WAVE EQUATION AND ITS APPLICATIONS

BY

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MATRICULATION NUMBER: 20183060

A PROJECT SUBMITTED TO THE

DEPARTMENT OF MATHEMATICS,

COLLEGE OF PHYSICAL SCIENCES,

FEDERAL UNIVERSITY OF AGRICULTURE, ABEOKUTA.

IN PARTIAL FULFILMENT FOR THE AWARD OF BACHELOR OF SCIENCE DEGREE IN MATHEMATICS.

NOVEMBER, 2023.

DECLARATION

I **OLALERE, BABATUNDE** hereby declare that the final year project Titled: "**Wave Equation and Its Applications**" submitted by me, in partial fulfillment of the requirements for the award of Bachelor of Science Degree in Mathematics, Federal University of Agriculture, Abeokuta. I declare that all external sources and references used in this project have been properly acknowledged and cited.

OLALERE, BABATUNDE

Date:.................................

CERTIFICATION

This is to certify that the work Titled: "**Wave Equation and Its Applications**" was carried out by **OLALERE, BABATUNDE** with **Matriculation Number: 20183060,** a student of the Department of Mathematics, College of Physical Sciences, Federal University of Agriculture, Abeokuta, Ogun State, Nigeria.

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DEDICATION

I dedicate this work to Almighty Allah, who made this project completion a reality. I dedicate this work also to my beloved parents, Late Mr Gbolagade Olalere and Mrs Kehinde Olalere, and to everyone that has been supportive and helpful in my education life.

ACKNOWLEDGMENT

All glory, honour and adoration is to the Almighty Allah who has made the success of this work and the completion of my BSc. programme at large a reality.

I would like to express my gratitude and appreciation to my supervisor, Prof. M. O. Omeike whose help in stimulating suggestion and encouragement helped in the process of completing this project.

I am grateful to the Acting Head of Department, Dr. E. O. Adeleke, and all lecturers of the Department of Mathematics.

My sincere appreciation also goes to my mother, Mrs Kehinde Olalere, for her full support, advice, prayer, love and care placed on me throughout this project period and my stay on campus at large.

My profound appreciation also goes to my wonderful siblings (Olalere Abidemi, Olalere Abimbola, Ahmed Firdaus, Abolade Babatunde, Olalere Abiola) for their contribution physically, spiritually, financially towards the success of my programme. I pray that almighty God take them to higher grounds.

Finally, my sincere gratitude also goes to all those who have contributed to my success in FUNAAB: my friends (Noibi Idris, Adeyemo Jumat, Raji Sabur, Omoleye Julius), departmental mates (Bello Quadri, Raji Muslimah, Oladeji Olayinka) and many others that i couldn't mention their names. Thank you all and God bless you. (AMEN).

ABSTRACT

This study investigates the applications of the wave equation with a primary focus on the vibrations of a stretched string, employing the method of separation of variables as the principal methodology. Through a comprehensive exploration, the research uncovers the fundamental characteristics and solutions provided by the wave equation in describing the transverse vibrations of a string. The study embarks on a detailed analysis, revealing the interplay between spatial and temporal components of the string's oscillations by employing the method of separation of variables.

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1.0 INTRODUCTION

1.1 Background to the Study

Waves are all around us, from ripples on water to sound in the air. They follow certain rules that scientists describe using something called the "wave equation." This special equation helps us understand how waves move and change through different things like strings, air, or even the ground.

From the gentle rustling of leaves in the wind to the roaring crescendo of ocean waves crashing on the shore, from the harmonious melodies of music to the transmission of information through wireless communication, waves are fundamental to our understanding of nature and the technologies that shape our lives.

The wave equation isn't just a bunch of confusing math. It's like a magic tool that scientists and engineers use to understand and solve problems in many areas. For example, it helps them design musical instruments, figure out how light travels, or even find resources underground using seismic waves.

In this exploration, we embark on a journey through the waves—both literal and figurative—that the wave equation unveils. We will delve into the mathematical intricacies of this equation, uncover its elegant solutions, and witness its transformative impact on fields as varied as physics, engineering, medicine, and more. The wave equation, at its essence, is a key to understanding the world around us and a tool to shape the world we imagine.

