EULER'S AND RUNGE-KUTTA METHODS FOR SOLVING SECOND ORDER DIFFERENTIAL EQUATIONS

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DECLARATION

I **KAREEM, SAMSON ADEBAYO** hereby declare that the final year project titled: "**Euler's and Runge-Kutta Methods for Solving Second Order Differential Equations**" submitted by me, in partial fulfillment of the requirements for the award of Bachelor of Science Degree in Mathematics, Federal University of Agriculture, Abeokuta. I declare that all external sources and references used in this project have been properly acknowledged and cited.

KAREEM, SAMSON ADEBAYO

DATE:................................

CERTIFICATION

This is to certify that the work titled: "**Euler's and Runge-Kutta Methods for Solving Second Order Differential Equations**" was carried out by **KAREEM, SAMSON ADEBAYO** with **Matriculation Number: 20183037,** a student of the Department of Mathematics, College of Physical Sciences, Federal University of Agriculture, Abeokuta, Ogun State, Nigeria.

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DEDICATION

This project is dedicated to the ALMIGHTY GOD, my source and anchor. I dedicate this work to my beloved parents, Mr and Mrs Kareem, as well as my ever-supportive siblings and to everyone that has been supportive and helpful in my education life. I dedicate this also to my support system in my department. Thank you for making this journey a beautiful one.

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ABSTRACT

Second-order differential equations are fundamental in modeling various natural instances and engineering systems. This project explores the application of numerical methods as tools for solving second-order differential equations, emphasizing their crucial role when analytical solutions are impractical. This research provides a comprehensive overview of second-order differential equations, and highlights the limitations of analytical techniques. Numerical methods, such as Euler`s and Runge-Kutta methods, are introduced and analyzed in-depth.

Table of Contents

Contents **Page**

Contents		Page
Title page		i
Declaration		ii
Certification		iii
Dedication		iv
Acknowledgment		v
Abstract		vi
Table of Contents		vii
CHAPTER ONE: Introduction		1
1.1	Background of the study	1
1.2	Motivation	2
1.3	Objectives	3
1.4	Preliminaries and Definitions of Terms	3
CHAPTER TWO: Literature Review		5
CHAPTER THREE: Methodology		7
3.1	Euler's Method	7

3.2 Runge-kutta Method 9

CHAPTER FOUR: Applications 11

4.1 Illustrative Examples 11

4.1.1 Example on Euler's Method 11

4.1.2 Example on Runge-kutta Method 17

CHAPTER FIVE: Conclusion and Recommendations 23

5.1 Conclusion 23

5.2 Recommendations 24

REFERENCES 25

CHAPTER ONE

1.0 INTRODUCTION

1.1 Background to the Study

Differential equations are mathematical tools that describe the fundamental relationships between quantities that vary continuously with respect to one or more independent variables. Among the diverse family of differential equations, second-order differential equations hold a special place due to their applications in various scientific and engineering disciplines. These equations, which involve the second derivative of an unknown function, arise in fields ranging from classical mechanics and electrical circuit analysis to fluid dynamics and quantum physics.

The solutions to second-order differential equations play a crucial role in understanding and predicting the behavior of complex physical systems. However, not all of these equations can be solved analytically using closed-form expressions. In fact, many real-world problems yield differential equations that defy analytical solution techniques. It is in the face of these challenges that numerical methods come to the forefront.

This project embarks on an exploration of numerical methods for solving second-order differential equations. Through this endeavor, we aim to equip the reader with the knowledge and skills necessary to tackle a wide array of problems that lean on the numerical solution of second-order differential equations.

In the pages that follow, we will journey through various numerical approaches, from the Euler's methods to Runge-Kutta Method. We will consider their applicability in different contexts, highlight their strengths and limitations, and provide illustrative examples to demonstrate their efficacy.

1.2 Motivation

Understanding the motivation behind the study of numerical methods for solving second-order differential equations is crucial to appreciating their significance and relevance across various scientific and engineering disciplines. At the core of our motivation lies the reality that many real-world problems are described by second-order differential equations. These equations capture the interplay of forces and variables that govern physical systems. Consider the structural engineer tasked with ensuring the safety of a newly designed bridge under diverse loading conditions, or the climate scientist striving to model the complex dynamics of atmospheric processes. In both cases, second-order differential equations arise as the mathematical framework for describing these phenomena. However, the majority of such equations resist analytical solutions, prompting the need for numerical techniques as our primary tools for exploration.

