# NUMERICAL METHODS FOR SOLVING SECOND ORDER DIFFERENTIAL EQUATIONS

**BY**

**KAREEM, SAMSON ADEBAYO**

**MATRICULATION NUMBER: 20183037**

**A PROJECT SUBMITTED TO THE**

**DEPARTMENT OF MATHEMATICS,**

**COLLEGE OF PHYSICAL SCIENCES,**

**FEDERAL UNIVERSITY OF AGRICULTURE, ABEOKUTA.**

**IN PARTIAL FULFILMENT FOR THE AWARD OF BACHELOR OF SCIENCE DEGREE IN MATHEMATICS.**

**NOVEMBER, 2023.**

# DECLARATION

I hereby declare that this research was written by me and is a correct record of my own research. It has not been presented in any previous application for any degree of this or any other University. All citations and sources of information are clearly acknowledged by means of references.

***KAREEM, SAMSON ADEBAYO***

***Date:.................................***

# CERTIFICATION

This is to certify that this research work entitled **Numerical Methods for Solving Second Order Differential Equations** is the outcome of the research work carried out by **Kareem Samson Adebayo** (20183037) in the Department of Mathematics, Federal University of Agriculture, Abeokuta, Ogun State.

.................. ..................

***PROF. M.O OMEIKE*** ***Date***

(**SUPERVISOR**)

.................. ..................

***DR. E.O. ADELEKE*** ***Date***

(Ag. **HEAD OF DEPARTMENT**)

# DEDICATION

This project work is dedicated to Almighty God, the creator of the universe and all mankind, who gave me this grace from the inception of this project work till its completion. And also to my wonderful family, starting from my beloved parents, Mr and Mrs Kareem as well as my ever-supportive siblings and to everyone that has been supportive and helpful in my education life.

## ACKNOWLEDGMENTS

All glory, honour and adoration is to the Almighty God who has made the success of the research work and the completion of my BSc. programme at large a reality.

I would like to express my gratitude and appreciation to my supervisor, Prof M.O OMEIKE whose help in stimulating suggestion and encouragement helped in the process of completing this project. I also sincerely thank him for the time spent proofreading and correcting my many mistakes.

I am grateful to the Ag. Head of Department, DR. E.O. Adeleke, immediate past Head of department Prof. B.I. Olajuwon and all lecturers of the Department of Mathematics, because all I have been taught from my first year in the Department made it possible for me to carry out this research work.

My sincere appreciation also goes to my parent Mr and Mrs Kareem for their full support, advice, prayer, love and care placed on me throughout this project period and my stay on campus at large. Daddy and Mummy, I pray to God that you live long to eat the fruit of your labor.

My profound appreciation also goes to my wonderful siblings: Victoria, Janet and Susan, for their contribution physically, spiritually, financially towards the success of my programme. I pray that almighty God take her to higher grounds.

Finally, my sincere gratitude also goes to all those who have contributed to my success in FUNAAB: my friend (Ayoola Tijesuni), departmental mates (Olalere Babatunde, Adebisi Adewunmi) and many others that I couldn`t mention their names. Thank you all and God bless you. (AMEN).

## ABSTRACT

Second-order differential equations are fundamental in modeling various natural phenomena and engineering systems. This project explores the application of numerical methods as indispensable tools for solving second-order differential equations, emphasizing their crucial role when analytical solutions are elusive or impractical. The research provides a comprehensive overview of second-order differential equations, including ordinary and partial forms, and highlights the limitations of analytical techniques. Numerical methods, such as finite difference, finite element, and Runge-Kutta methods, are introduced and analyzed in-depth, elucidating the principles behind each method, their strengths, and their limitations. Practical implementation of these methods is showcased through code examples, illustrating their efficacy in approximating solutions to differential equations. The project also delves into validation and accuracy aspects, addressing the critical issue of comparing numerical solutions with known analytical solutions or experimental data, ensuring confidence in the results. Real-world applications from diverse fields, including physics, engineering, biology, and economics, demonstrate the practical utility of these methods. In conclusion, this project underscores the significance of numerical methods in solving second-order differential equations and provides insights into their applicability, accuracy, and the broad range of challenges they can address. The findings contribute to the understanding of numerical techniques in scientific and engineering contexts and pave the way for future research and innovation in this field.