This research is all about exploring this cool wave equation and how it helps us in real life. We'll see how it explains things like vibrations in a guitar string and how it's used in technologies like ultrasound for medical imaging. It's like a secret code that tells us how waves behave and how we can use this knowledge to make amazing things.

1.2 Motivation

Understanding waves and their behavior through the study of the wave equation and its applications is really important. Waves are everywhere, from music to how earthquakes move through the ground. By exploring the math behind these waves, we can figure out how they work in things like music and technologies like ultrasounds used in hospitals.

The motivation to study the wave equation comes from how it helps us in real life. It's not just about the theory of waves—it's about how this knowledge can be used to create new technologies and understand things that affect our daily lives, like earthquakes or how doctors look inside our bodies. Learning about waves using the wave equation isn't just interesting; it's like a secret code that helps us understand and improve the world around us.

1.3 Objectives

- ✓ To explore the mathematical foundations of the wave equation, including its derivation, properties, and solutions in different physical contexts.
- ✓ To examine the diverse applications of the wave equation in various scientific disciplines, such as physics, engineering, medicine, and geophysics.
- ✓ To investigate and understand how the wave equation can be adapted to analyze complex wave phenomena in heterogeneous environments, such as biological tissues and geological formations.

1.4 Definition of Terms

Before delving into the intricacies of the wave equation and its applications, it's essential to establish a foundational understanding of key terms and concepts that will be central to our exploration.

- ✓ **Wave**: A wave is a disturbance or oscillation that propagates through a medium or space, transferring energy without the physical displacement of matter. Waves can take various forms, including mechanical waves (e.g., sound waves), electromagnetic waves (e.g., light and radio waves), and more.
- ✓ **Medium**: The medium refers to the substance or material through which a wave travels. It can be a solid (e.g., a rope for mechanical waves), a liquid (e.g., water for ocean waves), or even a vacuum (for electromagnetic waves).
- ✓ **Wave Equation**: At its core, the wave equation is a mathematical representation that describes how waves behave and propagate through a given medium or space. It is a partial differential equation that relates the second derivative of a wave function with respect to both time and space coordinates.
- ✓ **Time Domain and Spatial Domain**: When dealing with waves, it's common to analyze them in both the time domain (how the wave evolves over time) and the spatial domain (how the wave varies in space). These domains are essential for understanding the complete behavior of waves.
- ✓ **Wave Speed**: The wave speed (often denoted as "c") is a fundamental property of a wave and represents how fast the wave propagates through the medium. It depends on the properties of the medium, such as its density and elasticity.

- ✓ **Wavelength**: The wavelength (denoted as "λ") is the spatial period of a wave, representing the distance between two successive points in a wave that are in phase (e.g., two consecutive crests or troughs).
- ✓ **Frequency**: The frequency (denoted as "f") of a wave is the number of oscillations or cycles that occur per unit of time. It is inversely proportional to the wavelength and determines the pitch (for sound waves) or color (for light waves) of the wave.
- ✓ **Amplitude**: The amplitude of a wave (denoted as "A") represents the maximum displacement of a point on the wave from its equilibrium position. It is a measure of the wave's intensity or strength.
- ✓ **Phase**: The phase of a wave describes the relative position of a point on the wave with respect to a reference point. It is often expressed in radians or degrees.
- ✓ **Boundary Conditions**: In the context of the wave equation, boundary conditions are constraints that specify how the wave behaves at the boundaries of the medium or the region of interest. They play a crucial role in determining the solutions to the wave equation.

2.0 LITERATURE REVIEW

Introduction

The wave equation, a fundamental mathematical concept in physics and engineering, has long been a cornerstone for understanding and analyzing wave phenomena. This literature review explores the rich landscape of research and applications related to the wave equation, shedding light on its historical development, mathematical underpinnings, and multifaceted applications across diverse scientific disciplines.

Historical Evolution and Mathematical Foundations

The origins of the wave equation can be traced back to the pioneering work of Jean le Rond d'Alembert and Leonhard Euler in the 18th century. D'Alembert's principle and Euler's contributions laid the groundwork for formulating the wave equation as a partial differential equation (PDE) that describes the dynamics of waves propagating through space and time [(D'Alembert, 1747); (Euler, 1744)]. The wave equation's mathematical elegance and universality continue to captivate mathematicians and physicists to this day.

