^2k}{ds^2}$ >0, then it is a break in point and if $\frac{d^2k}{ds^2}$ <0, then it is a break-away point.

8. **Rule 8 Intersection with the imaginary axis**

The imaginary axis crossing is a point on the root locus that separates the stable operation of the system from the unstable operation. The value of ω at the axis crossing yields the frequency of oscillation. To find imaginary axis crossing, we can use the Routh-Hurwitz criterion and obtain the value of k by forcing a row of zeros in the Routh array. We can then use this value of k in the Auxiliary equation to find 's' corresponding to the crossing point on the imaginary axis.

# 8.4 EXAMPLE

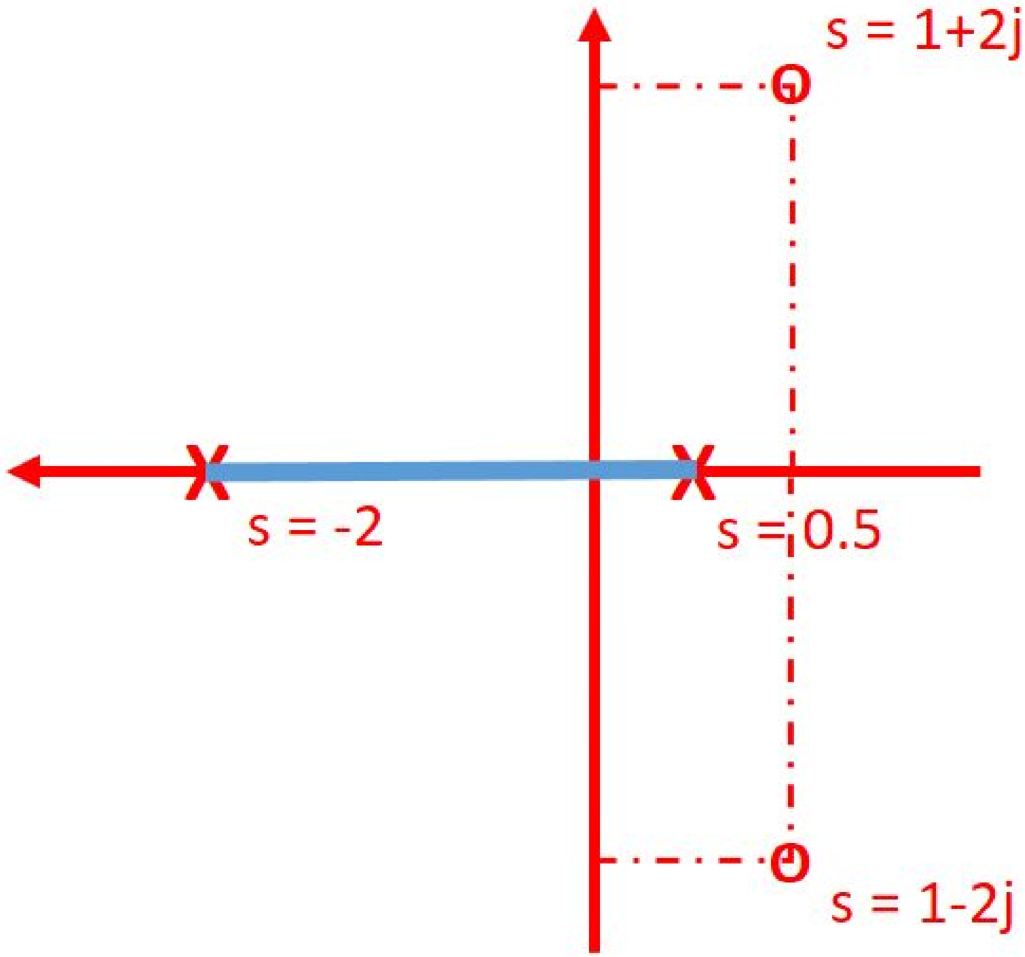
$$G(s) = \frac{k\ (s2 - 2s + 5)}{(s + 2)(s - 0.5)}$$

No. of poles = P = 2

No. of zeros = Z = 2

**Location of Poles and Zeros:**

Poles are at s = -2, s = 0.5 and Zeros are at s = 1+2j, s = 1-2j.




**Asymptotes:**

Since there are equal no. of poles and zeros i.e. P-Z =0, there are no asymptotes.

**Breakaway point:**

Characteristic equation of the system is,

$$(s+2)(s-0.5) + K(s^2-2s+5) = 0$$

$$\therefore K = \frac{-(s^2+0.5s-1)}{(s^2-2s+5)}$$

$$\frac{dK}{ds} = 0 => s = -0.4, 3.6$$

Here, since Breakaway point must lie between two consecutive poles so s = -0.4 is a valid Breakaway Point whereas s = 3.6 is an invalid point.

**Intersection with Imaginary axis:**

Routh array for the system is,

$$\begin{array}{c|cc|} s^2 & 1+k & 5k-1 \\ s^1 & 1.5-2k & 0 \\ s^0 & 5k-1 & 0 \end{array}$$

