Asymptotics of d-Dimensional Visibility

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Abstract

We focus on $\mathbb{N}^3$, imagined as a three dimensional, axis-aligned, grid world partitioned into $1 \times 1 \times 1$ unit cubes, each of which is considered to be empty, in which case a line of sight can pass through it, or obstructing, in which case no line of sight can pass through it. From a given position, some of these obstructing cubes block one's view of other obstructing cubes, motivating the following question: What is the largest number of obstructing cubes that can be simultaneously visible from the surface of an observer cube, given that all of the obstructing cubes lie within a cube of fixed size? We present a model through which the problem of visibility is turned into one of partially ordered sets, yielding an $\Omega(n^{\frac{8}{3}})$ lower bound, where n is the side length of the cube. Through the analysis of lattices corresponding to the elements of partially ordered sets modelling visibility in higher dimensions, the aforementioned lower bound is generalized to $\Omega(p^{d-\frac{1}{d}})$ for dimensions $d > 3$. The previous work along with additional analytic techniques are used to prove an $O(n^{\frac{d^2-1}{d}} \log n)$ upper bound in a reduced visibility setting. Finally, an $O(n^{\frac{d^2-1}{d}} \log n)$ upper bound considering the bottom faces of cubes and their higher dimensional analogues is presented.

1 Introduction

1.1 Visibility Problems

Visibility is a rich topic within discrete geometry, with strong connections to the fields of computer graphics and rendering, as well as digital representation of fundamental geometric shapes, such circles and lines. Central to the topic of visibility is the concept of a line of sight, which we will define as an unobstructed line segment between two points. Two objects are said to be visible from each other if there exists a line of sight between a point on the first object, and a point on the second. The preceding definition motivates the question of what can and cannot be seen from a particular point in space. This general question has been asked many times over the last century, although in varying visibility contexts. One famous family of questions, namely the Orchard Visibility Problem and its variants, considers an orchard in which a tree of radius

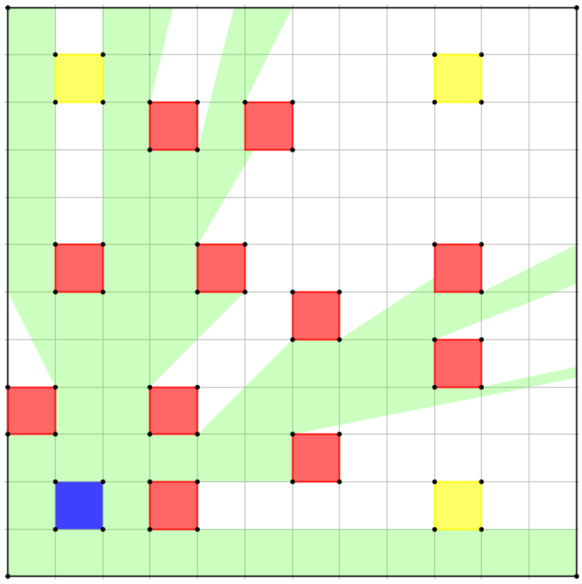

Figure 1: Locus of visible points from the darkened (blue) square is the lower left corner is shaded

$R < 1$ is centered at every lattice point and asks how this radius affects which trees are visible to an observer standing at a point within the orchard. When the observer is positioned at the origin and R is set to 0, the maximum number of trees visible is in bijection to the number of pairs $(x, y) \in \mathbb{N}^2$ such that $\gcd(x, y) = 1$, which for an infinitely large orchard is $\frac{6}{\pi^2}$ of the trees in the orchard. For $d > 2$ dimensions, this generalizes to $\frac{1}{\zeta(d)}$ of the trees, where ζ is the Riemann zeta function.

In [1], Brady considers an n by n axis-aligned grid composed of n^2 unit squares. Each square in the grid is in one of two states, either "obstructing", in which the entire square is opaqued, blocking any line of sight, or empty, in which a line of sight can pass the through the square, unblocked. Two obstructing squares C_1 and C_2 are said to be visible from each other if there exists a line of sight between the two squares. Figure 1 shows visibility from the darkened blue square in the lower left hand corner, with the obstructing squares visible from the blue square colored red, and the ones not visible colored yellow. The locus of points visible to the blue square are shown in shaded in green.

We then ask, if the number and placement of the obstructing squares in the grid is optimal, then what is the largest number of obstructing squares, as a function of n, the size of the grid, that can be visible to a given obstructing square? We will denote this value as $f_2(n)$. Brady demonstrates in [1] that $f_2(n)$ is $\Theta(n \log n)$. To prove the lower bound, he split the n by n grid into (mostly) disjoint parallelograms, each of which is independently analyzed. He then computed and summed the upper and lower bounds of the maximum number of obstructions in each parallelogram to bound $f_2(n)$ from above and below, respectively.