Analytical Solutions and Mathematical Techniques

Over the centuries, mathematicians have developed analytical solutions and mathematical techniques to solve the wave equation for various boundary conditions and media. The method of separation of variables has been pivotal in decomposing complex waveforms into simpler components. Fourier analysis, introduced by Jean-Baptiste Joseph Fourier, revolutionized our ability to understand wave behavior through spectral decomposition [(Fourier, 1822)]. These mathematical tools have become indispensable in diverse applications, from signal processing to quantum mechanics.

Applications in Physics and Engineering

The wave equation finds prolific applications in the physical and engineering sciences. Acoustics harnesses the wave equation to explore sound wave propagation, leading to innovations in audio technology and architectural acoustics [(Rayleigh, 1877)]. Electromagnetism relies on the wave equation to describe and predict the behavior of electromagnetic waves, thereby advancing wireless communication and optics [(Jackson, 1999)]. In structural engineering, the wave equation helps scrutinize vibrations in buildings and bridges, ensuring their stability and safety [(Den Hartog, 1985)].

Medical and Biological Applications

The wave equation's significance extends to medicine and biology. Ultrasound imaging, magnetic resonance imaging (MRI), and computed tomography (CT) scans all rely on the principles of wave propagation described by the wave equation. These medical imaging technologies have revolutionized diagnostics and patient care [(Hoskins & Martin, 2015)]. In biology, the wave equation is used to model various biological processes, including nerve impulse transmission and wave behavior within biological tissues [(Levin & Mangel, 2005)].

Oceanography and Geophysics

In oceanography and geophysics, the wave equation serves as a powerful tool for understanding and predicting a range of natural phenomena. It provides insights into ocean waves, tsunamis, seismic waves, and geological dynamics, contributing to disaster preparedness and environmental conservation [(Dziewonski & Anderson, 1981)].

Quantum Mechanics and Quantum Field Theory

At the quantum level, the wave equation takes on a distinct character. The Schrödinger equation, a type of wave equation, is central to understanding quantum particle behavior and the wave-particle duality [(Griffiths, 2005)]. In quantum field theory, the wave equation plays a foundational role in the study of quantum fields and their interactions [(Peskin & Schroeder, 1995)].

Emerging Trends and Future Directions

As scientific boundaries continue to expand, the wave equation remains a dynamic field of study. Emerging trends include the development of more efficient numerical methods, the application of wave-based technologies in quantum computing, and the exploration of novel materials with unique wave properties.

3.0 METHODOLOGY

3.1 Method of Separation of Variables

The method of separation of variables is a powerful mathematical technique used to solve partial differential equations (PDEs) by assuming a solution that is a product of simpler functions, each dependent on only one of the independent variables. This method is especially useful in solving linear partial differential equations, including the wave equation, heat equation, and Laplace's equation.

PRINCIPLE OF THE METHOD

The Form of the Solution: The method of separation of variables assumes that a solution to a partial differential equation can be expressed as a product of simpler functions, each depending on only one independent variable. For example, in the case of the wave equation, this assumption leads to a solution in terms of spatial and temporal functions.

Assuming a Separable Solution: Consider a partial differential equation

$$F(x, y) = X(x) \cdot Y(y)$$

This expression suggests that the solution F can be separated into two functions, $X(x)$ and $Y(y)$, each depending on a single independent variable.

APPLICATION TO THE WAVE EQUATION

The one-dimensional wave equation is expressed as:

$$\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2} \ .$$

To solve this equation using separation of variables, the assumption is made that the solution can be written as

$$u(x,t) = X(x) \cdot T(t) \ .$$

where $X(x)$ is a function of space and $T(t)$ is a function of time.

THE PROCEDURE

Applying the Separation: Substitute the assumed solution $u(x,t) = X(x) \cdot T(t)$ into the wave equation. Separate the variables by grouping terms related to x and t on either side of the equation.

Obtaining Two Ordinary Differential Equations: This separation leads to two separate ordinary differential equations: one involving the spatial variable $(X(x))$ and the other involving the temporal variable $(T(t))$.

Solving the Spatial Equation: Solve the spatial equation to find the spatial part of the solution $(X(x))$ by applying boundary or initial conditions, depending on the specific problem.

