

**APPROXIMATION METHODS FOR SOLVING  
FIRST ORDER PARTIAL DIFFERENTIAL  
EQUATIONS**

**A SEMINAR 2 PRESENTATION**

**BY**

**ADEBISI ADEWUNMI FAITH  
(MATRIC NUMBER: 20182972)**

**DEPARTMENT OF MATHEMATICS**

**COLLEGE OF PHYSICAL SCIENCES**

**IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE  
AWARD OF BACHELOR OF SCIENCE DEGREE IN  
MATHEMATICS,**

**FEDERAL UNIVERSITY OF AGRICULTURE, ABEOKUTA.**

**SUPERVISOR: PROF. J.A OGUNTUASE**

## **CERTIFICATION**

This is to certify that this report was undertaken and submitted by **ADEBISI ADEWUNMI FAITH** with matriculation number **20182972**, a student of the department of Mathematics, College of Physical Sciences, Federal University of Agriculture, Abeokuta, for SEMINAR 2.

.....

**PROF. J.A OGUNTUASE**

**(Supervisor)**

.....

**Date**

.....

**DR. E.O. ADELEKE**

**(Head of Department)**

.....

**Date**

## Table of Contents

CERTIFICATION .....	2
1.0 INTRODUCTION .....	4
1.1 Introduction .....	4
1.2 Preliminaries and Definitions of Terms.....	6
1.2.1 Preliminaries .....	6
1.2.2 Definitions of Key Terms .....	7
1.3 Problem Section .....	9
1.3.1 Statement of the Problem .....	9
1.3.2 Motivation.....	10
1.4 Literature Review .....	12
1.4.1 Past Literature .....	13
1.5 Benefits of Solving First Order PDEs Using Approximation Methods.....	16
1.6 Objectives.....	17
2.0 DISCUSSION.....	18
2.1 Euler Method .....	18
2.1.1 Example on Euler's Theorem .....	20
3.0 CONCLUSION AND RECOMMENDATION .....	32
3.1 Conclusion.....	32
3.2 Recommendation.....	32

## 1.0 INTRODUCTION

### 1.1 Introduction

Partial Differential Equations (PDEs) are omnipresent in the mathematical description of natural phenomena, governing diverse fields such as physics, engineering, economics, and environmental science. These equations, which involve functions of multiple variables and their partial derivatives, play a pivotal role in modeling dynamic processes and predicting real-world outcomes. While analytical solutions to PDEs are attainable for a select few cases, they often elude us when confronted with intricate nonlinearities or complex boundary conditions. This is where the art and science of approximation methods come to the fore.

Many important models in physical, biological or other sciences are based on partial differential equations (PDE)(Ghiocel and Maksen, 2018). Mathematical models for continuum dynamic phenomena involve one or more partial differential equations. Partial differential equations are used to mathematically formulate, and thus aid the solution of, physical and other problems involving functions of several variables, such as the propagation of heat or sound, fluid flow, elasticity, electrostatics, electrodynamics, etc. The nonlinear PDE are in a central position because they govern a large area of complex phenomena of motion, reaction, diffusion, equilibrium, conservation, and more (Kreyszig, 1999 and Tadmor, 2012).Partial differential equations (PDE) arise in connection with various physical and geometrical problems in which the functions involved depend on two or more independent variables, on time and on one or several space variables..

Approximation methods play a crucial role in solving first-order partial differential equations (PDEs) when analytical solutions are not readily available or difficult to obtain.

Approximation methods are extensively used in various branches of mathematics, including calculus, numerical analysis, and optimization. Approximation methods play a crucial role in solving partial differential equations (PDEs), especially when analytical solutions are difficult or impossible to obtain (Liu N., 2010). One class of PDEs that often requires approximation techniques is first-order partial differential equations. These equations involve the partial derivatives of a function with respect to one or more independent variables. There are various approximation methods available for solving first-order PDEs, and the choice of method depends on the specific problem at hand and the desired level of accuracy.

Approximation methods play a crucial role in solving first-order partial differential equations (PDEs) when analytical solutions are not readily available or difficult to obtain. These methods provide numerical approximations to the solutions of PDEs, allowing us to understand and analyze the behavior of physical systems or phenomena described by these equations (Groza G., 2013). Approximation methods are extensively used in various branches of mathematics, including calculus, numerical analysis, and optimization. Approximation methods play a crucial role in solving partial differential equations (PDEs), especially when analytical solutions are difficult or impossible to obtain (Liu N., 2010). One class of PDEs that often requires approximation techniques is first-order partial differential equations. These equations involve the partial derivatives of a function with respect to one or more independent variables. There are various approximation methods available for solving first-order PDEs, and the choice of method depends on the specific problem at hand and the desired level of accuracy.