1.3 Objectives

- ✓ **To Introduce Numerical Methods**: Introduce students or readers to numerical methods as a means to solve complex second-order differential equations that may not have analytical solutions.
- ✓ T**o Study Fundamental Numerical Techniques**: Provide an in-depth exploration of fundamental numerical techniques like Euler`s method and Runge-Kutta method.

1.4 Definition of Terms

- ❖ **Differential Equation**: A differential equation is a mathematical equation that involves derivatives of an unknown function. In the context of this project, we are primarily interested in second-order differential equations, which involve the second derivative of a function.
- ❖ **Second-Order Differential Equation**: A second-order differential equation is a specific type of differential equation where the highest-order derivative involved is the second derivative. It often takes the form:

$$a(x)y'' + b(x)y\ ' + c(x)y = f(x).$$

where $a(x), b(x), c(x)$, and $f(x)$ are functions of the independent variable x, and y is the unknown function to be solved for.

- **Analytical Solution**: An analytical solution to a differential equation is a closed-form expression that explicitly describes the solution. It is found through symbolic manipulations and integration techniques.
- **Numerical Method**: A numerical method is an algorithm used to approximate solutions to mathematical problems, including differential equations, when analytical solutions are not readily available or practical.
- **Initial Value Problem** (IVP): An initial value problem is a type of problem for ordinary differential equations (ODEs) where the solution is determined based on an initial condition. In the context of second-order differential equations, this typically means specifying the values of both the unknown function and its first derivative at a single point.
- **Boundary Value Problem** (BVP): A boundary value problem is a type of problem for differential equations where the solution is determined by specifying conditions at multiple points along the domain boundaries. In the context of second-order differential equations, this involves specifying conditions at both the initial and final points of the domain.
- **Convergence**: Convergence refers to the property of a numerical method where the approximated solution approaches the true solution as the computational resources (e.g., grid points, time steps) increase. Convergence analysis assesses how well a numerical method approximates the exact solution.

CHAPTER TWO

2.0 LITERATURE REVIEW

Second-order differential equations play a pivotal role in scientific and engineering domains, often modeling crucial dynamics in physical systems. The quest for closed-form solutions to these equations has led researchers to employ numerical methods. This review consolidates and analyzes existing literature related to numerical techniques applied in solving second-order differential equations, aiming to assess the methods' strengths, limitations, and practical applications.

Theoretical Foundations of Second-Order Differential Equations

The theoretical underpinnings of second-order differential equations have been extensively discussed by luminaries such as Coddington and Levinson (1955) in "Theory of Ordinary Differential Equations," providing insights into the classifications, characteristics, and significance of both ordinary and partial forms of these equations. This body of work emphasizes the challenges inherent in obtaining analytical solutions, warranting the development and application of numerical methods.

Numerical Methods for Solving Second-Order Differential Equations

Researchers, including Atkinson (2008) in "Numerical Analysis," have thoroughly explored numerous numerical methods such as finite difference, finite element, and Runge-Kutta methods. These methodologies have been studied for their theoretical foundations, computational implementation, advantages, and limitations, providing a deeper understanding of their applicability in diverse scenarios.

Validation and Accuracy

Validation strategies, error analysis, and convergence considerations have been addressed in studies by Dahlquist and Björck (2008) in "Numerical Methods." These validation techniques are crucial in ensuring the reliability and accuracy of numerical solutions when compared to known analytical solutions or experimental data.

Applications in Science and Engineering

Various studies, including those by Stetter (2002) in "The Analysis of Discretization Methods for Ordinary Differential Equations," have presented applications of numerical methods in solving second-order differential equations across multiple fields. Examples from physics, engineering, biology, and economics illustrate the practical utility of these methods in addressing real-world problems.

Challenges and Future Directions

This review identifies existing challenges and potential research directions. It suggests further exploration into advancements in numerical techniques, particularly in addressing gaps in methodologies and potential areas for improvement.

Conclusion

Synthesizing the reviewed literature emphasizes the significance of numerical methods in solving second-order differential equations, underlining their applicability across diverse domains. It also suggests directions for future research and development in this field.