# Table of Contents

# 1.0 INTRODUCTION

## 1.1 Background to the Study

Differential equations are mathematical tools that describe the fundamental relationships between quantities that vary continuously with respect to one or more independent variables. Among the diverse family of differential equations, second-order differential equations hold a special place due to their ubiquity in various scientific and engineering disciplines. These equations, which involve the second derivative of an unknown function, arise in fields ranging from classical mechanics and electrical circuit analysis to fluid dynamics and quantum physics.

The solutions to second-order differential equations play a crucial role in understanding and predicting the behavior of complex physical systems. However, not all of these equations can be solved analytically using closed-form expressions. In fact, many real-world problems yield differential equations that defy analytical solution techniques. It is in the face of these challenges that numerical methods come to the forefront, offering a powerful arsenal of computational techniques to approximate solutions and glean valuable insights.

This project embarks on an exploration of numerical methods for solving second-order differential equations. We will delve into the theoretical foundations of these methods, their practical implementation, and their significance in addressing real-world problems. Through this endeavor, we aim to equip the reader with the knowledge and skills necessary to tackle a wide array of problems that hinge on the numerical solution of second-order differential equations.

In the pages that follow, we will journey through various numerical approaches, from the venerable finite difference methods to sophisticated finite element techniques. We will consider their applicability in different contexts, highlight their strengths and limitations, and provide illustrative examples to demonstrate their efficacy. Additionally, we will delve into the realm of software tools and programming languages that facilitate the practical implementation of these methods.

As we embark on this voyage through the realm of second-order differential equations, we invite you to join us in discovering the art and science of numerical solutions—an indispensable skill for unraveling the mysteries of the physical world.

## 1.2 Motivation

Understanding the motivation behind the study of numerical methods for solving second-order differential equations is crucial to appreciating their significance and relevance across various scientific and engineering disciplines.

At the core of our motivation lies the inescapable reality that many real-world problems are described by second-order differential equations. These equations capture the intricate interplay of forces, phenomena, and variables that govern physical systems. Consider the structural engineer tasked with ensuring the safety of a newly designed bridge under diverse loading conditions, or the climate scientist striving to model the complex dynamics of atmospheric processes. In both cases, second-order differential equations arise as the mathematical framework for describing these phenomena. However, the majority of such equations resist analytical solutions, prompting the need for numerical techniques as our primary tools for exploration.

In the realm of scientific discovery, numerical methods enable us to unravel the mysteries of the natural world. They empower physicists to simulate quantum systems, astrophysicists to model celestial bodies, and biologists to understand the intricate dynamics of biological systems. These simulations can provide insights and predictions that not only deepen our understanding but also have practical applications, from optimizing drug delivery mechanisms to predicting the behavior of distant galaxies.

From an engineering standpoint, the motivation is equally compelling. The design of aircraft wings, the analysis of heat transfer in electronic devices, and the optimization of energy-efficient buildings all hinge on solving complex second-order differential equations. The ability to model these systems with precision is critical for innovation, efficiency, and safety in engineering endeavors.

Furthermore, as computational resources continue to advance, numerical methods play an increasingly pivotal role in solving larger and more complex problems. High-performance computing allows us to tackle simulations that were once inconceivable, such as weather forecasting at unprecedented resolutions and the design of advanced materials with tailored properties. The motivation to harness the full potential of these computational tools and methods is clear.

In conclusion, the study of numerical methods for solving second-order differential equations is motivated by the need to tackle complex problems that permeate science and engineering. From unveiling the mysteries of the cosmos to advancing the frontiers of technology, numerical methods are indispensable tools that empower us to explore, understand, and innovate in a rapidly evolving world. This project seeks to embrace this motivation, providing a comprehensive guide to these methods, their applications, and their transformative potential in addressing the challenges of our time.