Solving the Temporal Equation: Solve the temporal equation to find the temporal part of the solution $(T(t))$ by applying initial conditions.

Combining Solutions: The general solution is obtained by combining the spatial and temporal solutions, satisfying the initial or boundary conditions of the problem.

LIMITATIONS AND CONSIDERATIONS

Boundary Conditions: The method of separation of variables often requires specific boundary conditions to solve the separated ordinary differential equations, which can limit its application to certain well-defined problems.

Applicability: The method is primarily applicable to linear partial differential equations with certain characteristics that allow separation.

Assumptions: The assumption of separable solutions might not hold for all partial differential equations, especially for non-linear or non-separable equations.

SIGNIFICANCE AND APPLICATIONS

The method of separation of variables plays a pivotal role in solving a wide range of partial differential equations encountered in various fields, including physics, engineering, and mathematical modeling. Its applications extend to problems involving wave propagation, heat transfer, quantum mechanics, and more.

This method provides a systematic and structured approach to solving PDEs, offering valuable insights into the behavior of complex systems governed by partial differential equations.

4.0 APPLICATIONS

4.1 Illustrative Examples

4.1.1 Application of Wave Equation in a Stretched String

Analyze the vibrations of a stretched string using the wave equation

Method of Separation of Variables

Consider a one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{C^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{------------ (*)}$$

Let the solution to (*) be

$$U(x,t) = X(x)T(t).$$

Differentiating ….

$$U_x = X'(x)T(t).$$

$$U_{xx} = X''(x)T(t).$$

$$U_t = X(x)T'(t).$$

$$U_{tt} = X(x)T''(t).$$

Substituting back into (*), we have

$$X''(x)T(t) = \frac{1}{C^2} X(x)T''(t).$$

Equating our new equation to parameter β,

$$X''(x)T(t) = \frac{1}{C^2} X(x)T''(t) = \beta.$$

So, we have the below two equations:

$$C^2 \ \frac{X''(x)}{X(x)} = \beta \qquad ------- (**)$$

$$\frac{T''(t)}{T(t)} = \beta \qquad -------- \ (***)$$

Solving $(**)$

$$C^2 \, X''(x) - \beta \, X(x) \ = \ 0.$$

Auxiliary equation is

$$m^2 - \frac{\beta}{C^2} = 0$$

$$m \ = \ \pm\sqrt{\frac{\beta}{C^2}} = \ \pm \frac{1}{c}\sqrt{\beta} \, .$$

General Solution:

$$X(x) = \ Ae^{\frac{1}{c}\sqrt{\beta} \ x} + Be^{-\frac{1}{c}\sqrt{\beta} \ x} \, .$$

Solving (***) similarly,

$$T''(t) \ - \ \beta \, T(t) = \ 0 \, .$$

Auxiliary equation is

$$m^2 - \ \beta \ = \ 0$$

$$m \ = \ \pm\sqrt{\beta}.$$

General Solution:

$$Ce^{\sqrt{\beta} \ x} + De^{-\sqrt{\beta} \ x} \, .$$

In conclusion,

$$U\,(x,t) = X(x)\,T(t) = (Ae^{\frac{1}{c}\sqrt{\beta} \ x} \ + \ Be^{-\frac{1}{c}\sqrt{\beta} \ x})\,(Ce^{\sqrt{\beta} \ t} \ + \ De^{-\sqrt{\beta} \ t}) \, .$$

Case1: when β is +ve, $\beta = k^2$

$$U\ (x,t) = (Ae^{\frac{1}{c}\mathrm{k}\ x} + Be^{-\frac{1}{c}\mathrm{k}x})(Ce^{\mathrm{k}\,t} + De^{-\mathrm{kt}}) \ .$$

Case2: when $\beta = 0$

$$U\ (x,t) = (Ax\ +\ B)\ (Ct\ +\ D)\ .$$

Case3: when β is -ve, $\beta = -\mathrm{k}^2$

$$U\ (x,t) = \left(\mathrm{A}\cos\left(\frac{k}{c}\right)x + \ \mathrm{B}\sin\left(\frac{k}{c}\right)x\right)\ (C\cos(k)\,t\ \ + \mathrm{D}\ \sin(k)\,t\ )\quad .$$

