A COMPARISON OF SOME ITERATIVE METHODS OF SOLVING LINEAR SYSTEMS OF EQUATION

A SEMINAR 2 PRESENTATION

BY

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CERTIFICATION

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1.0 INTRODUCTION

1.1 Introduction

A linear system of equations is a set of equations that contain only linear terms. Each equation will have at least two variables, and the goal is to find the values of the variables that satisfy all of the equations in the system.

A linear equation is an equation that contains only terms with an exponent of 1. For example, $7x - 2y = 5$ is a linear equation, but $9x^2 + 4y^2 = 12$ is not, since the variable $x \; and \; y$ has an exponent of 2.

Linear equations are important because they are used in many real-world applications, such as solving optimization problems, finding the best fit line for a set of data points, and more.

Specifically, a linear equation in n variable is of the form:

$$a_0 + a_1x_1 + \cdots + a_nx_n = c$$

Where $x_1, \ldots, x_n$ are variables, $a_0, \ldots, a_n$ are coefficients and c is a constant.

System of linear equations plays an important and motivating role in the subject of linear algebra. In fact, many problems in linear algebra reduce to finding the solution of a system of linear equations.

The general form of systems of linear equations is:

$$a_{11}x + a_{12}y + \ldots + a_{1n}z = b_1$$

$$a_{21}x + a_{22}y + \ldots + a_{2n}z = b_2$$

$$\vdots$$

$$a_{m1}x + a_{m2}y + \ldots + a_{mn}z = b_m$$

Numerical analysis is the area of mathematics and computer science that create, analyses, and implements algorithms for solving numerically the problems of continuous mathematics.

Such problems originated generally from real-world application of algebra, geometry, calculus, and they involve variables which vary continuously.

The solution of system of Linear equations can be accomplished by a numerical method which falls in one of two categories: direct or iterative methods.

In this work, we will cover the iterative or indirect methods, which starts from an approximation to the true solution and, if convergent, derive the sequence of closer approximations-the cycle of computations being repeated till the required accuracy is obtained. This means that in a direct method the amount of computation is fixed while in an iterative method the amount of computation depends on the accuracy required.

In general, one should prefer a direct method for the solution a linear system, but in the case of matrices with a large number of zero elements, it will be advantageous to use iterative methods which present these elements.

1.2 Preliminaries and Definition of Terms

Iterative methods are numerical techniques used to approximate solutions to mathematical problems, particularly in cases where exact solutions are challenging or computationally expensive to obtain directly. Iterative methods work through a series of repeated steps, refining the approximation with each iteration until a satisfactory solution is reached. The iterative methods we are using are: Gauss-Seidel method, Jacobi and Successive over Relaxation (SOR) Methods.

A linear system of equations is a set of equations that contain only linear terms. Each equation will have at least two variables, and the goal is to find the values of the variables that satisfy all of the equations in the system.

1.3 Literature Review

In order to unfold the history of linear algebra, it is important that we first determine what Linear Algebra is. As such, this definition is not a complete and comprehensive answer, but rather a broad definition loosely wrapping itself around the subject.

First, linear algebra is the study of a certain algebraic structure called a vector shape. Secondly linear algebra is the study of linear sets of equations and their transformation properties. Finally, it is the branch of Mathematics charged with investigating the properties of finite dimensional vector spaces and linear mappings between such spaces. This study will discuss the history of linear algebra as it relates to linear sets of equations and their transformations and vector spaces. The project seeks to give a brief overview of the history of linear algebra and its practical applications touching on the various topics used in concordance with it.

Euler brought to light the idea that a system of equations does not necessarily need to have a solution (Perotti). He recognized the need for conditions to be placed upon unknown variables in order to find a solution. The initial work up until the period dealt with the concept of unique solutions and square matrices where the number of equations matched the number of unknowns.

With the turn into the 19th century, Gauss introduced a procedure to be used for solving a system of linear equations.

1.4 Motivation

The motivation for comparing iterative methods for solving linear systems of equations arises from the need to efficiently tackle complex real-world problems, where traditional direct methods may be impractical. This comparison serves to guide practitioners in selecting the most suitable iterative method for their specific scenarios, optimizing performance, and harnessing the advantages of iterative techniques in diverse fields. To prove that there exists a solution to every system of linear equations which can be more efficient and faster.