For stability $1+k > 0 \Rightarrow k > -1$

& $1.5-2k > 0 \Rightarrow k < 0.75$

& $5k-1 > 0 \Rightarrow k > 0.2$

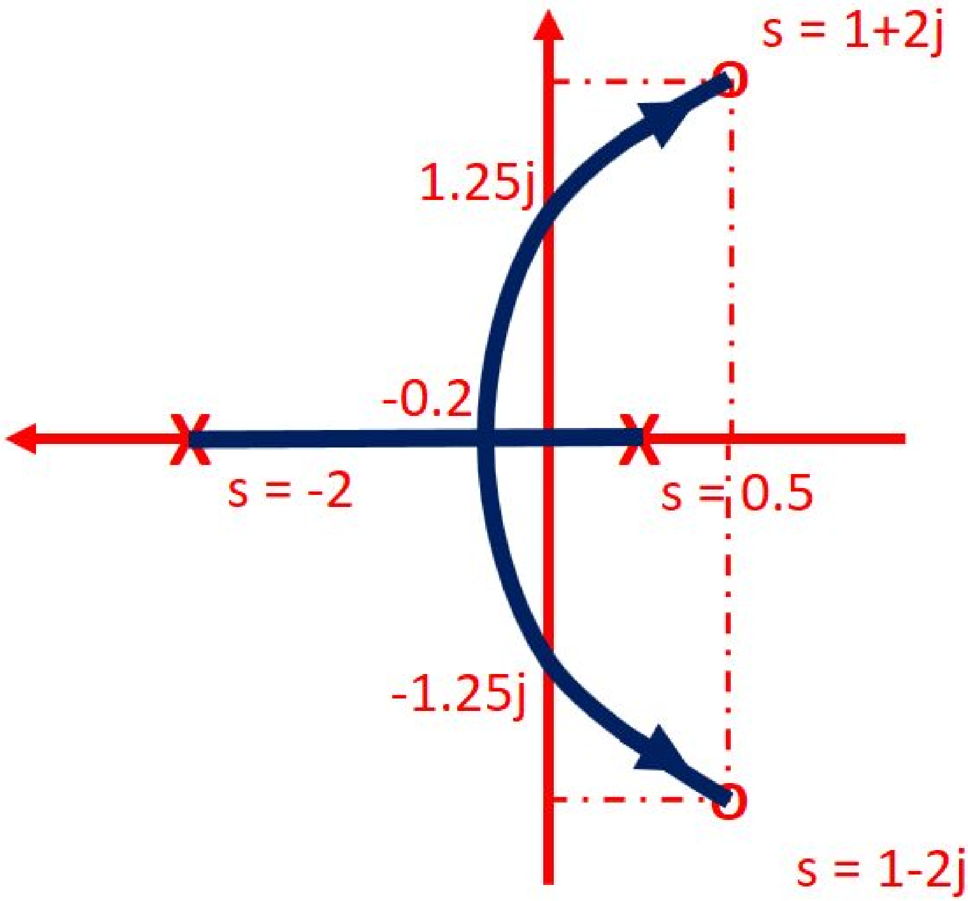
$\therefore$ range $0.2 < k < 0.75$

$\Rightarrow k = k_{max} = 0.75$

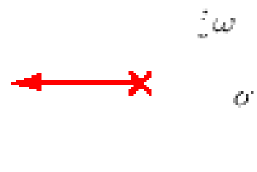
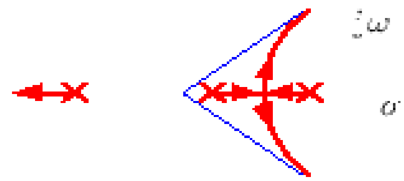
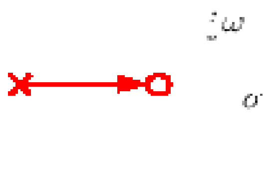
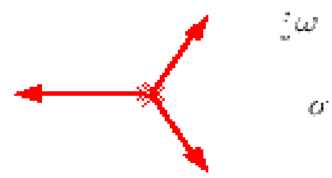
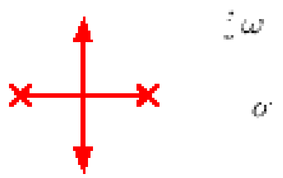
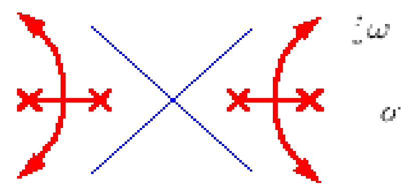
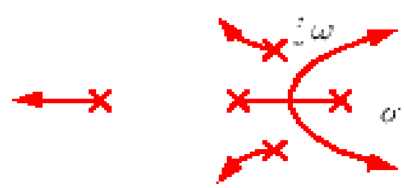
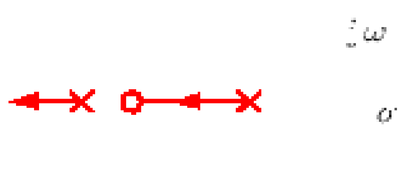
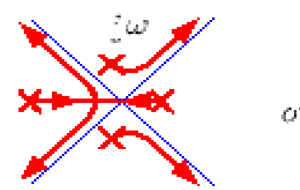
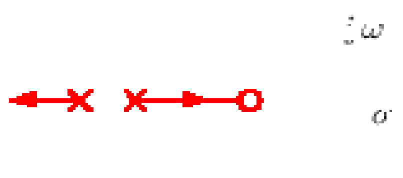
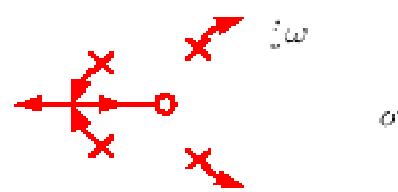
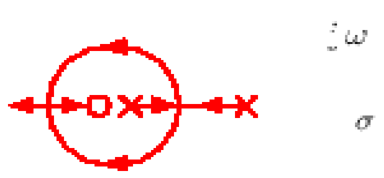
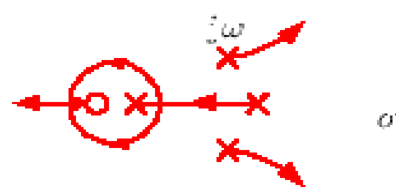
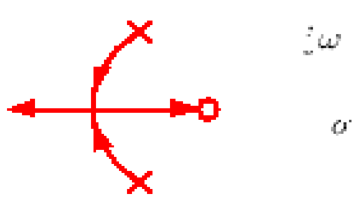
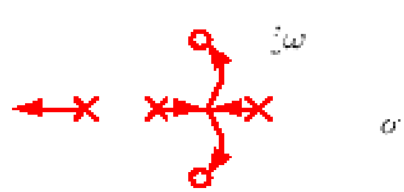
Auxiliary Equation: $(1+k)s^2 + (5k-1) = 1.75s^2 + 2.75 = 0$

$$s = \pm j1.25$$

Therefore the Root Locus intersects the Imaginary axis at s = $\pm$ j1.25.