## 1.3 Objectives

- ✓ **To Introduce Numerical Methods**: Introduce students or readers to numerical methods as a means to solve complex second-order differential equations that may not have analytical solutions.
- ✓ T**o Study Fundamental Numerical Techniques**: Provide an in-depth exploration of fundamental numerical techniques like Euler method and Runge-Kutta method.

## 1.4 Definition of Terms

- **Differential Equation**: A differential equation is a mathematical equation that involves derivatives of an unknown function. In the context of this project, we are primarily interested in second-order differential equations, which involve the second derivative of a function.
- **Second-Order Differential Equation**: A second-order differential equation is a specific type of differential equation where the highest-order derivative involved is the second derivative. It often takes the form:

$$a(x)\, y'' + b(x)\, y' + c(x)\, y = f(x)$$

where $a(x), b(x), c(x)$, and $f(x)$ are functions of the independent variable x, and y is the unknown function to be solved for.

- **Analytical Solution**: An analytical solution to a differential equation is a closed-form expression that explicitly describes the solution. It is found through symbolic manipulations and integration techniques.
- **Numerical Method**: A numerical method is an algorithm or computational technique used to approximate solutions to mathematical problems, including differential equations, when analytical solutions are not readily available or practical.
- **Finite Element Method**: The finite element method is a numerical approach primarily used for solving partial differential equations. It involves subdividing the problem domain into smaller finite elements, typically triangles or quadrilaterals in 2D or tetrahedra and hexahedra in 3D, and solving for the unknown function within each element.

- **Initial Value Problem** (IVP): An initial value problem is a type of problem for ordinary differential equations (ODEs) where the solution is determined based on an initial condition. In the context of second-order differential equations, this typically means specifying the values of both the unknown function and its first derivative at a single point.
- **Boundary Value Problem** (BVP): A boundary value problem is a type of problem for differential equations where the solution is determined by specifying conditions at multiple points along the domain boundaries. In the context of second-order differential equations, this involves specifying conditions at both the initial and final points of the domain.
- Convergence: Convergence refers to the property of a numerical method where the approximated solution approaches the true solution as the computational resources (e.g., grid points, time steps) increase. Convergence analysis assesses how well a numerical method approximates the exact solution.

## 2.0 LITERATURE REVIEW

Second-order differential equations play a pivotal role in scientific and engineering domains, often modeling crucial dynamics in physical systems. The quest for closed-form solutions to these equations has led researchers to employ numerical methods. This review consolidates and analyzes existing literature related to numerical techniques applied in solving second-order differential equations, aiming to assess the methods' strengths, limitations, and practical applications.

**Theoretical Foundations of Second-Order Differential Equations**

The theoretical underpinnings of second-order differential equations have been extensively discussed by luminaries such as Coddington and Levinson (1955) in "Theory of Ordinary Differential Equations," providing insights into the classifications, characteristics, and significance of both ordinary and partial forms of these equations. This body of work emphasizes the challenges inherent in obtaining analytical solutions, warranting the development and application of numerical methods.

**Numerical Methods for Solving Second-Order Differential Equations**

Researchers, including Atkinson (2008) in "Numerical Analysis," have thoroughly explored numerous numerical methods such as finite difference, finite element, and Runge-Kutta methods. These methodologies have been studied for their theoretical foundations, computational implementation, advantages, and limitations, providing a deeper understanding of their applicability in diverse scenarios.

## Validation and Accuracy

Validation strategies, error analysis, and convergence considerations have been meticulously addressed in studies by Dahlquist and Björck (2008) in "Numerical Methods." These validation techniques are crucial in ensuring the reliability and accuracy of numerical solutions when compared to known analytical solutions or experimental data.

## Applications in Science and Engineering

Various studies, including those by Stetter (2002) in "The Analysis of Discretization Methods for Ordinary Differential Equations," have presented applications of numerical methods in solving second-order differential equations across multiple fields. Examples from physics, engineering, biology, and economics illustrate the practical utility of these methods in addressing real-world problems.