CHAPTER THREE

3.0 METHODOLOGY

The Second Order Differential Equations can be solved using different methods, such as the Euler Method, Runge-kutta Method, Finite Difference Method, etc. Here, we will discuss the two most important techniques called the Euler Method and Runge-Kutta Method in detail.

3.1 Euler Method

The Euler Method, named after the Swiss mathematician Leonhard Euler, is a fundamental numerical technique used to approximate the solutions of ordinary differential equations (ODEs). This method is particularly useful when analytical solutions to ODEs are not readily available or when dealing with complex ODEs that cannot be solved algebraically. The Euler Method provides a straightforward and intuitive approach to solving ODEs, making it an excellent starting point for understanding numerical methods in differential equations.

Basic Concept:

At its core, the Euler Method is an iterative approach that discretizes the ODE into smaller time steps. It approximates the solution at each time step, incrementally building the solution over a specified interval. Here's a simplified overview of how the method works:

<u>Initial Conditions</u>: To start, you need initial conditions, which include the value of the dependent variable at a given initial time.

Discretization: The time interval over which you want to approximate the solution is divided into smaller time steps. The smaller the time step (often denoted as Δt), the more accurate the approximation.

Iterative Process: You iterate through each time step, updating the values of the dependent variable at each step based on its derivative (the rate of change) and the time step.

Approximation: At each step, you calculate the change in the dependent variable using the derivative at that time and the time step. This change is added to the previous value of the dependent variable, resulting in the new approximation for the dependent variable.

Repetition: The process is repeated until you reach the desired endpoint or time.

Euler's method can be expressed as:

$$y_{n+1} = y_n + \Delta t \cdot f(t_n, y_n) .$$

where:

y_n is the approximate value of y at time t_n

t_n is the time at step n.

y_{n+1} is the estimated value of y at time t_{n+1}

Δt is the time step size.

3.2 Runge-kutta Method

The Runge-Kutta method is a family of numerical techniques used for the approximate solution of ordinary differential equations (ODEs). It is named after the German mathematicians Carl Runge and Martin Kutta, who independently developed the method in the early 20th century. The Runge-Kutta method is widely employed in scientific and engineering applications for solving ODEs, especially when analytical solutions are not readily available.

Basic Concept:

The Runge-Kutta method is known for its accuracy and versatility. Unlike simple methods like the Euler method, it offers a more sophisticated approach to approximating the solution of ODEs. The key idea behind the Runge-Kutta method is to compute intermediate values of the dependent variable at several stages within a time step and then use a weighted combination of these values to update the solution.

Fourth-order Runge-Kutta method (RK4) can be expressed as follows for a single time step:

$$K_1 = \Delta t \cdot f(t_n, y_n) \ .$$

$$K_2 = \Delta t \cdot f\left(t_n + \frac{1}{2}\Delta t \ , y_n + \frac{1}{2} K_1\right) \ .$$

$$K_3 = \Delta t \cdot f\left(t_n + \frac{1}{2}\Delta t \ , y_n + \frac{1}{2} K_2\right) \ .$$

$$K_4 = \Delta t \cdot f(t_n + \Delta t, \ y_n + K_3) \ .$$

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 2\,K_2 + 2\,K_3 + K_2) \ .$$

where:

y_n is the approximate value of y at time t_n.

y_{n+1} is the estimated value of y at time t_{n+1}.

K_1, K_2, K_3, and K_4 are intermediate values representing the rate of change of y at different stages within the time step.

CHAPTER FOUR

4.0 APPLICATIONS

4.1 Illustrative Examples

4.1.1 Euler Method (Example)

Consider the second-order differential equation:

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0 \ .$$

With initial conditions: y (0) = 1, $\frac{dy}{dt}$ (0) = 0 .

We want to approximate the solution for $t = 0.3$ using the Euler method with a step size of $\Delta t = 0.1$.

Step 1: Define the differential equation:

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0 \ .$$

Step 2: Set up the initial conditions:

$$y(0) = 1, \frac{dy}{dt}(0) = 0 \ .$$

Step 3: Choose a step size, $\Delta t = 0.1$.

Step 4: Perform the iterations using the Euler method:

- ✓ Iteration 1 ($n = 0 \ to \ n = 1$):

Initialize: $t_0 = 0, y_0 = 1, \ and \ v_0 = 0$ (where v is the first derivative $\frac{dy}{dt}$).