# 8.5 TYPICAL ROOT LOCUS DIAGRAMS

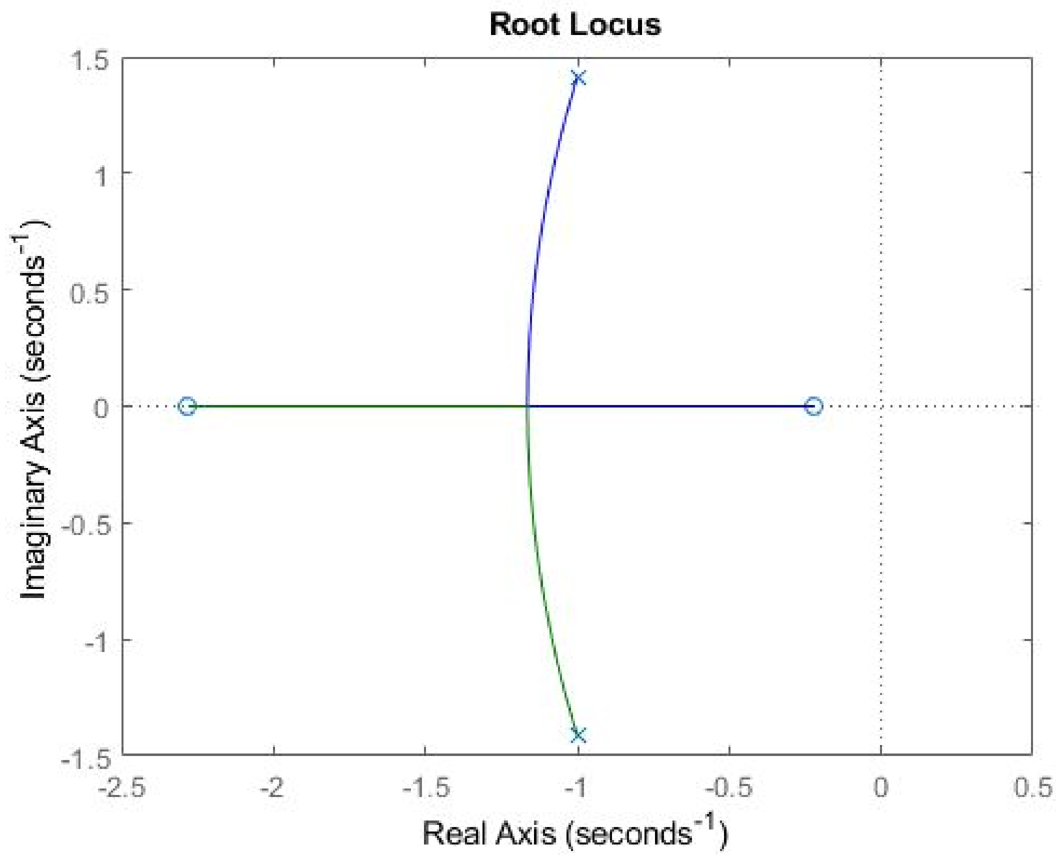
| No. | root locus | No. | root locus |
|---|---|---|---|
| 1 |   | 9 |   |
| 2 |   | 10 |   |
| 3 |   | 11 |   |
| 4 |   | 12 |   |
| 5 |   | 13 |   |
| 6 |   | 14 |   |
| 7 |   | 15 |   |
| 8 |   | 16 |   |

# 8.6 ROOT LOCUS IN MATLAB

Example:

$$h(s) = \frac{2s^2 + 5s + 1}{s^2 + 2s + 3}.$$

```
h = tf([2 5 1],[1 2 3]);
rlocus(h)
```




Matlab Documentation for Root Locus (Link)

# 8. FREQUENCY DOMAIN ANALYSIS

## 8.1 FREQUENCY RESPONSE

Earlier, we have discussed that there are several standard test signals used to study the performance of control systems. Out of these, the sinusoidal signal is perhaps the most important and the most useful. There's a good reason for that, and it's because of an interesting property of an LTI system, called sinusoidal fidelity. To explain sinusoidal fidelity, let's look at the mathematical operations associated with LTI systems. An LTI system can only be build using these or a combination of these operators:

Multiplication by a constant: $f(t) \times a$

Differentiation of the input signal: $\frac{d\ f(t)}{dt}$

Integration of the input signal: $\int f(t)\ dt$

Addition of two input signals: $f_1(t) + f_2(t)$

Why are these operations so special?? Let's try to find out, by providing sinusoidal inputs to these systems.

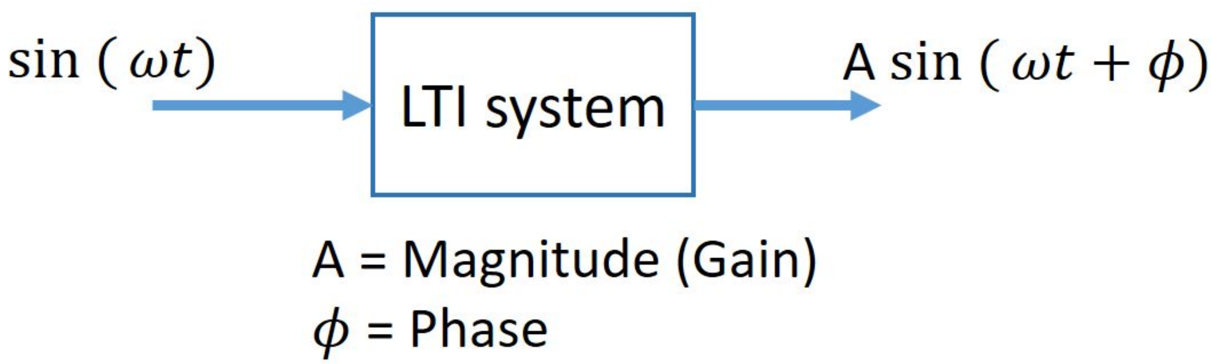
Let our sample sine input be: **A sin t** and the output of these operations will be,

Multiplication by a constant: **A sin t x a = Aa sint** …What do you know, the output is a sine wave. Just that the magnitude has just changed to **Aa**.

Differentiation of the input signal: $\frac{d\,(A \sin t)}{dt}$ **= A cos t = A sin(t +90)**

You gotta be kidding me! Again a sine wave output. This time the Magnitude doesn't change, but there is a phase shift. You know we are going with this. That’s right, the other two operations also produce sine wave outputs.

We have observed something very interesting here, these outputs are all sinusoids, just that they have different Magnitude or phase or both. So the shape of the input and the output waveforms are the same (both sinusoids) for LTI systems. The only things that change are the Amplitude and Phase of the signal. This property is called Sinusoidal fidelity. In fact, this property is used to determine whether a system is LTI or not, practically.




A critical inference can be made from this property, that both the input and the output signals will have the same frequency components i.e. No frequency components can be manufactured other than the one's already present in the input signal (ideally).