## Challenges and Future Directions

This review identifies existing challenges and potential research directions. It suggests further exploration into advancements in numerical techniques, particularly in addressing gaps in methodologies and potential areas for improvement.

## Conclusion

Synthesizing the reviewed literature emphasizes the significance of numerical methods in solving second-order differential equations, underlining their applicability across diverse domains. It also suggests directions for future research and development in this field.

## 3.0 METHODOLOGY

The Second Order Differential Equations can be solved using different methods, such as the Euler Method, Runge-kutta Method, Finite Difference Method, etc. Here, we will discuss the two most important techniques called the Euler Method and Runge-Kutta Method in detail.

### 3.1 Euler Method

The Euler Method, named after the Swiss mathematician Leonhard Euler, is a fundamental numerical technique used to approximate the solutions of ordinary differential equations (ODEs). This method is particularly useful when analytical solutions to ODEs are not readily available or when dealing with complex ODEs that cannot be solved algebraically. The Euler Method provides a straightforward and intuitive approach to solving ODEs, making it an excellent starting point for understanding numerical methods in differential equations.

**Basic Concept:**

At its core, the Euler Method is an iterative approach that discretizes the ODE into smaller time steps. It approximates the solution at each time step, incrementally building the solution over a specified interval. Here's a simplified overview of how the method works:

Initial Conditions: To start, you need initial conditions, which include the value of the dependent variable at a given initial time.

Discretization: The time interval over which you want to approximate the solution is divided into smaller time steps. The smaller the time step (often denoted as $\Delta t$), the more accurate the approximation.

Iterative Process: You iterate through each time step, updating the values of the dependent variable at each step based on its derivative (the rate of change) and the time step.

Approximation: At each step, you calculate the change in the dependent variable using the derivative at that time and the time step. This change is added to the previous value of the dependent variable, resulting in the new approximation for the dependent variable.

Repetition: The process is repeated until you reach the desired endpoint or time.

Euler's method can be expressed as:

$$y_{n+1} = y_n + \Delta t \cdot f(t_n, y_n)$$

where:

$y_n$ is the approximate value of y at time $t_n$

$t_n$ is the time at step n.

$y_{n+1}$ is the estimated value of y at time $t_{n+1}$

$\Delta t$ is the time step size.

## 3.2 Runge-kutta Method

The Runge-Kutta method is a family of numerical techniques used for the approximate solution of ordinary differential equations (ODEs). It is named after the German mathematicians Carl Runge and Martin Kutta, who independently developed the method in the early 20th century. The Runge-Kutta method is widely employed in scientific and engineering applications for solving ODEs, especially when analytical solutions are not readily available.

**Basic Concept**:

The Runge-Kutta method is known for its accuracy and versatility. Unlike simple methods like the Euler method, it offers a more sophisticated approach to approximating the solution of ODEs. The key idea behind the Runge-Kutta method is to compute intermediate values of the dependent variable at several stages within a time step and then use a weighted combination of these values to update the solution.

**Fourth-order Runge-Kutta method (RK4) can be expressed as follows for a single time step**:

$$K_1 = \Delta t \cdot f(t_n, y_n)$$

$$K_2 = \Delta t \cdot f\left(t_n + \frac{1}{2}\Delta t, y_n + \frac{1}{2}K_1\right)$$

$$K_3 = \Delta t \cdot f\left(t_n + \frac{1}{2}\Delta t, y_n + \frac{1}{2}K_2\right)$$

$$K_4 = \Delta t \cdot f(t_n + \Delta t, y_n + K_3)$$

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_2)$$

where:

$y_n$ is the approximate value of y at time $t_n$

$y_{n+1}$ is the estimated value of y at time $t_{n+1}$

$K_1$, $K_2$, $K_3$, and $K_4$ are intermediate values representing the rate of change of y at different stages within the time step.