Calculate v_1 using the first derivative:

$$v_1 = v_0 + \Delta t \left(\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y \right)$$

$$v_1 = 0 + 0.1(-3\, v_0 + 2y_0) = 0.1\left(-3(0) + 2(1)\right) = 0.2\,.$$

Calculate y_1 using the updated v_1 :

$$y_1 = y_0 + (\Delta t \cdot v_1)$$

$$y_1 = 1 + (0.1)(0.2) = 1.02\ .$$

Increment t by Δt : $t_1 = 0 + 0.1 = 0.1$.

✓ Iteration 2 (n = 1 to n = 2):

Continue from the values obtained at the end of Iteration 1:

$$t_1 = 0.1, \qquad y_1 = 1.02, \qquad v_1 = 0.2\,.$$

Calculate v_2 using the first derivative:

$$v_2 = v_1 + \Delta t\left(\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y\right)$$

$$v_2 = 0.2 + 0.1(-3v_1 + 2y_1) = 0.2 + 0.1\left(-3(0.2) + 2(1.02)\right) = 0.344\ .$$

Calculate y_2 using the updated v_2:

$$y_2 = y_1 + (\Delta t \cdot v_2)$$

$$y_2 = 1.02 + (0.1)(0.344) = 1.0544\,.$$

Increment t by Δt: $t_2 = 0.1 + 0.1 = 0.2$.

✓ Iteration 3 ($n = 2\ to\ n = 3$):

$$t_2 = 0.2, \qquad y_2 = 1.0544, \qquad v_2 = 0.344.$$

Calculate v_3 using the first derivative:

$$v_3 = v_2 + \Delta t\left(\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y\right)$$

$$v_3 = 0.344 + 0.1(-3v_2 + 2y_2) = 0.344 + 0.1\big(-3(0.344) + 2(1.0544)\big) = 0.45168 \ .$$

Calculate y_3 using the updated v_3:

$$y_3 = y_2 + (\Delta t \cdot v_3)$$

$$y_3 = 1.0544 + (0.1)(0.45168) = 1.099568 \, .$$

Increment t by Δt: $t_3 = 0.2 + 0.1 = 0.3$.

Now, we have completed Iteration 3, and the values are available up to $t = 0.3$. The numerical values of y and $\frac{dy}{dt}$ were calculated at t = 0.1, 0.2, 0.3 as follows:

$$At\ t = 0.1, \qquad y = 1.02, \ \ \frac{dy}{dt} = 0.2 \ .$$

$$At\ t = 0.2, \qquad y = 1.0544, \qquad \frac{dy}{dt} = 0.344 \ .$$

$$At\ t = 0.3, \qquad y = 1.099568, \ \ \frac{dy}{dt} = 0.45168 \ .$$

Comparison with Runge-Kutta Method

Step 1: Define the differential equation:

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0 \, .$$

Step 2: Set up the initial conditions:

$$y\,(0) = 1, \frac{dy}{dt}(0) = 0 \, .$$

Step 3: Choose a step size, $\Delta t = 0.1$.

Step 4: Perform the iterations using the Fourth-Order Runge-Kutta method:

✓ Iteration 1 (n = 0 to n = 1):

Initialize $t_1 = 0.1,\ y_0 = 1, and\ v_0 = 0$ (velocity).

Calculate $k1$ and $l1$:

$$k1 = (0.1)\, v_0 = 0 \ .$$

$$l1 = 0.1(-3v_0 + 2\, y_0) = 0.1\big(-3(0) + 2(1)\big) = 0.2 \, .$$

Calculate $k2$ and $l2$ at the midpoint:

$$k2 = 0.1\left(v_0 + \frac{1}{2}\, l1 \right) = 0.1 \left(0 + \frac{1}{2}\ (0.2)\right) = 0.01 \, .$$

$$l2 = 0.1\left(-3\left(v_0 + \frac{1}{2}\, l1\right) + 2\left(y_0 + \frac{1}{2}\, k1\right)\right)$$

$$= 0.1 \left(-3 \left(0 + \frac{1}{2}\, (-0.2)\right) - 2 \left(1 + \frac{1}{2} \cdot 0 \right)\right) = 0.17 \ .$$