This means that any linear system can be completely described by how it changes the amplitude and phase of cosine waves passing through it. This information is called the system's Frequency response. Just like our time domain transfer function, we can define a transfer function in frequency domain. The sinusoidal transfer function $G(j\omega)$ is a complex quantity and can be represented as magnitude and phase angle with frequency $\omega$ variable parameter. Such frequency domain transfer function can be obtained by substituting j $\omega$ for ‘s’ in our time domain transfer function G(s) of the system. When we are referring to frequency response, we are considering the steady state frequency response, so the term $\sigma$ in s becomes 0, making s = j $\omega$.

$$G(j\omega) = G(s)\,\big|_{s\,=\,j\omega}$$

Magnitude of output = Magnitude of input x $|G(j\omega)|$

Phase difference $\Phi = \angle G(j\omega)$

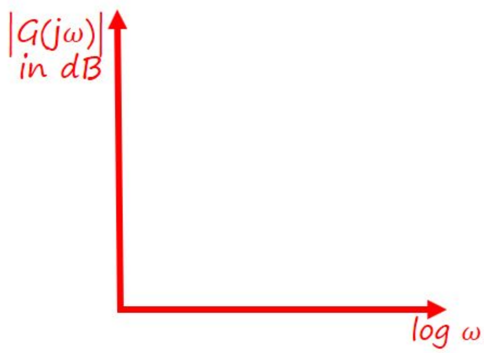
So to get the frequency response means to sketch the variation in magnitude and phase angle of $G(j\omega)$, when $\omega$ is varied from 0 to $\infty$. Many methods like Bode plot, Nyquist plot, Polar plot and M- $\Phi$ plot are used to sketch the frequency response of control systems. In the coming section, we’ll look into bode plots in detail.

## 8.2 BODE PLOT

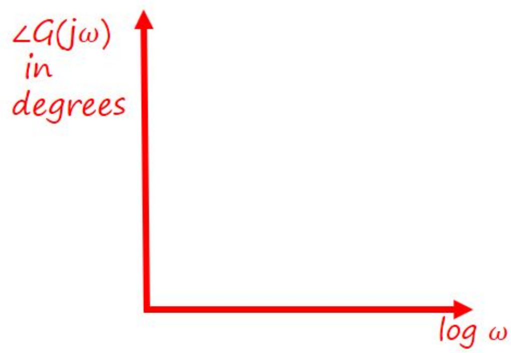
The basis of any frequency response plot is to plot the magnitude M and the phase $\Phi$ against the input frequency $\omega$. When $\omega$ is varied from 0 to $\infty$, there is a wide range of variations in M and $\Phi$ and hence it becomes difficult to accommodate all such variations with linear scale. So this guy, H.W. Bode suggested that it’s better to use the logarithmic values of Magnitude for plotting against the logarithmic values of $\omega$ and came up with the idea of Bode plots.

A Bode plot is essentially 2 separate plots:

1. Magnitude expressed in logarithmic scale against logarithmic values of frequency $\omega$. This is called the Magnitude plot.
2. Phase angle in degrees against logarithmic values of frequency $\omega$. This is called the Phase plot.




Magnitude Plot

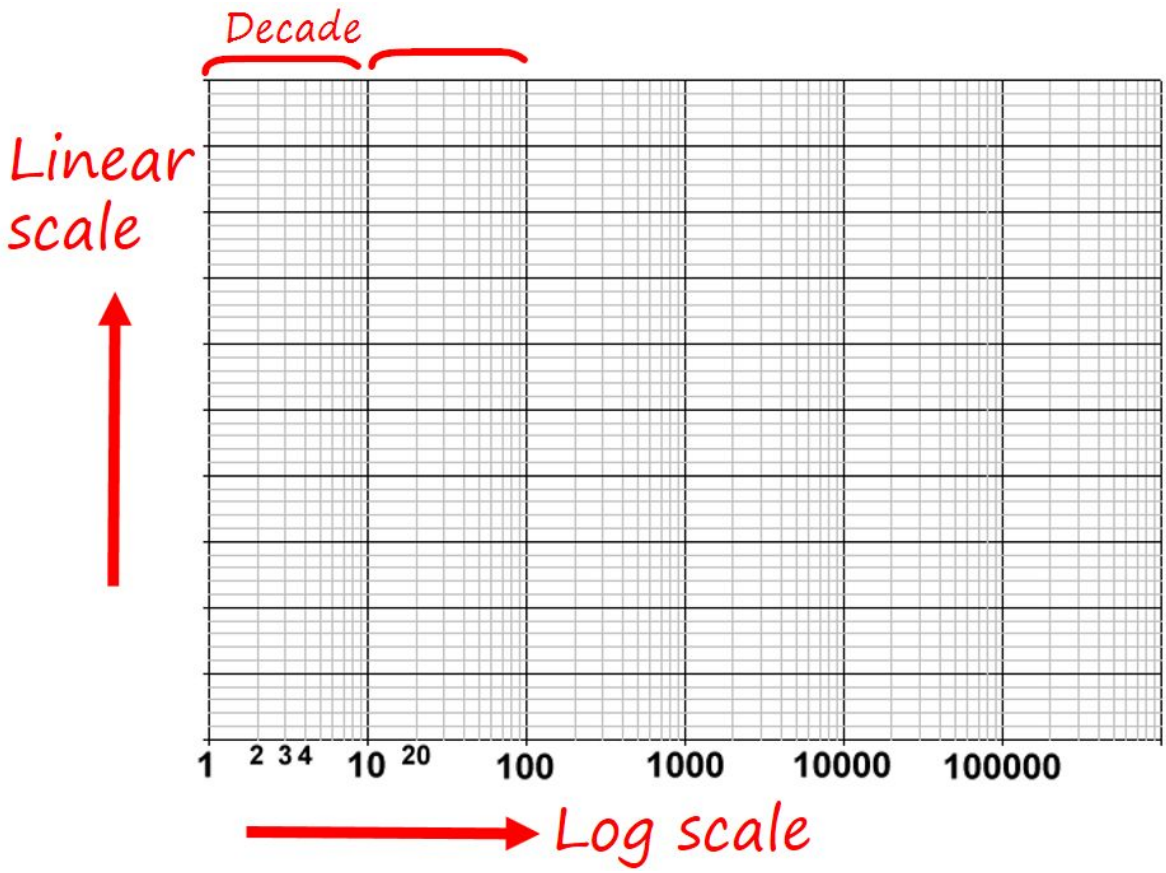



Phase Plot

# 8.3 SKETCHING THE BODE PLOT BY HAND

Sketching the bode plot by hand?? Yes, I'll give you this one, it is as crazy as it sounds. With the availability of software's like Matlab, nobody these day's bother to sketch bode plots by hand. The reason we included this section in this book is because, sketching a bode plot by hand (at least a few times) will give you a better insight into Frequency response of systems and help gain an intuitive understanding of the topic.