# 4.0 APPLICATIONS

## 4.1 Illustrative Examples

### 4.1.1 Euler Method (Example)

Consider the second-order differential equation:

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0$$

With initial conditions: $y$ (0) = 1, $\frac{dy}{dt}$ (0) = 0

We want to approximate the solution for $t = 0.3$ using the Euler method with a step size of $\Delta t = 0.1$

**Step 1**: Define the differential equation:

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0$$

**Step 2**: Set up the initial conditions:

$$y(0) = 1, \frac{dy}{dt}(0) = 0$$

**Step 3**: Choose a step size, $\Delta t = 0.1$

**Step 4**: Perform the iterations using the Euler method:

- ✓ Iteration 1 ($n = 0\ to\ n = 1$):

Initialize: $t_0 = 0, y_0 = 1,\ and\ v_0 = 0$ (where v is the first derivative $\frac{dy}{dt}$ ).

Calculate $v_1$ using the first derivative:

$$v_1 = v_0 + \Delta t\left(\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y\right)$$

$$v_1 = 0 + 0.1(-3\,v_0 + 2y_0) = 0.1\,(-3(0) + 2(1)\,) = 0.2$$

Calculate $y_1$ using the updated $v_1$ :

$$y_1 = y_0 + (\Delta t \cdot v_1)$$

$$y_1 = 1 + (0.1)(0.2) = 1.02$$

Increment t by $\Delta t$ : $t_1 = 0 + 0.1 = 0.1$

✓ Iteration 2 (n = 1 to n = 2):

Continue from the values obtained at the end of Iteration 1:

$$t_1 = 0.1, \qquad y_1 = 1.02, \qquad v_1 = 0.2$$

Calculate $v_2$ using the first derivative:

$$v_2 = v_1 + \Delta t(\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y\,)$$

$$v_2 = 0.2 + 0.1(-3v_1 + 2y_1) = 0.2 + 0.1(-3(0.2) + 2(1.02)) = 0.18$$

Calculate $y_2$ using the updated $v_2$:

$$y_2 = y_1 + (\Delta t \cdot v_2)$$

$$y_2 = 1.02 + (0.1)(0.18) = 1.038$$

Increment t by $\Delta t$: $t_2 = 0.1 + 0.1 = 0.2$

✓ Iteration 3 ($n = 2\ to\ n = 3$):

$$t_2 = 0.2, \qquad y_2 = 1.038, \qquad v_2 = 0.18$$

Calculate $v_3$ using the first derivative:

$$v_3 = v_2 + \Delta t(\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y)$$

$$v_3 = 0.18 + 0.1(-3v_2 + 2y_2) = 0.18 + 0.1(-3(0.18) + 2(1.038)) = 0.1704$$

Calculate $y_3$ using the updated $v_3$:

$$y_3 = y_2 + (\Delta t \cdot v_3)$$

$$y_3 = 1.038 + (0.1)(0.1704) = 1.05504$$

Increment t by $\Delta t$: $t_3 = 0.2 + 0.1 = 0.3$

Now, we have completed Iteration 3, and the values are available up to $t = 0.3$. The numerical values of y and $\frac{dy}{dt}$ were calculated at t = 0.1, 0.2, 0.3 as follows:

$$At\ t = 0.1, \qquad y = 1.02, \quad \frac{dy}{dt} = 0.2$$

$$At\ t = 0.2, \qquad y = 1.038, \qquad \frac{dy}{dt} = 0.18$$

$$At\ t = 0.3, \qquad y = 1.05504, \quad \frac{dy}{dt} = 0.1704$$

### 4.1.2 Runge-Kutta Method (Example)

Consider the second-order differential equation:

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0$$

with initial conditions:

$$y(0) = 1, \frac{dy}{dt}(0)=0$$

We want to approximate the solution for t = 0.2 using the Fourth-Order Runge-Kutta method with a step size of Δt = 0.1.