## 1.2 Preliminaries and Definitions of Terms

### 1.2.1 Preliminaries

- ✓ **Partial Differential Equation (PDE):** A partial differential equation is a mathematical equation that involves partial derivatives of a function with respect to multiple independent variables. PDEs are used to describe how a quantity or field varies in space and time. They are classified into different orders based on the highest order of derivative present.
- ✓ **First-Order Partial Differential Equation (First-Order PDE):** A first-order PDE is a PDE in which the highest-order derivative involved is a first partial derivative. It can be written in the form  $F(x, y, u, u_x, u_y) = 0$ , where  $u$  is the unknown function,  $u_x$  and  $u_y$  are its first partial derivatives with respect to  $x$  and  $y$ , and  $F$  represents the equation relating these quantities.
- ✓ **Analytical Solutions:** Analytical solutions are exact mathematical expressions or functions that satisfy a given PDE. These solutions are obtained through mathematical manipulation, such as separation of variables, the method of characteristics, or other mathematical techniques, resulting in a closed-form solution.
- ✓ **Numerical Approximation Methods:** Numerical approximation methods are computational techniques used to approximate the solutions of PDEs. These methods are employed when analytical solutions are not readily available or practical. Numerical methods divide the problem domain into discrete points and use algorithms to calculate approximate values at these points.

### 1.2.2 Definitions of Key Terms

- ❖ **Finite Difference Method (FDM):** The finite difference method is a numerical technique for solving PDEs that approximates derivatives using the differences between values at neighboring points in a grid. It discretizes the spatial domain and time, allowing for the calculation of approximate solutions by iterating through the grid.
- ❖ **Finite Element Method (FEM):** The finite element method is a numerical technique commonly used to solve PDEs by dividing the domain into smaller elements. It represents the solution as a piecewise approximation over these elements and employs variational methods to derive a system of equations that approximate the PDE.
- ❖ **Finite Volume Method (FVM):** The finite volume method is a numerical technique used to solve PDEs by dividing the domain into control volumes or cells. It focuses on the conservation of quantities within these volumes and discretizes the PDE by considering fluxes of quantities across the control volume boundaries.
- ❖ **Initial and Boundary Conditions:** PDEs typically require initial conditions (specifying the values of the solution at a starting time) and boundary conditions (describing how the solution behaves at the boundaries of the problem domain) to formulate a well-posed problem.
- ❖ **Accuracy and Convergence:** Accuracy refers to how closely a numerical solution approximates the true solution of a PDE. Convergence refers to the property of a numerical method where the solution approaches the exact solution as the grid or computational resources are refined.

- ❖ **Stability:** Stability is a crucial property of numerical methods, ensuring that small errors do not grow uncontrollably during the computation. Unstable methods can yield inaccurate and unreliable results.
- ❖ **Computational Tools:** Various software and programming languages, such as MATLAB, Python (with libraries like NumPy and SciPy), and dedicated finite element or finite difference software, are commonly used for implementing numerical methods to solve PDEs.



## 1.3 Problem Section

### 1.3.1 Statement of the Problem

Partial Differential Equations (PDEs) serve as essential tools for modeling and understanding a wide range of physical, biological, and engineering processes. Among these, first-order partial differential equations (First-Order PDEs) play a foundational role in representing phenomena such as heat conduction, fluid flow, wave propagation, and transport phenomena. While analytical methods offer exact solutions for some First-Order PDEs, their applicability becomes limited when faced with complex geometries, nonlinearities, or intricate boundary conditions.

The problem at hand arises from the need to efficiently and accurately address the following key challenges:

- I. **Lack of Analytical Solutions:** Many practical problems involving First-Order PDEs defy closed-form analytical solutions, making it imperative to develop numerical approximation methods that provide reliable results in such scenarios.
- II. **Complex Boundary Conditions:** Real-world problems often require the consideration of complex boundary conditions, which can significantly complicate the process of finding exact solutions. Numerical methods must be capable of accommodating and handling these intricacies.
- III. **Computational Efficiency:** The efficient utilization of computational resources is a critical concern, particularly when dealing with large-scale problems. Developing numerical techniques that strike a balance between accuracy and computational cost is essential.