1.5 Problem Statement

Linear equations are approximately one of the difficult equations in sciences and less attention is given to them compared to the other forms of equations.

1.6 Objectives

This seminar work seeks to achieve the following:

i. To deduce some methods for solving systems of linear equation

ii. Using iterative methods to solve some linear systems; and

iii. Testing for convergence rate with respect to their iterative method i.e. to verify the iterative method that converges faster.

2.0 DISCUSSION

2.1 SUCCESSIVE OVER RELAXATION (SOR) METHOD

The relaxation matrix L_ω seems a lot more complicated than the Gauss-Seidel, but the iterative system associated with relaxation method is very similar to the method of Gauss-Seidel and quite simple.

The method of Successive Over-Relaxation (SOR) is a variant of Gauss-Seidel method for solving a linear system of equations, resulting in faster convergence. It was devised simultaneously by David M. Young Jr. and Stanley P. Frankel in 1950.

ω Over relaxation parameter satisfying $1 \leq \omega \leq 2$

If $\omega = 1$, the SOR reduces to the Gauss-Seidel's method

If $\omega > 1$: Successive Over relaxation

If $\omega < 1$: Successive under relaxation

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

Successive over relaxation method

$$x = \omega(c_1 - k_{12}y - k_{13}z) + (1-\omega)x$$

$$y = \omega(c_2 - k_{21}x - k_{23}z) + (1-\omega)y$$

$$z = \omega(c_3 - k_{31}x - k_{32}y) + (1-\omega)z$$

2.2 Example Using Successive Over-Relaxation Method

$$5x + 2y + z = 12$$

$$x + 4y + 2z = 15$$

$$x + 2y + 5z = 20$$

Adding parameter,

$$x = \omega\left(\frac{12}{5} - \frac{2}{5}y - \frac{1}{5}\mathrm{z}\right) + (1-\omega)x$$

$$\mathrm{y} = \omega\left(\frac{15}{4} - \frac{1}{4}\mathrm{x} - \frac{2}{4}z\right) + (1-\omega)y$$

$$z = \omega\left(4 - \frac{1}{5}x - \frac{2}{5}\mathrm{y}\right) + (1-\omega)z$$

Let $\omega = 1.05$

$$x = 1.05\left(\frac{12}{5} - \frac{2}{5}y - \frac{1}{5}\mathrm{z}\right) + (0.05)x$$

$$\mathrm{y} = 1.05\left(\frac{15}{4} - \frac{1}{4}\mathrm{x} - \frac{2}{4}z\right) + (0.05)y$$

$$z = 1.05\left(4 - \frac{1}{5}x - \frac{2}{5}\mathrm{y}\right) + (0.05)z$$

First Iteration: For the first equation, set $x = y = z = 0$

$$x = 1.05\left(\frac{12}{5} - \frac{2}{5}(0) - \frac{1}{5}(0)\right) + (0.05)(0) = 2.52$$

For the second equation set $x = 2.52, y = 0 \; and \; z = 0$

$$\mathrm{y} = 1.05(\frac{15}{4} - \frac{1}{4}(2.52) - \frac{2}{4}(0) + (0.05)(0) = 3.28$$

For the second equation set $x = 2.52, y = 3.28 \; and \; z = 0$

$$z = 1.05\left(4 - \frac{1}{5}(2.52) - \frac{2}{5}(3.28)\right) + (0.05)(0) = 2.29$$

By continuing the iteration like this, we get

No of iterations	X	y	z
0	0	0	0
1	2.52	3.28	2.29
2	0.53	2.44	2.95
3	0.84	2.05	3.01
4	0.98	2.00	3.00
5	1.00	2.00	3.00
6	1.00	2.00	3.00

Because the last two columns in the Table are identical, we can conclude that the solutions are:

$$x = 1, y = 2 \text{ and } z = 3$$

Therefore, the SOR method converges with 6 iterations.