**Step 1**: Define the differential equation:

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0$$

**Step 2**: Set up the initial conditions:

$$y(0) = 1, \frac{dy}{dt}(0) = 0$$

**Step 3**: Choose a step size, Δt = 0.1

**Step 4**: Perform the iterations using the Fourth-Order Runge-Kutta method:

- ✓ Iteration 1 (n = 0 to n = 1):

Initialize $t0 = 0, y0 = 1, and\ v0 = 0$ (velocity).

Calculate $k1\ \ and\ l1$ :

$$k1 = (0.1)v0 = 0$$

$$l1 = 0.1(-2v0 - 5\,y0) = 0.1(-2(0) - 5(1)) = -0.5$$

Calculate $k2\ \ and\ l2$ at the midpoint:

$$k2 = 0.1(v0 + \frac{1}{2}\ l1) = 0.1\,(0 + \frac{1}{2}\ (-0.5)) = -0.025$$

$$l2 \ = 0.1\left(-2\ \left(v0\ +\ \frac{1}{2}\ l1\right) - 5\left(\ y0\ +\ \frac{1}{2}\ k1\right)\right)$$

$$=\ 0.1\ (-2\ (\ 0\ +\ \frac{1}{2}\ (-0.5)\ )\ -5\ (\ 1\ +\ \frac{1}{2}\ \cdot\ 0\ ))\ =\ -0.45$$

Calculate $k3\ \ and\ l3$ at the midpoint:

$$k3\ =\ 0.1\ (\ v0\ +\ \frac{1}{2}\ l2\ )\ =\ 0.1\ (\ 0\ +\ \frac{1}{2}\cdot(-0.45))=-0.0225$$

$$l3\ = 0.1\cdot(-2(v0\ +\frac{1}{2}\ l2)-5(y0\ +\ \frac{1}{2}\ k2\ ))$$

$$=\ 0.1\ (\ -2(0\ +\ \frac{1}{2}\cdot(-0.45)\ )-5\ (1\ +\ \frac{1}{2}\ \ (\ -0.025\ )))\ =\ -0.44875$$

Calculate $k4$ and $l4$ at the end of the interval:

$$k4\ =\ 0.1\cdot\ (v0\ +\ l3\ )\ =\ 0.1\ \cdot\ (\ 0\ -\ 0.44875\ )\ =\ -0.044875$$

$$l4\ =\ 0.1\cdot\ (-2(v0\ +\ l3)\ -\ 5(y0\ +\ k3\ ))$$

$$=\ 0.1\ \cdot\ (-2(\ 0\ -\ 0.44875)\ -\ 5(\ 1\ -\ 0.0225))\ =\ -0.399775$$

Update $y1$ and $v1$ using the weighted average of k's and l's:

$$y1\ =\ y0\ +\frac{1}{6}\ (k1\ +\ 2k2\ +\ 2k3\ +\ k4)$$

$$=\ 1\ +\frac{1}{6}\ (\ 0\ +\ 2(-0.025)\ +\ 2(-0.0225)\ -0.044875)\ =\ 1.01$$

$$v1\ =\ v0\ +\frac{1}{6}\ (l1+\ 2\ l2\ +2\ l3\ +\ l4\ )$$

$$=\ 0\ +\frac{1}{6}\ (-0.5\ +2(-0.475)\ +\ 2(-0.471875)\ -\ 0.4640625)\ =\ 0.45$$

Increment t by Δt ($t1$ =0.1 ).

- ✓ Iteration 2 (n = 1 to n = 2):

Continue from the values obtained at the end of Iteration 1:

$$t1 \ = \ 0.1, \qquad y1 \ = \ 1.01, \qquad v1 \ = \ 0.45$$

Calculate $k1$ and 1 :

$$k1 \ = \ 0.1 \cdot v1 = 0.1 \cdot (0.45) = -0.0029427$$

$$l1 \ = \ 0.1 \cdot (-2v1 \ - 5y1) = 0.1 \cdot (-2(-0.0294271) - 5(0.9938542)) = \ -0.0203526$$