In our paper, we generalize the two dimensional bounds Brady obtained to $d > 2$ dimensions.

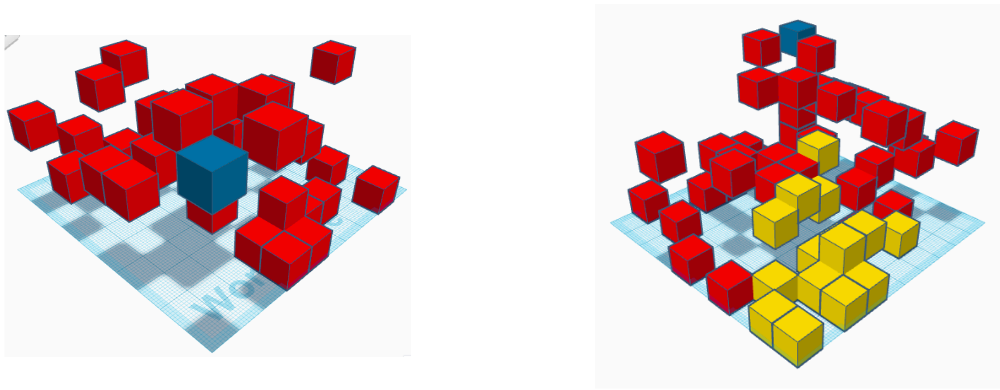

Figure 2: Visibility is taken from perspective of the dark blue cube. The cubes that are both obstructing and visible to the blue cube are painted red while the non-visible obstructing cubes are painted yellow.

In doing so, the two dimensional n by n grid of squares becomes a d-hypercube of side length n consisting of n^d unit hypercubes, each of which is either empty or obstructing. Within this larger d-hypercube of side length n, we denote the maximum possible number of obstructing unit d-hypercubes visible from a given obstructing unit d-hypercube as $f_d(n)$. It is easiest to visualize this question when $d = 3$, and so we shall go about analyzing $f_3(n)$ before extending our results to higher numbers of dimensions. Figure 2 illustrates visibility from the dark blue cube, in the case of $d = 3$.

1.2 Main Results

Adopting a similar argument to that taken in the two-dimensional case (see [1]), we divide the space into parallelepipeds centered at the origin and arrive at a lower bound via construction. By projecting the possible obstructing cubes intersecting the parallelepipeds onto the bottom faces of the parallelepipeds, we construct a partially ordered set that characterizes the conditions by which obstructions can block each other, through which the task of constructing sets of simultaneously visible cubes from the origin is transformed into that of constructing an antichain of our partially ordered set of maximal size. We demonstrate the existence of small vectors modulo n that span a lattice corresponding to an antichain of the partially ordered set, ultimately yielding an $\Omega(n^{\frac{8}{3}})$ lower bound on $f_3(n)$.

The partially ordered set presented above is then modified to model visibility in $d > 3$ dimensions. We extend the techniques used for $f_3(n)$ to construct a $d-1$ dimensional antichain of the generalized partially ordered set, yielding an $\Omega(n^{\frac{d^2-1}{d}})$ bound on $f_d(n)$.

In approaching an upper bound on $f_d(n)$, we first consider a reduced visibility environment in which we restrict visibility to only lines of sight parallel to the edges of the d-paralleletope in consideration. Using the same partially ordered set as used in the lower bound in three dimensions, we demonstrate that there exists a chain cover of sufficiently small size, yielding an

$O(n^{\frac{d^2-1}{d}} \log n)$ upper bound in the reduced visibility environment.

The task of bounding visibility from above in an unrestricted setting is the topic of future publication.

1.3 Organization of Material

In Section 2, a brief introduction is given to partially ordered sets. In Section 3, a lower bound on $f_3(n)$ is proven. In Section 4, the bound presented in Section 3 is generalized to $d > 3$ dimensions. Section 5 provides an introduction to the so called Toy Upper Bound, a simplification of the true upper bound. In Section 6, a brief introduction is given to the discrete Fourier transform. In Section 7, the results of Sections 5 and 6 are combined to present a bound on visibility in the restricted setting of the toy upper bound in $d > 2$ dimensions. Finally in Section 8, an upper bound is presented in an unrestricted visibility environment where cubes, and their higher dimensional analogues are replaced with their bottom "faces".