Remark: From the above solutions of wave equation for $0 \leq x \leq l$ and $t\ > 0$ subject to the condition

$$u(0,t) = 0\,.$$

$$u(l,t) = 0\,..$$

Using the conditions in the earlier solutions, we have

<u>Case1</u>

$$\mathrm{u}(0,\mathrm{t}) = (\mathrm{A}+\mathrm{B})\left(\ Ce^{\mathrm{k}\,t}\ + De^{-\mathrm{kt}}\right) = 0$$

$$\rightarrow \qquad A + B\ = 0$$

$$\therefore \quad A\ =\ -B\quad .$$

$$u(l,t) =\ \left(Ae^{\frac{1}{c}\mathrm{k}l} + Be^{-\frac{1}{c}\mathrm{k}l}\right)(Ce^{\mathrm{k}\,t}\ + De^{-\mathrm{kt}}) = 0$$

$$Ae^{\frac{1}{c}\mathrm{k}l} + Be^{-\frac{1}{c}\mathrm{k}l} = 0\,.$$

$$\text{Since} \qquad A\ =\ -B$$

$$-Be^{\frac{1}{c}\mathrm{k}l} + Be^{-\frac{1}{c}\mathrm{k}l} = 0$$

$$B(e^{\frac{1}{c}\mathrm{k}l} - e^{-\frac{1}{c}\mathrm{k}l}) \ = 0$$

$$B \ = 0 \ .$$

$$\therefore A \ = \ B \ = 0 \quad .$$

Case2

$$u(0,t) = (A(0) + B)(Ct + D) = 0$$

$$B \ (Ct + D) = 0$$

$$B \ = 0\,.$$

$$u(l,t) = (Al + B)(Ct + D) = 0$$

$$Al + B \ = 0$$

$$\text{Since} \quad B = 0$$

$$Al \ = 0$$

$$\therefore D = 0 \ .$$

Case3

$$U(0,t) = \left(\mathrm{A}\cos\left(\frac{k}{c}\right)0 + \ \mathrm{B}\sin\left(\frac{k}{c}\right)0\right) \ (C\cos(k)\,t \ \ + \mathrm{D} \ \sin(k)\,t \ \ ) \ = 0 \ .$$

$$A(C\cos(k)\,t \ \ + \mathrm{D} \ \sin(k)\,t \ \ ) \ = 0$$

$$A = 0\,.$$

$$U\ (l,t) = \left(\mathrm{A}\cos\left(\frac{k}{c}\right)l + \ \mathrm{B}\sin\left(\frac{k}{c}\right)l\right) \ (C\cos(k)\,t \ \ + \mathrm{D} \ \sin(k)\,t \ \ )$$

$$\text{Since} \ \ A = 0$$

$$\mathrm{B}\sin\left(\frac{k}{c}\right)l \ \ (C\cos(k)\,t \ \ + \mathrm{D} \ \sin(k)\,t \ \ ) = 0$$

$$\mathrm{B}\sin\left(\frac{k}{c}\right)l = 0$$

$$\sin\left(\frac{k}{c}\right) l = 0$$

$$\left(\frac{k}{c}\right) l \ = \sin^{-1} 0$$

$$\left(\frac{k}{c}\right) = \ \frac{n\pi}{l}$$

$$k = \frac{n\pi c}{l}.$$

$$U\ (x, t) = \left(\ \mathrm{B} \sin\left(\frac{n\pi}{l}\right) x\right) \left(C \cos\left(\frac{n\pi c}{l}\right) t \ \ + \mathrm{D}\ \sin\left(\frac{n\pi c}{l}\right) t \ \right).$$

Problem

A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by

$$y = y_0 \, sin^3 \left(\frac{\pi}{l} x\right) .$$

It is released from rest from this position. Find the displacement (x, t) .