Calculate $k3$ and $l3$ at the midpoint:

$$k3 = 0.1 \left(v_0 + \frac{1}{2}\, l2 \right) = 0.1 \left(0 + \frac{1}{2} \cdot (0.17)\right) = 0.0085 \ .$$

$$l3 = 0.1 \cdot \left(-3\left(v_0 + \frac{1}{2}\, l2\right) - 2\left(y_0 + \frac{1}{2}\, k2 \right)\right)$$

$$= 0.1 \left(-3\left(0 + \frac{1}{2} \cdot (0.17)\right) + 2 \left(1 + \frac{1}{2}\ (0.01)\right)\right) = 0.1755 \, .$$

Calculate $k4$ and $l4$ at the end of the interval:

$$k4 = 0.1 \cdot (v_0 + l3) = 0.1 \cdot (0 + 0.1755) = 0.01755 \, .$$

$$l4 = 0.1 \cdot \big(-3(v_0 + l3) + 2(y_0 + k3)\big)$$

$$= 0.1 \cdot \big(-3(0 + 0.0085) + 2(1 + 0.1755)\big) = 0.23255 \, .$$

Update $y1$ and $v1$ using the weighted average of k's and l's:

$$y1 = y_0 + \frac{1}{6}(k1 + 2k2 + 2k3 + k4)$$

$$= 1 + \frac{1}{6}(0 + 2(0.01) + 2(0.0085) + 0.01755) = 1.0091 .$$

$$v1 = v_0 + \frac{1}{6}(l1 + 2\,l2 + 2\,l3 + l4)$$

$$= 0 + \frac{1}{6}(0.2 + 2(0.17) + 2(0.1755) + 0.23255) = 0.1872583 .$$

Increment t by Δt ($t2 = 0.2$).

✓ Iteration 2 (n = 1 to n = 2):

Continue from the values obtained at the end of Iteration 1:

$$t2 = 0.2, \qquad y1 = 1.0091, \qquad v1 = 0.1872583 .$$

Calculate $k1$ and 1 :

$$k1 = 0.2 \cdot v1 = 0.2 \cdot (0.1872583) = 0.03745 .$$

$$l1 = 0.2 \cdot (-3v1 + 2y1) = 0.2\big(-3(0.1872583) + 2(1.0091)\big) = 0.29129.$$

Calculate $k2$ and $l2$ at the midpoint:

$$k2 = 0.2 \cdot \left(v1 + \frac{1}{2}\,l1\right) = 0.2 \cdot \left(0.1872583 + \frac{1}{2} \cdot (0.29129)\right) = 0.06658 .$$

$$l2 = 0.2 \cdot \left(-3\left(v1 + \frac{1}{2}\,l1\right) + 2\left(y1 + \frac{1}{2}\,k1\right)\right)$$

$$= 0.2 \cdot \left(-3\left(0.1872583 + \frac{1}{2}(0.29129)\right) + 2\left(1.0091 + \frac{1}{2}(0.03745)\right)\right)$$

$$= 0.21139 .$$

Calculate $k3$ and $l3$ at the midpoint:

$$k3 \;=\; 0.1 \cdot \left(v1 \;+ \frac{1}{2}\, l2 \;\right) = 0.2 \cdot \left(0.1872583 \;+ \frac{1}{2} \cdot (0.21139)\right) = \;0.05859 \;.$$

$$l3 \;=\; 0.2 \cdot \left(-3\left(v1 \;+ \frac{1}{2} l2 \;\right) - 2\left(\; y1 \;+ \frac{1}{2} k2 \;\right)\right)$$

$$= \;0.2 \cdot \left(-3 \;\left(0.1872583 \;+\; \frac{1}{2} \cdot (\;0.21139)\right) + 2\left(1.0091 + \frac{1}{2} \cdot (0.06658)\right)\right)$$

$$= 0.24118 \;.$$

Calculate $k4$ and $l4$ at the end of the interval:

$$k4 \;=\; 0.2 \cdot (\;v1 \;+\; l3\,) = 0.2 \cdot (0.1872583 + 0.24118) = 0.08568\,.$$

$$l4 \;=\; 0.2 \cdot (-3(v1 \;+\; l3\,) + 2(y1 \;+\; k3\,))$$

$$= 0.2 \cdot \big(-3\,(0.1872583 + 0.24118) + 2(1.0091 + 0.05859)\big) = \;0.17\,.$$