- IV. **Adaptability to Diverse Applications:** First-Order PDEs appear in diverse fields, ranging from physics and engineering to finance and environmental science. A versatile set of approximation methods is needed to address the specific requirements of each application domain.
- V. **Stability and Convergence:** Ensuring the stability and convergence of numerical methods is paramount to prevent the amplification of errors during the computation and to guarantee the accuracy of the obtained solutions.

### 1.3.2 Motivation

Partial Differential Equations (PDEs) have proven to be indispensable tools for modeling and understanding complex, dynamic systems in various scientific and engineering disciplines. These equations describe how quantities evolve in space and time, making them invaluable for simulating physical, chemical, biological, and economic processes. Among the many types of PDEs, first-order partial differential equations (First-Order PDEs) hold a special place as fundamental building blocks for modeling a wide range of phenomena.

The motivation for this research project stems from several pressing needs and challenges in the realm of solving First-Order PDEs:

- A. **Real-World Applications:** First-Order PDEs find applications in countless real-world scenarios. For instance, they describe heat diffusion in materials, the flow of fluids through porous media, the propagation of electromagnetic waves, and the dynamics of financial markets. Accurate solutions to these equations are essential for understanding and predicting outcomes in diverse fields.

- B. Complexity and Nonlinearity:** In practical applications, the mathematical models that involve First-Order PDEs often exhibit intricate geometries, nonlinear behavior, and complex boundary conditions. These complexities render the derivation of analytical solutions infeasible or impractical, necessitating the development and utilization of numerical approximation methods.
- C. Efficiency and Scalability:** As computational resources continue to advance, there is a growing need for efficient and scalable numerical techniques. Researchers and engineers require methods that can handle increasingly large and complex problem domains while maintaining a balance between accuracy and computational cost.
- D. Interdisciplinary Impact:** First-Order PDEs are not confined to a single domain but transcend disciplinary boundaries. Whether in the context of engineering, environmental science, medical imaging, or risk analysis, these equations are ubiquitous. Therefore, advancements in approximation methods for solving First-Order PDEs have the potential to impact a wide array of fields.
- E. Innovation in Numerical Methods:** The continuous advancement of numerical methods has opened doors to innovative techniques for solving First-Order PDEs. Investigating and understanding these methods can lead to breakthroughs in computational science and engineering, ultimately benefiting society through improved problem-solving capabilities.

## 1.4 Literature Review

The problem of finding exact solutions to partial differential equations has been deeply studied in the literature. However, there is not a general method to be followed when handling a specific equation (Grasegger, et al.,2016). The authors in (Grasegger, et al.,2016) present a procedure for solving first-order autonomous algebraic partial differential equations in an arbitrary number of variables. The exact solutions for Fisher, Burger–Fisher, Benjamin–Bona–Mahony–Burgers and Modified Benjamin–Bona–Mahony are obtained in (Daghan, et al., 2010) by using -expansion method.

Numerical solutions of the ordinary differential equations (ODE) by using Taylor se-ries method have been investigated by many authors(Barrio, 2006, Barrio, et al.,2011. Neidinger, 2004, Neidinger,1992. Razzaghi and Razzaghi, 1988, Razzaghi et al.,1989) and references therein). However, there are few references on the solution of the partial differential equations (PDE) by using Taylor series method (B`ulb`ul, 2011, Groza G, 2013). One advantage of the method of using Taylor series or the polynomial approximation is that a differentiable approximate solution is obtained, which can be replaced into the equation and the initial or boundary conditions. In this manner, the accuracy of solution can be evaluated directly and the problem is reduced to that of solving a system of algebraic.

Numerical methods were first put into use as an effective tool for solving partial differential equations (PDEs) by John von Neumann in the mid-1940s. In a 1949 letter von Neumann wrote “the entire computing machine is merely one component of a greater whole, namely, of the unity formed by the computing machine, the mathematical problems that go with it, and the type of planning which is called by both.” The “greater whole” is viewed today as

scientific computation: over the past sixty years, scientific computation has emerged as the most versatile tool to complement theory and experiments, and numerical methods for solving PDEs are at the heart of many of today's advanced scientific computations. Numerical solutions found their way from financial models on Wall Street to traffic models on Main Street.