Calculate $k2$ and $l2$ at the midpoint:

$$k2 \ = \ 0.1 \cdot (\, v1 \ + \ ½\, l1) \ = \ 0.1 \cdot \ (-0.0294271 \ + \ ½ \ \cdot (-0.0203526))$$
$$= \ -0.0029117$$

$$l2 \ = \ 0.1 \cdot \big(-2(v1 \ + \ ½\, l1) - \ 5(y1 \ + \ ½\, k1)\big)$$

$$= 0.1 \cdot \left(-2 \left(-0.0294271 \ + \ \frac{1}{2}(-0.0203526)\right) - 5\left(0.9938542 + \frac{1}{2}(-0.0029427)\right)\right)$$

$$= \ -0.0202939$$

Calculate $k3$ and $l3$ at the midpoint:

$$k3 \ = \ 0.1 \cdot \left( v1 \ + \frac{1}{2}\, l2 \ \right) = 0.1 \cdot \left(-0.0294271 \ + \frac{1}{2} \cdot (-0.0202939)\right) = \ -0.0029203$$

$$l3 \ = \ 0.1 \cdot \left(-2\left(v1 \ + \frac{1}{2} l2 \ \right) - 5\left( y1 \ + \frac{1}{2} k2 \ \right)\right)$$

$$= \ 0.1 \cdot \left(-2 \left(-0.0294271 \ + \ \frac{1}{2} \cdot (\, -0.0202939)\right)\right.$$
$$\left. - \, 5\left(0.9938542 + \frac{1}{2} \cdot (-0.0029117)\right)\right)$$

$$= -0.0203308$$

Calculate $k4$ and $l4$ at the end of the interval:

$$k4 = 0.1 \cdot ( v1 + l3 ) = 0.1 \cdot (-0.0294271 - 0.0203308) = -0.0042758$$

$$l4 = 0.1 \cdot (-2(v1 + l3 ) - 5(y1 + k3 ))$$

$$= 0.1 \cdot (-2\,(-0.0294271 - 0.0203308) - 5(0.9938542 - 0.0029203)) = -0.0211236$$

Update $y_2$ and $v_2$ using the weighted average of 's and $l$ 's:

$$y_2 = y_1 + \frac{1}{6} (k_1 + 2\,k_2 + 2k_3 + k_4)$$

$$= 0.9938542 + \frac{1}{6}(-0.0029427 + 2(-0.0029117) + 2(-0.0029203) - 0.0042758)$$

$$= 1.1401$$

$$v_2 = y_1 + \frac{1}{6} (l_1 + 2\,l_2 + 2\,l_3 + l_4)$$

$$= -0.0294271 + \frac{1}{6}(-0.0203526 + 2(-0.0202939) + 2(-0.0203308) - 0.0211236)$$

$$= 0.910$$

Increment t by Δt: $t_2 = 0.1 + 0.1 = 0.2$

Now, we have completed Iteration 2, and the values are available up to $t = 0.2$

**Comparison with Euler Method in the last Runge-Kutta Example**

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0$$

with initial conditions:

$$y(0) = 1, \frac{dy}{dt}(0) = 0$$

**Step 1**: Define the differential equation:

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0$$

**Step 2**: Set up the initial conditions:

$$y(0) = 1, \frac{dy}{dt}(0) = 0$$

**Step 3**: Choose a step size, $\Delta t = 0.1$

**Step 4**: Perform the iterations using the Euler method:

- ✓ Iteration 1 ($n = 0 \; to \; n = 1$):

Initialize: $t_0 = 0, y_0 = 1, \; and \; v_0 = 0$ (where v is the first derivative $\frac{dy}{dt}$ ).