First things first you will need a semi-log graph paper. In such a paper, the X axis is divided in logarithmic scale (non-linear scale) and the Y axis in linear scale like a normal graph sheet. The interesting part about the X axis is that the distance between 1 and 2 is greater than the distance between 2 and 3 and so on. The distance between 1 and 10 (or 10 and 100 etc.) is called a decade. Note that the first set of numbers goes as 1,2,3... the second as 10, 20, 30... and the next as 100, 200, 300...The best part about using a semi-log graph sheet is that a wide range of frequencies can be accommodated on a single sheet.




If you are still confused about semi-log graphs, it may be good idea to download a semi log template (link) and write in your own numbers as practice.

Consider an Open loop transfer function,

$$G(s) = \frac{K(1+sT_1)}{s(1+sT_2)(1+sT_3)}$$

You need to express the TF in this form

$$G(j\omega) = \frac{K(1+j\omega T_1)}{j\omega(1+j\omega T_2)(1+j\omega T_3)}$$

Replace s by jω

Complex no.'s in polar form

$$= \frac{K\angle 0 \sqrt{1+j\omega T_1}\angle \tan^{-1} \omega T_1}{\omega\angle 90\sqrt{1+j\omega T_2}\angle \tan^{-1} \omega T_2 \sqrt{1+j\omega T_3}\angle \tan^{-1} \omega T_3}$$

Therefore the Magnitude and Phase angle of G(j $\omega$ ) is given by,

$$\text{Magnitude of } G(j\omega) = \frac{K\sqrt{1+\omega^2 {T_1}^2}}{\omega\sqrt{1+\omega^2 {T_2}^2}\sqrt{1+\omega^2 {T_3}^2}}$$

$$\text{Phase angle of } G(j\omega) = \tan^{-1} \omega T_1 - 90^0 - \tan^{-1} \omega T_2 - \tan^{-1} \omega T_3$$

Because Magnitude is expressed in decibels (**|G(j $\omega$ )| in dB = 20 log |G(j $\omega$ )|** ), and for that we are taking log of individual magnitude terms, multiplication is converted to addition.

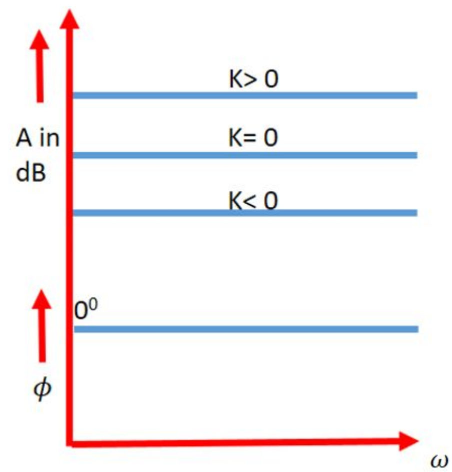
$$G(j\omega)| \text{ in dB} = 20\log\frac{K}{\omega} + 20\log\sqrt{1+\omega^2 {T_1}^2} - 20\log\sqrt{1+\omega^2 {T_2}^2} - 20\log\sqrt{1+\omega^2 {T_3}^2}$$

Hence in the Magnitude plot, the dB magnitudes of individual factors of G(j $\omega$) can be added. Therefore, to sketch the magnitude plot, the knowledge of magnitude variations of individual factors is essential.

Basic factors that are frequently found in G(j $\omega$) are:

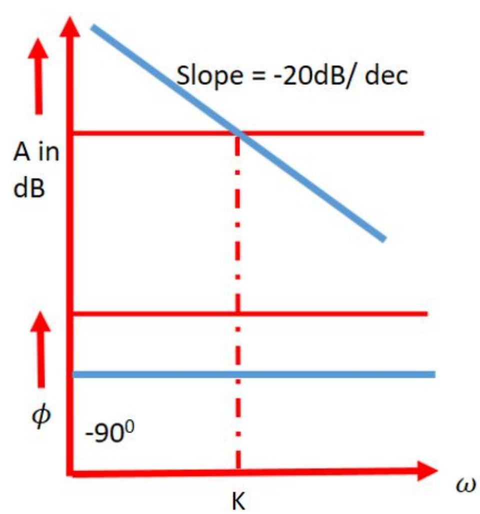
**1. Constant Gain K:**

$$A = 20 \log K$$
$$\phi = 0^0$$



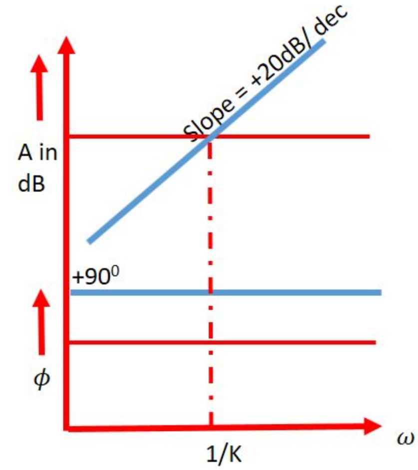

**2. Integral factor (K/s)**

$$A = 20 \log K/\omega$$
$$\phi = -90^0$$




**3. Derivative Factor (Ks):**

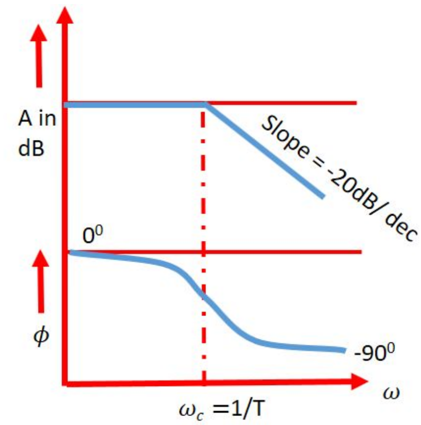
$$A = 20 \log K\omega$$
$$\phi = +90^0$$