Update y_2 and v_2 using the weighted average of 's and l 's:

$$y_2 \;=\; y_1 \;+ \frac{1}{6}\;\;(k_1 \;+ 2\;k_2 +\; 2k_3 +\; k_4)$$

$$= 1.0091 + \frac{1}{6}(0.03745 + \;2(0.06658) + 2(0.05859) + \;0.08568)$$

$$= \;1.071345 \;.$$

$$v_2 \;=\; y_1 \;+ \frac{1}{6}\;(l_1 \;+ 2\;l_2 \;+ 2\;l_3 +\; l_4\,)$$

$$= \;1.0091 + \frac{1}{6}(0.29129 \;+\; 2(0.21139) + \;2(0.24118) + 0.17)$$

$$= \;1.23684 \;.$$

Now, we have completed Iteration 2, and the values are available up to $t = 0.2$

In conclusion while solving using Euler`s Method $\mathbf{y_2 = 1.0544}$ and while solving using Runge-Kutta Method $\mathbf{y_2 = 1.071345}$ which shows the accuracy of the two methods.

4.1.2 Runge-Kutta Method (Example)

Consider the second-order differential equation:

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0 .$$

with initial conditions:

$$y(0) = 1, \frac{dy}{dt}(0) = 0 .$$

We want to approximate the solution for t = 0.2 using the Fourth-Order Runge-Kutta method with a step size of Δt = 0.1.

Step 1: Define the differential equation:

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0 .$$

Step 2: Set up the initial conditions:

$$y(0) = 1, \frac{dy}{dt}(0) = 0 .$$

Step 3: Choose a step size, Δt = 0.1 .

Step 4: Perform the iterations using the Fourth-Order Runge-Kutta method:

- ✓ Iteration 1 (n = 0 to n = 1):

Initialize $t_1 = 0.1,\ y_0 = 1, and\ v_0 = 0$ (velocity).

Calculate $k1\ \ and\ l1$:

$$k1 = (0.1)\, v_0 = 0 .$$

$$l1 = 0.1(-2v_0 + 5\, y_0) = 0.1\big(-2(0) + 5(1)\big) = 0.5 .$$

Calculate $k2\ \ and\ l2$ at the midpoint:

$$k2 \ = \ 0.1(v_0 \ + \ \frac{1}{2} \ l1 \ ) \ = \ 0.1 \left(0 \ + \frac{1}{2} \ \ (0.5) \right) \ = \ 0.025 \, .$$

$$l2 \ \ = 0.1 \left(-2 \ \left(v_0 \ + \ \frac{1}{2} \ l1 \right) + 5 \left(\ y_0 + \frac{1}{2} \ k1 \right) \right)$$

$$= \ 0.1 \ \left(-2 \ \left(\ 0 \ + \ \frac{1}{2} \ (0.5) \right) + 5 \ \left(\ 1 \ + \frac{1}{2} \ \cdot \ 0 \ \right) \right) = \ 0.45 \ \ .$$

Calculate $k3$ and $l3$ at the midpoint:

$$k3 \ \ = \ 0.1 \ \left(\ v_0 \ \ + \ \frac{1}{2} \ l2 \ \right) = \ 0.1 \ \left(\ 0 \ + \frac{1}{2} \cdot (0.45) \right) = 0.0225.$$

$$l3 \ \ = 0.1 \cdot \left(-2 \left(v_0 \ \ + \frac{1}{2} \ l2 \right) + 5 \left(y_0 + \frac{1}{2} \ k2 \ \right) \right)$$

$$= \ 0.1 \ \left(\ -2 \left(0 \ + \frac{1}{2} \cdot (0.45) \right) + 5 \ \left(1 \ + \frac{1}{2} \ \ (\, 0.025 \,) \right) \right) = \ 0.46875 \, .$$