#### 1.4.1 Past Literature

- Gamet, L. and Ducros (1999) in their research paper they studied development of a fourth order compact scheme for approximation of first derivatives on non uniform grids. They present numerical analysis of truncation error. Convection equation for first derivative and diffusion equation for second derivative is considered. The ability of nonuniform mesh generalization of compact schemes is demonstrated to reproduce result.
- Abarbanel, S. and Ditkowski, A. (2000) in their research paper, temporal behavior and rate of convergence of error bounds of finite difference approximations to partial differential equations is studied. They determined dependence of the error bounds on mesh size and time .For this purpose hyperbolic and parabolic partial differential equations are used.
- Mickens, R. (2001) this paper is an introduction to non standard finite difference methods, which are useful to construct differential equations. In his paper, he described exact finite difference scheme, also rules for constructing non standard scheme with its application.

- Fukagata, K. and Kesagi, N. (2002) they developed highly energy conservative finite difference method for cylindrical system. They proved that when approximate interpolation schemes are used then energy conservation in discretized space is satisfied. This holds for both equally and unequally spaced mesh on cylindrical coordinate system but not on Cartesian coordinates.
- Farjadpur, A. and Roundy (2006) finite difference time domain method suffer from reduced accuracy due to discretization, for modeling discontinuous dielectric materials. They show that accuracy can be improved by using sub pixel smoothing, if it is properly designed. Also this scheme attains quadratic convergence.
- Zhong, Q. and Zhi, L. (2006) In their research paper, They proposed numerical methods for solving non-linear Poisson-Boltzmann equation  $\Delta\psi = \sinh\psi$ , where  $\psi$  is the electrostatic potential. A monotone iterative method was given for semi-linear partial differential equation of elliptic type. The modified central finite difference scheme is introduced. Numerical solutions agree with solutions obtained by adaptive finite element method.
- Thankane, K.S. and Stys, T. (2009) in their research article, they present effective algorithms based on finite difference method for linear and non linear beam equations. Also they give the analysis of convergence of the algorithms. Solution of number of beam equations is given by designing Mathematica Module.
- McGee, S. and Seshaiyer, P. (2009) in their research paper application of finite difference methods for coupled flow interaction transport models are given. They considered a coupled two dimensional model with transient Navier-Stokes equation to model the blood flow in the vessel and Darcy's flow to model the plasma flow through the vessel wall. The advection –diffusion equation is coupled with the velocities from the flows in the

vessel and wall. The coupled chemical transport equations are discretized by the finite difference method and solved by using additive Schwarz method.

- Dolicanin, C.B. and Nicolice, V.B. (2010) in their research paper, finite difference method is used to study of phenomenon in the theory of thin plates. FDM based on replacing differential equation into difference equation. This method can efficiently solve the problem of bending of thin plates. It is used to find solutions for the plate deflection, moments, stress, strain etc.
- Islam, M.R. and Alias, N. (2010) Finite difference method is used to discretise a parabolic partial differential equation. They presented a mathematical simulation model using one dimensional parabolic equation. This model is regarding to moisture and temperature behavior of tropical herbs during dehydration. here Jacobi, Gauss seidal and Red black Gauss seidal iterative methods are studied. It has proved that dehydration model is capable to simulate mass and temperature distribution through numerical methods approach. This mathematical simulation is time consuming and capable to reduce the risk of real experiments in actual process.

## 1.5 Benefits of Solving First Order PDEs Using Approximation Methods

Solving First Order Partial Differential Equations (First Order PDEs) using approximation methods offers numerous benefits across various fields of science and engineering. These benefits make approximation methods a valuable tool for researchers, engineers, and practitioners. Here are some of the key advantages:

- ✓ Versatility: Approximation methods are versatile and can be applied to a wide range of problems involving First Order PDEs. They are not limited to specific geometries or boundary conditions, making them suitable for various real-world scenarios.
- ✓ Applicability: Many real-world problems are described by First Order PDEs, including heat conduction, fluid flow, wave propagation, and transport phenomena. Approximation methods allow for the effective modeling and analysis of these phenomena, facilitating problem-solving in diverse fields.
- ✓ Complex Geometries: Approximation methods, such as the Finite Element Method (FEM), excel in handling problems with complex geometries or irregular boundaries. They can adapt to the intricacies of the physical domain, enabling accurate simulations and analyses.
- ✓ Numerical Stability: When implemented correctly, numerical approximation methods are designed to be numerically stable. This means that they can produce reliable results even when dealing with problems that involve small-scale variations or challenging boundary conditions.
- ✓ Computational Efficiency: Approximation methods can leverage the power of modern computational resources, allowing for efficient numerical solutions. This is particularly



important for solving large-scale or time-dependent problems that may be infeasible to tackle analytically.