**4. First order factor in denominator (1/1+sT):**

$$A = -20 \log \sqrt{1 + \omega^2 T^2}$$
$$\phi = -\tan^{-1} \omega T$$

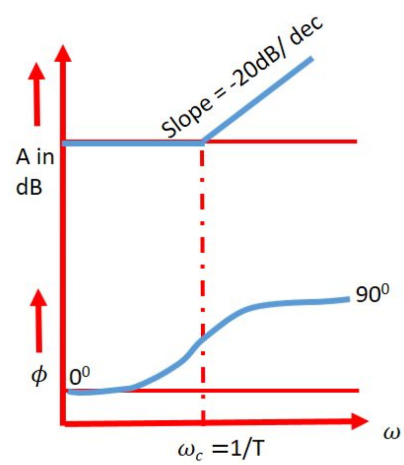
$+90^0$




**5. First order factor in denominator (1+sT):**

$$A = 20 \log \sqrt{1 + \omega^2 T^2}$$
$$\phi = \tan^{-1} \omega T$$

$+90^0$



## 8.3 EXAMPLE

$$G(s) = \frac{Ks^2}{(1+0.2s)(1+0.02s)}$$

- **Replace s with j $\omega$ to obtain sinusoidal transfer function.**

$$G(j\omega) = \frac{K(j\omega^2)}{(1+0.2j\omega)(1+0.02j\omega)}$$

- **Find the corner frequencies and the slopes contributed by each term.**

| Term | Corner Frequency | Slope | Change in slope |
|---|---|---|---|
| $(j\omega)^2$ | - | +40 | |
| $\frac{1}{(1+j0.2\omega)}$ | $\omega_{c1} = \frac{1}{0.2} = 5$ | -20 | +40- 20= 20 |
| $\frac{1}{(1+j0.02\omega)}$ | $\omega_{c1} = \frac{1}{0.02} = 50$ | -20 | +20- 20= 0 |

- **Choose a frequency $\omega_l$ lower than $\omega_{c1}$ (Say $\omega_l$ = 0.5) and $\omega_h$ higher than $\omega_{c2}$ (Say $\omega_h$ = 100) and calculate the gains at these frequencies.**

At $\omega = \omega_l$,

A =20 log $|(j\,\omega)^2|$ = 20 log $(\omega)^2$ = 20 log $(0.5)^2$ = -12 dB

At $\omega = \omega_{C1}$,

A =20 log $|(j\,\omega)^2|$ = 20 log $(\omega)^2$ = 20 log $(5)^2$ = 28 dB

At $\omega = \omega_{C2}$,

$$A = [\text{slope from } \omega_{C1} \text{ to } \omega_{C2} \text{ x } \log \frac{\omega_{C2}}{\omega_{C1}}]$$

$$+ A(\text{at } \omega = \omega_{C1}) = [20 \text{ x } \log(\frac{50}{5}) + 28] = 48\text{dB}$$

At $\omega = \omega_h$,

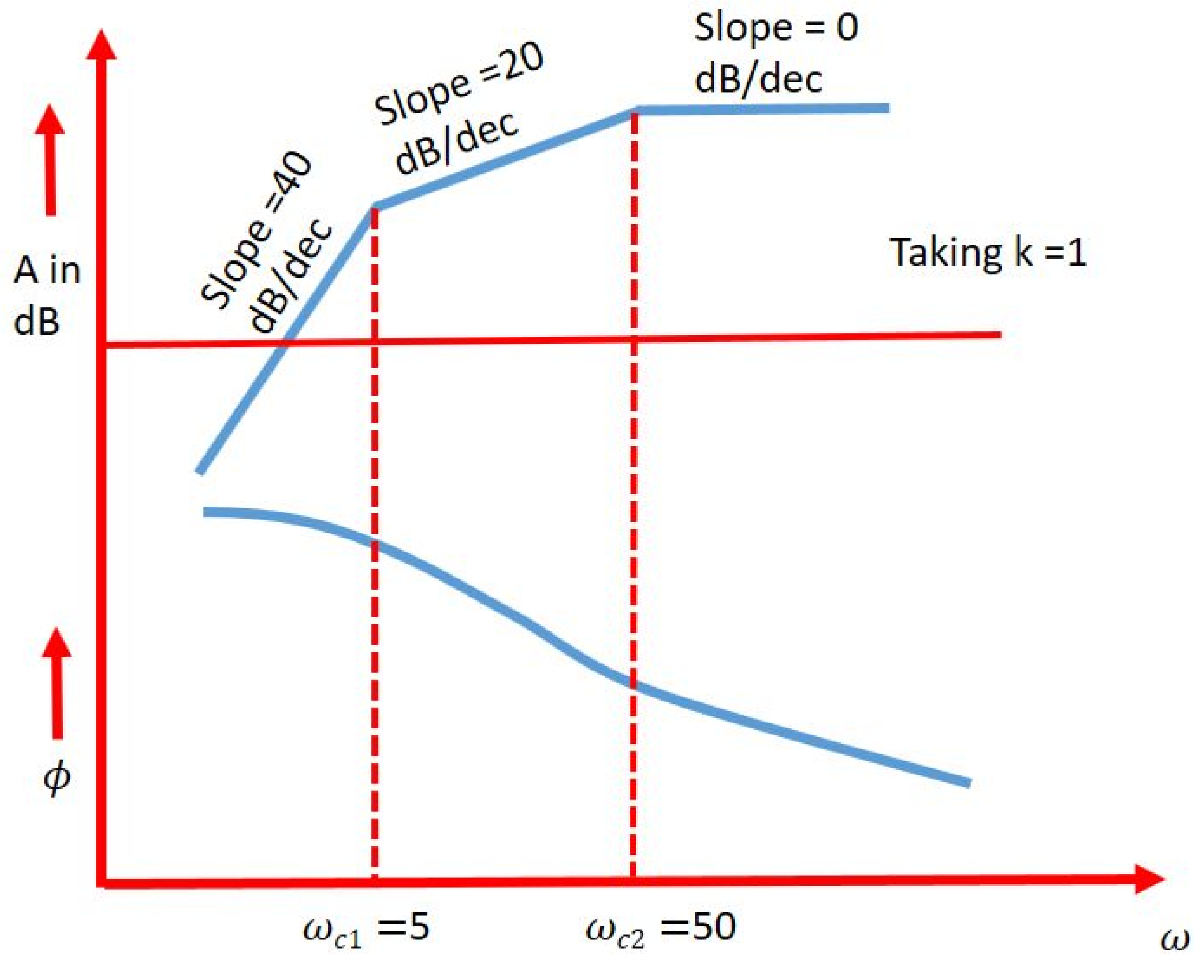
$$A = [\text{slope from } \omega_{C2} \text{ to } \omega_h \text{ x } \log \frac{\omega_h}{\omega_{C2}}]$$

$$+ A(\text{at } \omega = \omega_{C2}) = [0 \text{ x } \log(\frac{100}{50}) + 48] = 48\text{dB}$$

- **Plotting the Phase plot is straightforward.**

$$\phi = \angle G(j\omega) = 180 - \tan^{-1} 0.2\omega - \tan^{-1} 0.02\omega$$

| $\omega$ | $\angle G(j\omega)$ |
|---|---|
| 0.5 | 174 |
| 1 | 168 |
| 5 | 130 |
| 10 | 106 |
| 50 | 50 |
| 100 | 30 |