Calculate $k4$ and $l4$ at the end of the interval:

$$k4 \ \ = \ 0.1 \cdot \ (v_0 + \ l3 \ ) = \ 0.1 \ \cdot \ (\, 0 + 0.46875 \,) = \ 0.046875 \, .$$

$$l4 \ \ = \ 0.1 \cdot \ \left(-2(v_0 + \ l3) + \ 5(y_0 \ + \ k3 \ ) \right)$$

$$= \ 0.1 \ \cdot \ \left(-2(\, 0 + \ 0.46875) + \ 5(\, 1 + \ 0.0225) \right) = \ 0.4175 \, .$$

Update $y1$ and $v1$ using the weighted average of k's and l's:

$$y1 \ \ = \ y_0 + \frac{1}{6} \ \ (k1 \ + \ 2k2 \ \ + \ 2k3 \ + \ k4)$$

$$= \ 1 \ + \frac{1}{6} \ \ (\, 0 \ + \ 2(0.025) + \ 2(0.0225) + 0.4175) = \ 1.08542 \ \ .$$

$$v1 \ \ = \ v_0 + \frac{1}{6} \ \ (l1 + \ 2 \, l2 \ + 2 \, l3 \ + \ l4 \,)$$

$$= 0 + \frac{1}{6} \ (0.5 + 2(0.45) + 2(0.46875) + 0.4175) = 0.4592 .$$

Increment t by Δt ($t2 = 0.2$).

✓ Iteration 2 (n = 1 to n = 2):

Continue from the values obtained at the end of Iteration 1:

$$t1 = 0.2, \qquad y1 = 1.08542, \qquad v1 = 0.4592 .$$

Calculate $k1$ and 1 :

$$k1 = 0.2 \cdot v1 = 0.2 \cdot (0.4592) = 0.09184 .$$

$$l1 = 0.2 \cdot (-2v1 + 5y1) = 0.2 \cdot \left(-2(0.4592) + 5(1.08542)\right) = 0.90174 .$$

Calculate $k2$ and $l2$ at the midpoint:

$$k2 = 0.1 \cdot \left(v1 + \frac{1}{2}\, l1 \right) = 0.2 \cdot \left(0.0294271 + \frac{1}{2} \cdot (0.0203526) \right) = 0.0029117 .$$

$$l2 = 0.2 \cdot \left(-2\left(v1 + \frac{1}{2}\, l1 \right) + 5\left(y1 + \frac{1}{2}\, k1 \right) \right)$$

$$= 0.2 \cdot \left(-2 \left(0.0294271 + \frac{1}{2}(0.0203526) \right) + 5\left(0.9938542 + \frac{1}{2}(0.0029427) \right) \right)$$

$$= 0.0202939 .$$

Calculate $k3$ and $l3$ at the midpoint:

$$k3 = 0.2 \cdot \left(v1 + \frac{1}{2}\, l2 \right) = 0.2 \cdot \left(0.0294271 + \frac{1}{2} \cdot (0.0202939) \right) = 0.0029203 .$$

$$l3 = 0.2 \cdot \left(-2\left(v1 + \frac{1}{2} l2 \right) + 5\left(y1 + \frac{1}{2} k2 \right) \right)$$

$$= 0.2 \cdot \left(-2 \left(-0.0294271 + \frac{1}{2} \cdot (-0.0202939)\right) + 5\left(0.9938542 + \frac{1}{2} \cdot (-0.0029117)\right)\right)$$

$$= 0.0203308 .$$

Calculate $k4$ and $l4$ at the end of the interval:

$$k4 = 0.2 \cdot (v1 + l3) = 0.2 \cdot (0.0294271 + 0.0203308) = 0.0042758 .$$

$$l4 = 0.2 \cdot (-2(v1 + l3) + 5(y1 + k3))$$

$$= 0.1 \cdot \left(-2 (0.0294271 + 0.0203308) + 5(0.9938542 + 0.0029203)\right) = -0.0211236 .$$

Update y_2 and v_2 using the weighted average of k 's and l 's:

$$y_2 = y_1 + \frac{1}{6} (k_1 + 2 k_2 + 2k_3 + k_4)$$

$$= 0.9938542 + \frac{1}{6}(0.0029427 + 2(0.0029117) + 2(0.0029203) + 0.0042758)$$

$$= 1.1401 .$$

$$v_2 = y_1 + \frac{1}{6} (l_1 + 2 l_2 + 2 l_3 + l_4)$$

$$= -0.0294271 + \frac{1}{6}(0.0203526 + 2(0.0202939) + 2(0.0203308) + 0.0211236)$$

$$= 0.910 .$$

Now, we have completed Iteration 2, and the values are available up to $t = 0.2$.