- ✓ Scalability: These methods are scalable, meaning they can handle problems of varying sizes and complexities. Researchers can adapt them to suit the computational resources available and the specific requirements of their problem.

## 1.6 Objectives

- To Explore Various Numerical Approximation Methods
- To Compare and Contrast Numerical Methods
- To Develop Computational Skills
- Gain To Investigate Stability and Convergence

## 2.0 DISCUSSION

### 2.1 Euler Method

#### Euler's Method: An Introduction

Euler's method, named after the Swiss mathematician Leonhard Euler, is a simple yet fundamental numerical technique used to approximate solutions to ordinary differential equations (ODEs). ODEs are essential in modeling a wide range of dynamic systems in science and engineering, from physics and biology to economics and engineering. Euler's method provides an iterative approach to estimate the values of an unknown function at discrete points in time or space.

#### Key Concepts of Euler's Method

- ✓ **First-Order ODEs:** Euler's method is primarily applicable to first-order ODEs, which involve the derivative of an unknown function with respect to one independent variable.
- ✓ **Discretization:** To apply Euler's method, we discretize the independent variable (e.g., time) into small time steps ( $\Delta t$ ). The smaller the time step, the more accurate the approximation.
- ✓ **Approximation of Derivatives:** Euler's method estimates the derivative of the function at a given point by evaluating it at that point. This approximation assumes that the derivative remains relatively constant over the small time step.

- ✓ **Iterative Updates:** The method iteratively updates the function's value based on the previous value and the estimated derivative. It "steps" through the domain of interest, accumulating the values of the function at each time step.

### **Mathematical Formulation**

For a first-order ODE of the form:

$$\frac{dy}{dt} = f(t, y)$$

where:

$y$  is the unknown function.

$t$  is the independent variable (e.g., time).

$f(t, y)$  is a function that defines the rate of change of  $y$  at a given point.

Euler's method can be expressed as:

$$y_{n+1} = y_n + \Delta t \cdot f(t_n, y_n)$$

where:

$y_n$  is the approximate value of  $y$  at time  $t_n$

$t_n$  is the time at step  $n$ .

$y_{n+1}$  is the estimated value of  $y$  at time

$\Delta t$  is the time step size.

### 2.1.1 Example on Euler's Theorem

#### **Euler's Method for Population Growth**

##### **Problem Statement:**

Suppose we have a population of bacteria that grows at a rate proportional to its current size. We want to model the population's growth over time using the following first-order ODE:

$$\frac{dP}{dt} = k \cdot P$$

where:

P is the population size.

t is time.

k is the growth rate constant.

#### **Euler's Method Implementation**

Let's assume:

Initial population,  $P(0) = 100$

Growth rate constant,  $k = 0.2$

Time step size,  $\Delta t = 0.1$

**Iteration 1 (t = 0.1 seconds):**

Using Euler's method:

$$P(0.1) = P(0) + \Delta t \cdot (k \cdot P(0))$$

$$= 100 + 0.1 \cdot (0.2 \cdot 100)$$

$$= 120$$

So, at  $t = 0.1$  seconds, the estimated population is 120.

**Iteration 2 (t = 0.2 seconds):**

$$P(0.2) = P(0.1) + \Delta t \cdot (k \cdot P(0.1))$$

$$= 120 + 0.1 \cdot (0.2 \cdot 120)$$

$$= 144$$

At  $t = 0.2$  seconds, the estimated population is 144.

**Iteration 3 (t = 0.3 seconds):**

Next, we calculate the population at  $t=0.3$  seconds:

$$P(0.3) = P(0.2) + \Delta t \cdot (k \cdot P(0.2))$$

$$= 144 + 0.1 \cdot (0.2 \cdot 144)$$

$$= 172.8$$

At  $t=0.3$  seconds, the estimated population is approximately 172.8.

**Iteration 4 (t = 0.4 seconds):**

We continue by calculating the population at t = 0.4 seconds:

$$\begin{aligned}P(0.4) &= P(0.3) + \Delta t \cdot (k \cdot P(0.3)) \\&= 172.8 + 0.1 \cdot (0.2 \cdot 172.8) \\&= 207.36\end{aligned}$$

At t = 0.4 seconds, the estimated population is approximately 207.36.

**Iteration 5 (t = 0.5 seconds):**

Now, we calculate the population at t = 0.5 seconds:

$$\begin{aligned}P(0.5) &= P(0.4) + \Delta t \cdot (k \cdot P(0.4)) \\&= 207.36 + 0.1 \cdot (0.2 \cdot 207.36) \\&= 248.832\end{aligned}$$

At t = 0.5 seconds, the estimated population is approximately 248.832.