More Bode Plot Examples: Link

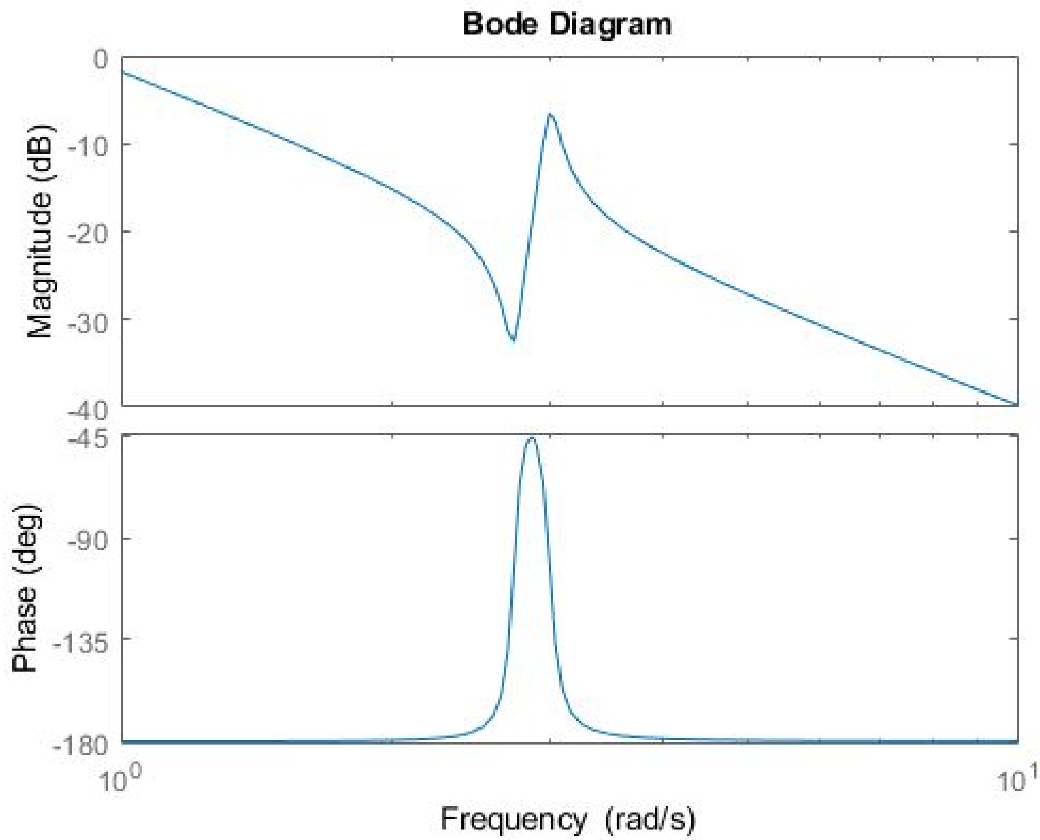
# 8.4 BODE PLOT IN MATLAB

You can plot a very accurate Bode plot using just few lines of code in Matlab.

Example:

$$H(s) = \frac{s^2 + 0.1s + 7.5}{s^4 + 0.12s^3 + 9s^2}.$$

```
H = tf([1 0.1 7.5],[1 0.12 9 0 0]);
bode(H)
```

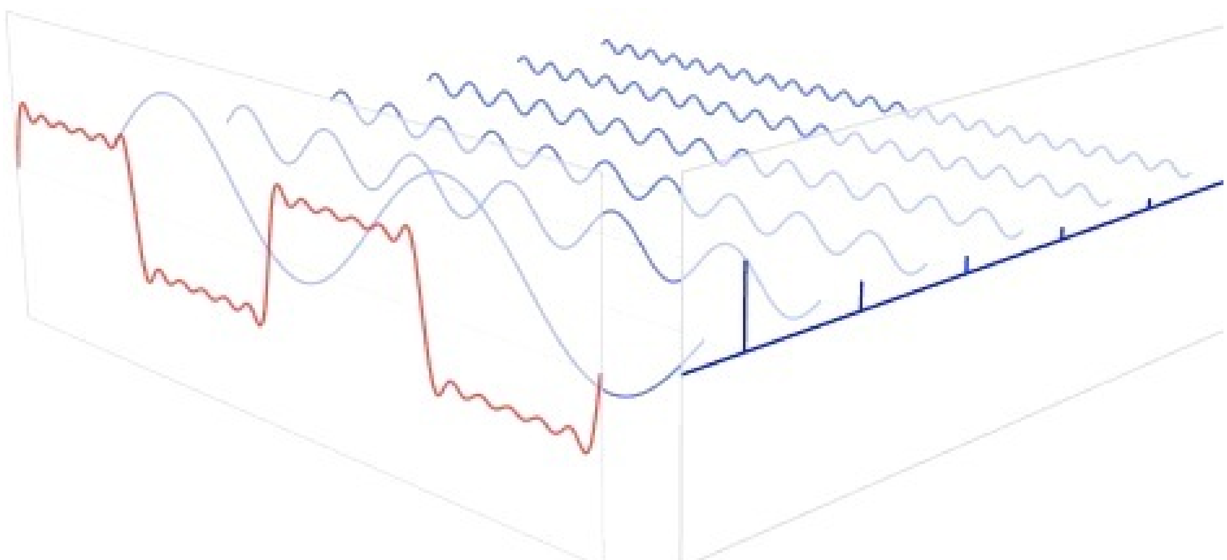



Matlab documentation for Bode Plots ([Link]())

# APPENDIX

## 1. FOURIER SERIES

Fourier series is a basic mathematical tool for representing periodic signals. Using Fourier series, it is possible to express periodic signals in terms of carefully chosen sinusoids. So every periodic signal in this world can be expressed used some combination of sinusoids. Isn't this cool??