Comparison with Euler Method in the last Runge-Kutta Example

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0 .$$

with initial conditions:

$$y(0) = 1, \frac{dy}{dt}(0) = 0 .$$

Step 1: Define the differential equation:

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0 .$$

Step 2: Set up the initial conditions:

$$y(0) = 1, \frac{dy}{dt}(0) = 0 .$$

Step 3: Choose a step size, $= 0.1$.

Step 4: Perform the iterations using the Euler method:

- ✓ Iteration 1 ($n = 0 \; to \; n = 1$):

Initialize: $t_0 = 0, y_0 = 1, \; and \; v_0 = 0$ (where v is the first derivative $\frac{dy}{dt}$).

Calculate v_1 using the first derivative:

$$v_1 = v_0 + \Delta t \left(\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y \right)$$

$$v_1 = 0 + 0.1(-2\,v_0 + 5y_0) = 0.1\left(-2(0) + 5(1)\right) = 0.5 .$$

Calculate y_1 using the updated v_1 :

$$y_1 = y_0 + (\Delta t \cdot v_1)$$

$$y_1 = 1 + (0.1)(0.5) = 1.05 .$$

Increment t by Δt : $t_1 = 0 + 0.1 = 0.1$.

- ✓ Iteration 2 (n = 1 to n = 2):

Continue from the values obtained at the end of Iteration 1:

$$t_1 = 0.1, \qquad y_1 = 1.05, \qquad v_1 = 0.5\,.$$

Calculate v_2 using the first derivative:

$$v_2 = v_1 + \Delta t\left(\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y\right)$$

$$v_2 = 0.5 + 0.1(-2v_1 + 5y_1) = 0.5 + 0.1\big(-2(0.5) + 5(1.05)\big) = 0.925\,.$$

Calculate y_2 using the updated v_2:

$$y_2 = y_1 + (\Delta t \cdot v_2)$$

$$y_2 = 1.05 + (0.1)(0.925) = 1.1425\,.$$

In conclusion while solving using Runge Kutta y_2 = 1.1401 and while solving using Euler Method y_2 = 1.1425 which shows the accuracy of the two methods.

CHAPTER FIVE

5.0 CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

This work is a fundamental and essential area of study in mathematics and various scientific and engineering disciplines. Here are some key points from the discussion:

- **Numerical Methods as Problem-Solving Tools**: Numerical methods, including the Euler`s method and Runge-Kutta methods, provide powerful problem-solving tools for approximating solutions to second-order differential equations. These methods involve the discretization of time and the iterative calculation of values to estimate the solution.
- **Euler Method**: The Euler method is a straightforward numerical technique that serves as a foundation for understanding more complex methods. It involves taking small time steps and updating the solution based on the derivative of the dependent variable.
- **Runge-Kutta Method**: The Runge-Kutta method is a more accurate and versatile approach that employs a series of stages to calculate intermediate values and then combines them to update the solution. It is particularly well-suited for a wide range of ODEs.

5.2 Recommendation

I recommend that other researchers should further their research and innovation in the field of numerical methods for ODEs, especially the development of adaptive methods that can automatically adjust time step sizes for better accuracy. Also, I encourage learners to gain hands-on experience by implementing numerical methods in programming languages like Python or MATLAB. Provide exercises and projects that involve coding and simulating solutions to real-world problems.

REFERENCES

[1] Ascher, U. M., and Petzold, L. R. (1998). Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations, Society for Industrial and Applied Mathematics (SIAM) pp. 85-95.

[2] Burden, R. L., and Faires, J. D. (2016). Numerical Analysis (3rd edition), Cengage Learning pp. 145-165.

[3] Butcher, J. C. (2008). Numerical Methods for Ordinary Differential Equations, John Wiley and Sons pp. 81-87.

[4] Hairer, E., and Wanner, G. (2002). Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems, Springer pp. 108-122.

[5] Lambert, J. D. (1973). Computational Methods in Ordinary Differential Equations (volume_2), John Wiley and Sons pp. 83-95.

[6] Stoer, J., and Bulirsch, R. (2002). Introduction to Numerical Analysis, Springer pp. 285-292.