**Iteration 6 (t = 0.6 seconds):**

Finally, we calculate the population at t = 0.6 seconds:

$$\begin{aligned}P(0.6) &= P(0.5) + \Delta t \cdot (k \cdot P(0.5)) \\&= 248.832 + 0.1 \cdot (0.2 \cdot 248.832) \\&= 298.5984\end{aligned}$$

At  $t = 0.6$  seconds, the estimated population is approximately 298.5984.

You can use these detailed iterations to understand how Euler's method approximates the population growth at each time step. This technique is particularly useful for modeling dynamic systems when analytical solutions are not readily available.

## 2.2 Runge-Kutta Method

The Runge-Kutta method is a numerical technique used for solving ordinary differential equations (ODEs) and is particularly effective for solving initial value problems. It's a family of numerical integration methods that are widely used because of their accuracy and ease of implementation. The method was developed by German mathematicians Carl Runge and Martin Kutta in the late 19th and early 20th centuries.

### Here's an overview of the Runge-Kutta method

**Background:** The Runge-Kutta method is used to approximate the solution of a first-order ordinary differential equation of the form:

$$\frac{dy}{dt} = f(t, y)$$

where :

t is the independent variable (usually time),

y is the dependent variable, and

f(t,y) is a known function that describes the rate of change of y with respect to t.

**K4 can be expressed as follows for a single time step:**

$$K_1 = \Delta t \cdot f(t_n, y_n)$$

$$K_2 = \Delta t \cdot f\left(t_n + \frac{1}{2} \Delta t, y_n + \frac{1}{2} K_1\right)$$

$$K_3 = \Delta t \cdot f\left(t_n + \frac{1}{2} \Delta t, y_n + \frac{1}{2} K_2\right)$$

$$K_4 = \Delta t \cdot f(t_n + \Delta t, y_n + K_3)$$



$$y_{n+1} = y_n + \frac{1}{6} (K_1 + 2 K_2 + 2 K_3 + K_4)$$

where:

$y_n$  is the approximate value of  $y$  at time  $t_n$

$y_{n+1}$  is the estimated value of  $y$  at time  $t_{n+1}$

$K_1$ ,  $K_2$ ,  $K_3$ , and  $K_4$  are intermediate values representing the rate of change of  $y$  at different stages within the time step.

**General Idea:** The method works by breaking down the time interval into discrete steps and approximating the change in  $y$  over each step. It then updates the value of  $y$  at each step to iteratively compute the solution.

**Accuracy:** RK4 is a fourth-order method, which means that its error decreases with step size to the fourth power. This makes it more accurate than simpler methods like the Euler method for the same step size.

**Advantages:**

- RK4 is relatively easy to implement and is suitable for a wide range of differential equations.
- It provides good accuracy, making it a popular choice for numerical simulations.
- The method is stable for many types of problems.

**Limitations:**

- RK4 can be computationally expensive for very small step sizes, especially in high-dimensional systems.

- It may not be suitable for stiff differential equations, where the solution changes rapidly.

In summary, the Runge-Kutta method, particularly the fourth-order RK4 variant, is a versatile and widely used technique for numerically solving ordinary differential equations. It offers a good balance between accuracy and computational efficiency, making it a valuable tool in various scientific and engineering applications

### 2.2.1 Modeling The Cooling Of A Hot Cup Of Coffee

#### **Problem Statement:**

Suppose we have a cup of coffee initially at a temperature of 80°C, and it's placed in a room with a constant temperature of 25°C. The rate at which the coffee cools down follows the first-order ODE:

$$\frac{dT}{dt} = -k \cdot (T - T_{\text{room}})$$

where:

T is the temperature of the coffee at time t.

T<sub>room</sub> is the room temperature (25°C).

k is the cooling rate constant.

#### **RK4 Implementation:**

Initialization:

T (0) = 80 °C (initial temperature)

T<sub>room</sub> = 25 °C (room temperature)

$K = 0.1$  (cooling rate constant)

$\Delta t = 0.5$  (time step size)

**Iteration 1 (t = 0.5 seconds):**

At  $t = 0.5$  seconds, we estimate  $T(0.5)$  using the RK4 method:

$$K_1 = \Delta t \cdot (-k \cdot (T(0) - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (80 - 25)) = -2.75$$

$$K_2 = \Delta t \cdot (-k \cdot (T(0) + 0.5 \cdot K_1 - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (80 + 0.5 \cdot (-2.75) - 25)) = -2.205$$

$$K_3 = \Delta t \cdot (-k \cdot (T(0) + 0.5 \cdot K_2 - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (80 + 0.5 \cdot (-2.205) - 25)) = -2.09175$$

$$K_4 = \Delta t \cdot (-k \cdot (T(0) + K_3 - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (80 - 2.09175 - 25)) = -2.0041625$$

Update  $T(0.5)$  using these values:

$$T(0.5) = T(0) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 54.0375$$

At  $t = 0.5$  seconds, the estimated coffee temperature is approximately  $54.04^\circ\text{C}$ .