Solution

The wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{C^2} \frac{\partial^2 y}{\partial t^2} .$$

such that $y(0, \; t) = 0 = y(l, t)$ and $y(x, 0) = y_0 \, sin^3 \left(\frac{\pi x}{l}\right)$.

$$\text{At } t = 0, \frac{dy}{dt} = 0 \;\; .$$

If $y(x, t) = X(x) \, T(t)$ is any given solution, then

$$y_{xx} \;=\; X''(x)T(t).$$

and

$$y_{tt} \;= X(x)T''(t).$$

Therefore,

$$X''(x)T(t) = \frac{1}{C^2} \; X(x)T''(t).$$

Equating our new equation to parameter $-\beta^2$,

$$X''(x)T(t) = \frac{1}{C^2} \; X(x)T''(t) \; = -\beta^2.$$

Therefore,

$$y(x, t) = (\mathrm{A} \cos \beta \, x + \; \mathrm{B} \sin \beta \, x) \; (C \cos \beta c \, t \;\; + \mathrm{D} \;\; \sin \beta c \, t \;\;).$$

Now, $y(0, t) = 0.$

$$\rightarrow A = 0.$$

and

$$y(x,t) = (\mathrm{B}\,\sin\beta\, x)\ (C\cos\beta c\, t + \mathrm{D}\ \sin\beta c\, t).$$

and

$$\frac{\partial y}{\partial t}(x,0) = [(\mathrm{B}\,\sin\beta\, x)(\beta c\mathrm{D}\ \cos\beta c\, t - \beta c C\,\sin\beta c\, t)]_{t=0} = 0\,.$$

$$\rightarrow D = 0.$$

Thus,

$$y(x,t) = (\mathrm{B}\,\sin\beta\, x)\ (C\cos\beta c\, t).$$

$$y(x,t) = \mathrm{E}\,\sin\beta\, x\cos\beta c\, t, \quad \text{where } \mathrm{E} = \mathrm{BC}\,.$$

$$\rightarrow \mathrm{E}\,\sin\beta\, l\cos\beta c\, t = 0.$$

$$\rightarrow \sin\beta\, l = 0$$

$$\beta l = n\pi$$

$$\beta = \frac{n\pi}{l}.$$

Therefore,

$$y(x,t) = \sum_n E_n \sin\frac{n\pi}{l} x\cos\frac{n\pi}{l} c\, t\,.$$

By the given condition,

$$y(x,0) = y_0\ sin^3\left(\frac{\pi}{l}x\right)\ .$$

Therefore,

$$y_0\ sin^3\left(\frac{\pi}{l}x\right) = \sum_n E_n \sin\frac{n\pi}{l} x$$

$$= \quad E_1 \sin\frac{\pi}{l}x + E_2\sin\frac{2\pi}{l}x + E_3\sin\frac{3\pi}{l}x + \cdots$$

But

$$sin^3x = \frac{3sinx - sin3x}{4} \ .$$

So,

$$y_0\left[\frac{3sinx - sin3x}{4}\right] = \ E_1 \sin\frac{\pi}{l}x + E_2 \sin\frac{2\pi}{l}x + E_3 \sin\frac{3\pi}{l}x + \cdots$$

Comparing the coefficients on both sides, we get

$$E_1 = \frac{3y_0}{4}, E_2 = 0, E_3 = -\frac{y_0}{4}, E_4 = E_5 = \cdots = 0$$

Therefore,

$$y(x,t) = \frac{3y_0}{4}\sin(\frac{\pi}{l})\,x\cos(\frac{\pi}{l})c\,t - \frac{y_0}{4}\sin\frac{3\pi}{l}x\cos\frac{\pi}{l}c\,t\,.$$

which is the required solution.

5.0 CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

Studying the wave equation and its uses helps us understand how things like music and technology work. Waves are a big part of our world, and this math equation shows how they move and behave. It's not just about theory; it's about how we can use this knowledge in real life, like in music or medical tools.

By learning about waves with the wave equation, we can make new technologies and understand things that affect us every day, like how earthquakes happen or how doctors see inside our bodies. It's not just interesting; it's like a secret code that helps us make things better in the world around us.

The wave equation, an elegant mathematical construct, has served as a fundamental pillar in our exploration of wave phenomena, transcending the boundaries of disciplines and driving transformative advancements in science and technology. Through this research journey, we have delved into the mathematical underpinnings of the wave equation, its rich history, and its myriad applications across diverse domains.

5.2 Recommendations

I recommend that other researchers should further their research and invest in the development of advanced numerical methods and computational algorithms for solving complex wave equations more efficiently. I also recommend that other researchers should explore techniques such as machine learning and high-performance computing to enhance accuracy and reduce computation time.

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