2.3 Another Example Using Successive Over-Relaxation Method

$$3x - y + z = -1$$

$$-x + 3y - z = 7$$

$$x - y + 3z = -7$$

We will first write down the equations in Gauss Seidel Method

$$x_{k+1} = \frac{1}{3}(-1 + y_k - z_k)$$

$$y_{k+1} = \frac{1}{3}(7 + x_{k+1} + z_k)$$

$$z_{k+1} = \frac{1}{3}(-7 - x_{k+1} + y_{k+1})$$

Adding parameter,

$$x_{k+1} = (1 - \omega). x_k + \omega. \frac{1}{3}(-1 + y_k - z_k)$$

$$y_{k+1} = (1 - \omega). y_k + \omega. \frac{1}{3}(7 + x_{k+1} + z_k)$$

$$z_{k+1} = (1 - \omega). z_k + \omega. \frac{1}{3}(-7 - x_{k+1} + y_{k+1})$$

Initial guess $(x, y, z) = (0,0,0)$ and $\omega = 1.25$

1st approximation

$$x_1 = (1 - 1.25).\,0 + 1.25\,.\frac{1}{3}(-1 + 0 - 0) = -0.41667$$

$$y_1 = (1 - 1.25).\,0 + 1.25\,.\frac{1}{3}(7 - 0.41667 + 0) = 2.74306$$

$$z_1 = (1 - 1.25).\,0 + 1.25\,.\frac{1}{3}(-7 - (-0.41667) + 2.74306) = -1.60012$$

2nd approximation,

$$x_2 = (1 - 1.25).-0.41667 + 1.25\,.\frac{1}{3}\big(-1 + 2.74306 - (-1.60012)\big) = 1.49715$$

$$y_2 = (1 - 1.25).\,2.74306 + 1.25\,.\frac{1}{3}\big(7 + 1.49715 + (-1.60012)\big) = 2.188$$

$$z_2 = (1 - 1.25).-1.60012 + 1.25\,.\frac{1}{3}(-7 - 1.49715 + 2.188) = -2.22878$$

By continuing the iteration like this, we get

No of iterations	x	y	z
0	0	0	0
1	-0.41667	2.74306	-1.60012
2	1.49715	2.188	-2.22878
3	1.04937	1.87824	-2.01411
4	0.9428	2.00073	-1.97234
5	1.00308	2.01263	-2.00294

6	1.00572	1.998	-2.00248
7	0.99877	1.99895	-1.9993
8	0.99958	2.00038	-1.99984
9	1.0002	2.00005	-2.0001

Approximately, we conclude that the solutions are:

$$x = 1, y = 2 \; and \; z = -2$$

Therefore, the SOR method converges with 9 iterations.

3.0 CONCLUSION AND RECOMMENDATION

3.1 Conclusion

The comparison of iterative methods for solving linear systems of equations has provided valuable insights into their performance characteristics .This research serves as a practical resource, aiding researchers, engineers and students in making informed choices and efficiently addressing complex real-world problems with varying computational demands.

3.2 Recommendation

I recommend that future research on the "comparison of some iterative methods of solving linear systems of equations" should consider the following:

- ✓ Expanded Method Selection: Include a broader range of iterative methods beyond the basic ones like Jacobi, Gauss-Seidel, and SOR. Investigate more advanced techniques, such as Krylov subspace methods (e.g., GMRES, BiCGStab), preconditioning strategies, and domain decomposition approaches, to provide a comprehensive overview of available methods.
- ✓ Robustness Testing: Assess the robustness of these methods for a wider variety of linear systems, including those that are ill-conditioned, sparse, or exhibit specific structural characteristics. Analyze how each method's performance varies with different problem types.
- ✓ Parallel and Distributed Computing: Explore the scalability and efficiency of these methods on modern parallel and distributed computing architectures, reflecting the growing importance of high-performance computing in solving large-scale linear systems.

- ✓ Error Analysis: Conduct a thorough error analysis to quantify the accuracy and numerical stability of each method, considering factors like round-off errors and machine precision.
- ✓ Comparison Metrics: Use a range of performance metrics, including computational time, memory usage, and convergence rates, to provide a more comprehensive evaluation of the methods' suitability for different scenarios.
- ✓ Real-World Applications: Apply the compared methods to real-world applications and problems in diverse fields such as engineering, physics, finance, and data science, showcasing their practical relevance.
- ✓ User-Friendly Tools: Develop user-friendly software tools or libraries that implement the compared methods, making them accessible to a wider audience of researchers and practitioners.
- ✓ Parameter Optimization: Investigate automated or data-driven approaches for tuning method-specific parameters (e.g., relaxation factors for SOR) to achieve optimal convergence.

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