2 Partially Ordered Sets

As we will make extensive use of them later on, we now take the time to discuss the fundamental notions connected with partially ordered sets. We start with a definition.

Definition 2.1. A *partial order* on a set P is a binary relation $\leq$ satisfying

1. For all $a \in P$, $a \leq a$ ($\leq$ is reflexive)
2. For $a, b \in P$, $a \leq b$ and $b \leq a$ imply that $a = b$ ($\leq$ is antisymmetric)
3. For $a, b, c \in P$, $a \leq b$ and $b \leq c$ imply that $a \leq c$ ($\leq$ is transitive)

Definition 2.2. A *partially ordered set*, or *poset* for short, is a set P along with a partial order $\leq$ on the set.

In other words, a partially ordered set is a set equipped with a rule for comparing elements. It is important to note that not all pairs of elements in a poset are necessarily comparable under $\leq$, hence the name partially ordered set.

Example 2.1 (Examples of Partially Ordered Sets)**.** Listed below are several examples of naturally occurring partially ordered sets:

1. The natural numbers under the relation of divisibility (i.e. $a \leq b$ if and only if $a \mid b$)
2. The sets of subsets of a set S under the relation of containment (i.e. $S_1 \leq S_2$ i and only if $S_1 \subseteq S_2$)

We now define several important structures present within posets, namely chains and antichains.

Definition 2.3. Let P be a partially ordered set with relation $\leq$. A *chain* of P is a subset $S \subseteq P$ such that for all $a, b \in S$, either $a \leq b$ or $b \leq a$.

The set $\{1, 2, 4, 12, 24\}$ is a chain under the first poset mentioned in Example 2.1.

Definition 2.4. Let P be a partially ordered set with relation $\leq$. A *antichain* of P is a subset $S \subseteq P$ such that for all $a, b \in S$, neither $a \leq b$ nor $b \leq a$.

The set $\{3, 4, 7, 22\}$ is an antichain under the first poset mentioned in Example 2.1.

Definition 2.5. The *width* of a partially ordered set is the size of the partially ordered set's largest antichain.

We now state a theorem due to Dilworth [3], relating the width of a partially ordered set to partitions of the same poset into chains. Dilworth's Theorem is crucial in establishing our general upper bound on $f_d(n)$.

Theorem 2.1 (Dilworth's Theorem)**.** *The width of a partially ordered set P is equal to the minimum number of chains into which P can be partitioned.*

We now examine a special case of partially ordered sets, namely those in which any two elements are comparable.

Definition 2.6. A *totally ordered set* P is a set P, along with a binary relation $\leq$, such that for all $a, b \in P$, at least one of $a \leq b$ or $b \leq a$ holds. We say that $\leq$ is a *total ordering* of P.

In light of this, we say that a total ordering $\leq_1$ on a partially ordered set P is *compatible* with a partial ordering $\leq_2$ if for all $a, b \in P$, $a \leq_2 b$ implies $a \leq_1 b$ and $b \leq_2 b$ implies $b \leq_1 a$.

Definition 2.7. A *linear extension* of a partially ordered set P is a total ordering of P for which the partial order on P is compatible.

Let $R = (\leq_1, \leq_2, \cdots, \leq_d)$ be a family of linear extensions of a partially ordered set P. We say that the intersection of the linear extensions in R is the partially ordered set P' compatible with each of the $\leq_i$, and with the minimal possible number of linear extensions. We may now define the *dimension* of a partially ordered set.

Definition 2.8. The dimension of a partially ordered set P, with partial order $\leq$ is the least integer d for which their exists a family $R = (\leq_1, \leq_2, \cdots, \leq_d)$ of linear extensions of P such that

$$P = \bigcap_{i=1}^{d} \leq_i$$

Proposition 2.1. *The partially ordered set $S_d = \{(x_1, x_2, \cdots, x_d) \mid x_i \in \mathbb{R}\}$ under product order has dimension d.*

To show the above, it suffices to note that the linear extensions of a product ordered partially ordered set are in bijection with the equivalence classes of maps to $\mathbb{R}^d$. In light of this equivalence, we use the two terms interchangeably, referring to posets whose elements are d-tuples as being d-dimensional.

As will soon be discussed, the question of visibility can be reduced to one of analyzing certain partially ordered sets. In light of this, we conclude this section with a result due to Brightwell [2], which provides upper and lower bounds on the width of a product-ordered random d-dimensional tuple.