In the above figure, notice how a series of sinusoids (sine and cosine waves) combine to form the resultant signal, which looks nothing like a sinusoid. (Note that the different components have different amplitudes and different frequencies)

The General expression for Fourier series is:

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

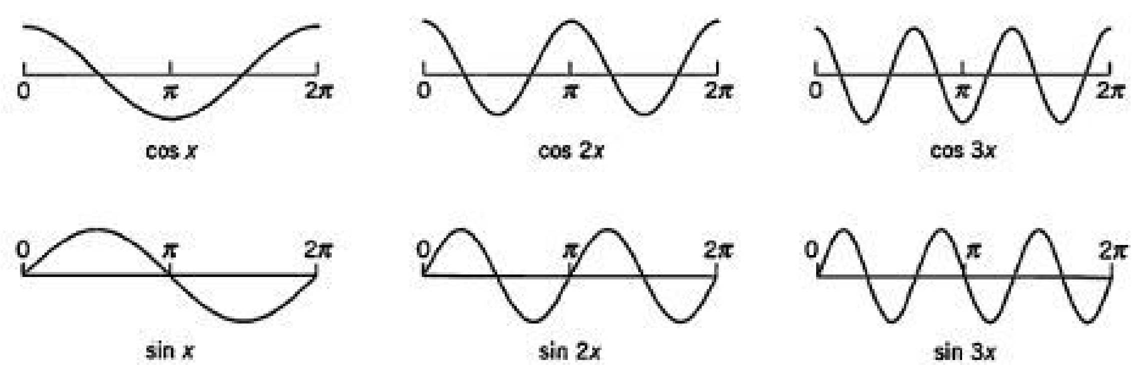
here **a0,a1,a2...,b1,b2,b3....**are the Fourier coefficients. They tell us how much a sine or cosine wave of a particular frequency is contributing to the resultant signal.

The value of **a0** tells us how much a cosine of zero frequency (**cos 0 =1**, so basically DC) is present in the final wave. **a0** is also called the DC value or the Average value or the DC offset. Since all the other terms in the expansion are pure sinusoids, their individually average to zero, so the average value solely depends on **a0**.

Since **sin 0** = **0**, there can't be any contribution from zero frequency sine wave, so **b0** is always **0**.

The value of **a1** tells us how much a cosine of fundamental frequency is present in the final wave.

Similarly, contribution from each sinusoid in the main signal can be found out separately. This information is very useful and it can be used to manipulate signals in a lot of ways.




Fourier series can be expressed in a more compact form using complex notation. Using the complex notation, we can represent the contributions from both sine and cosine waves of the same frequency by a single coefficient.

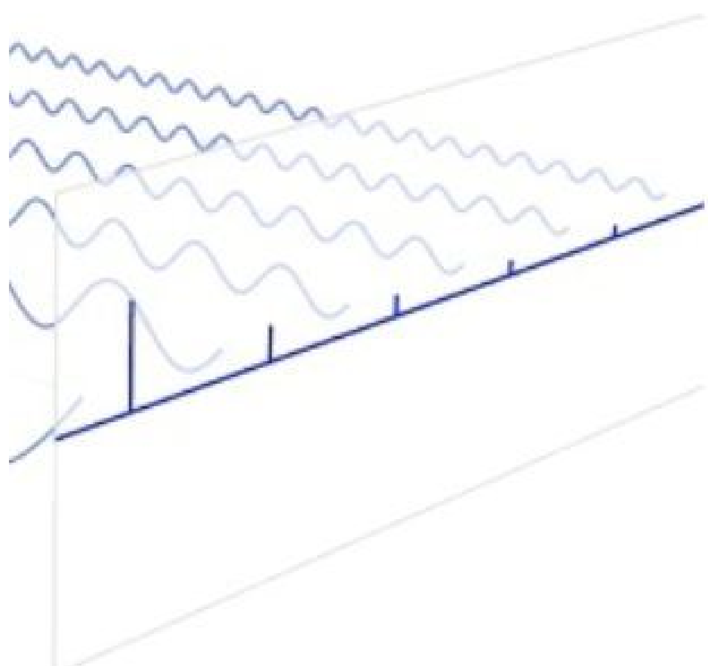
$$x(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega t}$$

This is called the synthesis equation. Here the Fourier coefficients are complex. This notation has its advantages, it is possible to calculate all Fourier coefficients using a single expression. Electrical engineers use **j** instead of **i**, since **i** is frequently used to denote electric current.

The values of $\mathbf{c_n}$ can be obtained using the expression:

$$c_n = 1/T \int_0^T x(t)\, e^{-jn\omega t}\, dt$$

This expression is called the analysis equation and the plot of $|\mathbf{c_n}|$ vs $\mathbf{n}$ is called the frequency spectrum of the signal.



Notice the lines corresponding to each frequency component in the above picture. This is exactly the Frequency spectrum. It tells us how much each frequency component contributes to the original signal. This information is invaluable to us.

# 2. FOURIER TRANSFORM

We have now seen how the Fourier series is used to represent a periodic function by a discrete sum of complex exponentials. But how often are natural signals periodic?? Now that's a problem. Too bad we can't apply Fourier series to non periodic signals.

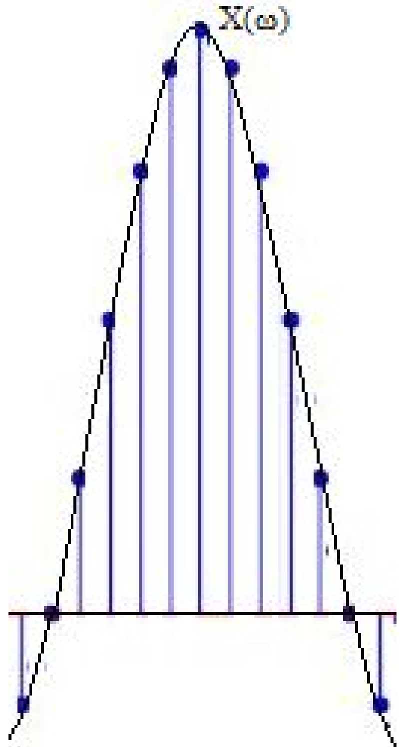
Why don't we assume an Aperiodic signal to be a periodic signal with infinite time period. Why don't we assume that the same pattern exists after infinite time. This is where we introduce the Fourier transform.

The Fourier transform is used to represent a general, non periodic function by a continuous superposition or integral of complex exponentials. The Fourier transform can be viewed as the limit of the Fourier series of a function when the period approaches to infinity, so the limits of integration change from one period to $(-\infty,\infty)$.

The expression for the Fourier Transform is given by:

$$X(\omega) = \int_{t=-\infty}^{\infty} x(t)\, e^{-j\omega t}$$

**X(ω)** is a continuous function of **ω**. The Fourier series coefficients are basically the sampled values of **X(ω)** or in other words, **X(ω)** forms the envelope for the Fourier series coefficients.


X(ω)

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Youtube Channel by Brian Douglas
https://www.youtube.com/channel/UCq0imsn84ShAe9PBOFnoIrg?ab_channel=BrianDouglas

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