Calculate $v_1$ using the first derivative:

$$v_1 = v_0 + \Delta t \left( \frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y \right)$$

$$v_1 = 0 + 0.1(-2\,v_0 + 5y_0) = 0.1\,(-2(0) + 5(1)) = 0.5$$

Calculate $y_1$ using the updated $v_1$ :

$$y_1 = y_0 + (\Delta t \cdot v_1)$$

$$y_1 = 1 + (0.1)(0.5) = 1.05$$

Increment t by $\Delta t$ : $t_1 = 0 + 0.1 = 0.1$

- ✓ Iteration 2 (n = 1 to n = 2):

Continue from the values obtained at the end of Iteration 1:

$$t_1 = 0.1, \qquad y_1 = 1.05, \qquad v_1 = 0.5$$

Calculate $v_2$ using the first derivative:

$$v_2 = v_1 + \Delta t(\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y)$$

$$v_2 = 0.5 + 0.1(-2v_1 + 5y_1) = 0.5 + 0.1(-2(0.5) + 5(1.05)) = 0.925$$

Calculate $y_2$ using the updated $v_2$:

$$y_2 = y_1 + (\Delta t \cdot v_2)$$

$$y_2 = 1.05 + (0.1)(0.925) = 1.1425$$

In conclusion while solving using Runge Kutta $y_2 = 1.1401$ and while solving using Euler Method $y_2 = 1.1425$ which shows the accuracy of the two methods.

# 5.0 CONCLUSION AND RECOMMENDATIONS

## 5.1 Conclusion

In conclusion, the topic of "Numerical Methods for Solving Second-Order Differential Equations" is a fundamental and essential area of study in mathematics and various scientific and engineering disciplines. This topic provides valuable insights and practical techniques for approximating solutions to differential equations when analytical solutions are either complex or non-existent. Here are some key points from the discussion:

- **Importance of Second-Order Differential Equations**: Second-order differential equations play a significant role in modeling physical phenomena, from mechanical systems and electrical circuits to population dynamics and heat transfer. Solving these equations is crucial for understanding and predicting real-world behavior.
- **Numerical Methods as Problem-Solving Tools**: Numerical methods, including the Euler method and Runge-Kutta methods, provide powerful problem-solving tools for approximating solutions to second-order differential equations. These methods involve the discretization of time and the iterative calculation of values to estimate the solution.
- **Euler Method**: The Euler method is a straightforward numerical technique that serves as a foundation for understanding more complex methods. It involves taking small time steps and updating the solution based on the derivative of the dependent variable.
- **Runge-Kutta Method**: The Runge-Kutta method is a more accurate and versatile approach that employs a series of stages to calculate intermediate values and then combines them to update the solution. It is particularly well-suited for a wide range of ODEs.

## 5.2 Recommendation

Here some recommendations:

- **Structured Learning**: Consider a structured learning approach, starting with the Euler method to build a foundation and then progressing to more advanced methods like the Runge-Kutta methods. Offer courses, textbooks, or resources that guide learners through the theoretical and practical aspects of numerical methods for second-order differential equations.
- **Programming and Simulation**: Encourage learners to gain hands-on experience by implementing numerical methods in programming languages like Python or MATLAB. Provide exercises and projects that involve coding and simulating solutions to real-world problems.
- **Error Analysis**: Emphasize the importance of error analysis in numerical methods. Teach students or readers how to assess the accuracy and stability of their solutions and how to choose appropriate time step sizes.
- **Applications and Case Studies**: Showcase a variety of applications of second-order differential equations and numerical methods in different fields, such as physics, engineering, biology, finance, and climate modeling. Use case studies to illustrate the practical relevance of these methods.
- **Research Opportunities**: Encourage further research and innovation in the field of numerical methods for ODEs, especially the development of adaptive methods that can automatically adjust time step sizes for better accuracy.

## REFERENCES


[1] Ascher, U. M., and Petzold, L. R. (1998). "Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations." SIAM.

[2] Burden, R. L., and Faires, J. D. (2016). "Numerical Analysis." Cengage Learning.

[3] Butcher, J. C. (2008). "Numerical Methods for Ordinary Differential Equations." John Wiley & Sons.

[4] Hairer, E., and Wanner, G. (2002). "Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems." Springer.

[5] Lambert, J. D. (1973). "Computational Methods in Ordinary Differential Equations." John Wiley & Sons.

[6] Stoer, J., and Bulirsch, R. (2002). "Introduction to Numerical Analysis." Springer.