**Iteration 2 (t = 1.0 seconds):**

At  $t = 1.0$  seconds, we estimate  $T(1.0)$  using the RK4 method:

$$K_1 = \Delta t \cdot (-k \cdot (T(0.5) - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (54.0375 - 25)) = -1.9443125$$

$$K_2 = \Delta t \cdot (-k \cdot (T(0.5) + 0.5 \cdot K_1 - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (54.0375 + 0.5 \cdot (-1.9443125) - 25)) = -1.56458125$$

$$K_3 = \Delta t \cdot (-k \cdot (T(0.5) + 0.5 \cdot K_2 - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (54.0375 + 0.5 \cdot (-1.56458125) - 25)) = -1.4971315625$$

$$K_4 = \Delta t \cdot (-k \cdot (T(0.5) + K_3 - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (54.0375 - 1.4971315625 - 25)) = -1.45001296875$$

Update T(1.0) using these values:

$$T(1.0) = T(0.5) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 44.22698703125$$

At t=1.0 seconds, the estimated coffee temperature is approximately 44.23 °C.

### Iteration 3 (t = 1.5 seconds):

At t = 1.5 seconds, we estimate T(1.5) using the RK4 method:

$$K_1 = \Delta t \cdot (-k \cdot (T(1.0) - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (80 - 25)) = -2.75$$

$$K_2 = \Delta t \cdot (-k \cdot (T(1.0) + 0.5 \cdot K_1 - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (80 + 0.5 \cdot (-2.75) - 25)) = -2.205$$

$$K_3 = \Delta t \cdot (-k \cdot (T(1.0) + 0.5 \cdot K_2 - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (80 + 0.5 \cdot (-2.205) - 25)) = -2.09175$$

$$K_4 = \Delta t \cdot (-k \cdot (T(1.0) + K_3 - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (80 - 2.09175 - 25)) = -2.0041625$$

Update T(1.5) using these values:

$$T(1.5) = T(1.0) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 30.3968375$$

At t = 1.5 seconds, the estimated coffee temperature is approximately 30.40 °C.

**Iteration 4 (t = 2.0 seconds):**

At t = 2.0 seconds, we estimate T (2.0) using the RK4 method:

$$K_1 = \Delta t \cdot (-k \cdot (T(1.5) - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (30.3968375 - 25)) = -0.29016375$$

$$K_2 = \Delta t \cdot (-k \cdot (T(1.5) + 0.5 \cdot K_1 - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (30.3968375 + 0.5 \cdot (-0.29016375) - 25)) = -0.217929625$$

$$K_3 = \Delta t \cdot (-k \cdot (T(1.5) + 0.5 \cdot K_2 - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (30.3968375 + 0.5 \cdot (-0.217929625) - 25)) = -0.2064840625$$

$$K_4 = \Delta t \cdot (-k \cdot (T(1.5) + K_3 - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (30.3968375 - 0.2064840625 - 25)) = -0.19896453125$$

Update T (2.0) using these values:

$$T(2.0) = T(1.5) + \frac{1}{6} (K_1 + 2 K_2 + 2 K_3 + K_4) = 28.6826828125$$

At t = 2.0 seconds, the estimated coffee temperature is approximately 28.68 °C.

**Iteration 5 (t = 2.5 seconds):**

At t = 2.5 seconds, we estimate T (2.5) using the RK4 method:

$$K_1 = \Delta t \cdot (-k \cdot (T(2.0) - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (28.6826828125 - 25)) = -0.1706828125$$

$$K_2 = \Delta t \cdot (-k \cdot (T(2.0) + 0.5 \cdot K_1 - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (28.6826828125 + 0.5 \cdot (-0.1706828125) - 25)) = -0.127470390625$$

$$K_3 = \Delta t \cdot (-k \cdot (T(2.0) + 0.5 \cdot K_2 - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (28.682682 + 0.5 \cdot (-0.127470390625) - 25)) = -0.1207666828125$$

$$K_4 = \Delta t \cdot (-k \cdot (T(2.0) + k_3 - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (28.68268 - 0.1207666 - 25)) = -0.116407670703125$$

Update T(2.5) using these values:

$$T(2.5) = T(2.0) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 27.85704515625$$

At t=2.5 seconds, the estimated coffee temperature is approximately 27.86 °C.

### **Iteration 6 (t = 3.0 seconds):**

Finally, at t=3.0 seconds, we estimate T (3.0) using the RK4 method:

$$K_1 = \Delta t \cdot (-k \cdot (T(2.5) - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (27.85704515625 - 25)) = -0.139535228125$$

$$K_2 = \Delta t \cdot (-k \cdot (T(2.5) + 0.5 \cdot K_1 - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (27.85704 + 0.5 \cdot (-0.13953) - 25)) = -0.10482392109375$$

$$K_3 = \Delta t \cdot (-k \cdot (T(2.5) + 0.5 \cdot K_2 - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (27.857045 + 0.5 \cdot (-0.10482392) - 25)) = -0.0993200546875$$

$$K_4 = \Delta t \cdot (-k \cdot (T(2.5) + K_3 - T_{room}))$$

$$= 0.5 \cdot (-0.1 \cdot (27.857045 - 0.0993200 - 25)) = -0.095923176171875$$

Update T(3.0) using these values:

$$T(3.0) = T(2.5) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 27.3998568328125$$

At t=3.0 seconds, the estimated coffee temperature is approximately 27.40 °C.

These calculations provide a detailed understanding of how the coffee's temperature decreases over time due to its cooling rate.



## **3.0 CONCLUSION AND RECOMMENDATION**

### **3.1 Conclusion**

In conclusion, the study of approximation methods for solving first-order partial differential equations (PDEs) provides valuable insights into their practical applications, and the detailed example of Euler's method demonstrates its utility in solving real-world problems involving dynamic processes.

### **3.2 Recommendation**

- Researchers and practitioners should explore advanced numerical techniques, such as the fourth-order Runge-Kutta method, to enhance the accuracy of approximations for first-order PDEs.
- The development of user-friendly software tools for solving first-order PDEs can facilitate their widespread use in various scientific and engineering disciplines.
- Educational institutions should offer courses and training in numerical methods for solving PDEs to equip students and professionals with essential skills.



## REFERENCES

- Kreyszig E.(1999) Advanced Engineering Mathematics (eighth ed.), Wiley.
- Tadmor, E. (2012). A review of numerical methods for nonlinear partial differential equations. Bulletin of the American Mathematical Society, 49(4), 507-554.
- Lastra, A., Sendra, J. R., & Sendra, J. (2023). Symbolic Treatment of Trigonometric Parameterizations: The General Unirational Case and Applications. Communications in Mathematics and Statistics, 1-25.
- Barrio, R. (2006). Sensitivity analysis of ODEs/DAEs using the Taylor series method. SIAM Journal on Scientific Computing, 27(6), 1929-1947.
- Barrio, R., Rodr'iguez, M., Abad, A., & Blesa, F. (2011). Breaking the limits: the Taylor series method. Applied mathematics and computation, 217(20), 7940-7954.
- Neidinger, R. (2005). Directions for computing truncated multivariate Taylor series. Mathematics of computation, 74(249), 321-340.
- Groza G., Razzaghi M. (2013). A Taylor series method for the solution of the linear initial-boundary-value problems for partial differential equations.Comput. Math. Appl., 66 (7) , pp. 1329-1343
- Dehghan M.(2005). On the solution of an initial-boundary value problem that combines Neumann and integral condition for the wave

equation Numer. Methods Partial Differential Equations, 21 (1) , pp. 24-40

- Dehghan M., Shokri A.(2008). A numerical method for one-dimensional nonlinear Sine-Gordon equation using collocation and radial basis functions Numer. Methods Partial Differential Equations, 24 (2) , pp. 687-698
- Llavona J.G.(1986). Approximation of Continuously Differentiable Functions North-Holland Mathematics Studies 130, New York
- He Ji-Huan.(1999) Variational iteration method - a kind of non-linear analytical technique: some examples Internat. J. Non-Linear Mech., 34 , pp. 699-708
- Momani S., Odibat Z.(2008). A novel method for nonlinear fractional partial differential equations: Combination of DTM and generalized Taylor's formula J. Comput. Appl. Math., 220 , pp. 85-95