Theorem 2.2 (Brightwell)**.** *There exists a constant C such that, for each fixed d, almost every $P_d(n)$ satisfies*

$$\left(\frac{1}{2}\sqrt{d} - C\right)n^{1-\frac{1}{d}} \leq W_d(n) \leq \frac{7}{2}dn^{1-\frac{1}{d}}$$

where $P_d(n)$ denotes the poset that consists of a random d-dimensional tuple under product order, and where $W_d(n)$ is the width of such a set.

Definition 2.9. Let $f_d(n)$ denote the maximum possible number of obstructing unit d-hypercubes visible from a given obstructing unit d-hypercube.

If $f_d(n)$ were to behave in the same stochastic manner as Theorem 2.2, we would expect $f_d(n)$ to be $\Omega(n^{d-\frac{1}{d}})$ and $O(n^{d-\frac{1}{d}})$.

3 A lower bound on $f_3(n)$

In this section, we show that $f_3(n) = \Omega(n^{\frac{8}{3}})$ by proving the existence of a sufficiently large set of obstructing cubes, all of which are simultaneously visible from a cube centered at the origin. In order to simplify some of the number theoretic computation, we will assume that n is some prime p by rounding all $f_3(n)$ to $f_3(p)$, where p is the largest prime number less than n. From Bertrand's postulate, such a simplification will affect the lower bound by at most a constant factor, which is in this case between $2^{-\frac{8}{3}}$ and 1.

3.1 Setup

We first consider the set of parallelepipeds with opposite and parallel square faces, one of which is a unit square whose vertices have integer coordinates on the upper face of the cube, and the other the unit square on the bottom face of the cube with one vertex at the origin. There

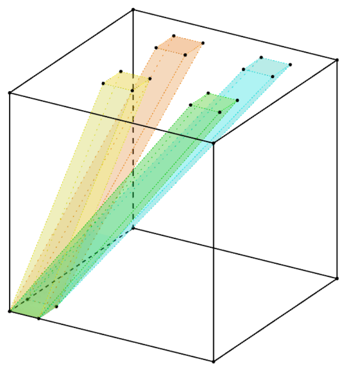

Figure 3: Example parallelepipeds

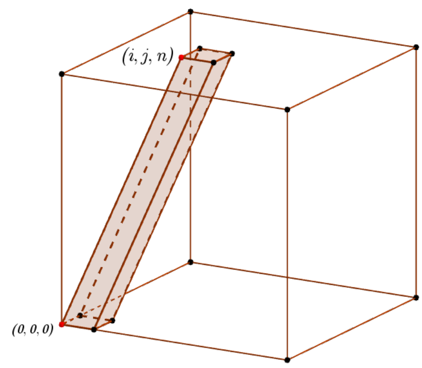

Figure 4: The parallelepiped $\mathcal{P}$, defined by the vertex at (i, j, p).

are p^2 such parallelepipeds, one for each unit square on the cube's upper face. Specifically, we are considering parallelepipeds with coordinates $(0,0,0), (0,1,0), (1,0,0), (1,1,0), (i,j,p), (i+1,j,p), (i,j+1,p), (i+1,j+1,p)$ where p is the size of the grid, p is a prime number, and $0 \leq i,j \leq p-1$. It follows that each parallelepiped can be represented by an ordered pair (i,j) for integers $0 \leq i,j \leq p$.

We now shift our focus to one of these parallelepipeds, call it $\mathcal{P}$, whose top face has vertex (i,j,p) closest to the origin. We will refer to (i,j,p) as the *characteristic vertex* of $\mathcal{P}$. Let E be the edge of $\mathcal{P}$ containing both (i,j,p) and the origin. Denote E, as well as the other three edges of $\mathcal{P}$ congruent to E as the *lateral edges* of $\mathcal{P}$. Within $\mathcal{P}$, we only consider the set of possible obstructing cubes intersecting E. We restrict visibility to lines of sight parallel to the lateral edges of $\mathcal{P}$. As a result, a line of sight passing through the observer cube is equivalent to the line of sight passing through the bottom face of the observer cube, so we may flatten the observer cube to merely its bottom face without sacrificing any visibility. The crucial observation is that from the perspective of the observer face, all possible obstructing cubes intersecting E can be seen only by their bottom face. It follows that a set of obstructing cubes, all of which intersect E, are simultaneously visible to the observer's face if and only if the same statement is true of their bottom faces.

Note that obstructing cubes near the origin have the potential to be counted as intersecting the lateral edge containing the characteristic vertex for more than some fixed constant number of times. For this reason, we will further restrict candidates for obstructing cubes to those intersecting the E edge of $\mathcal{P}$ that are in the upper half of the parallelepiped, ensuring that each obstructing cube is counted in only a fixed constant number of parallelepipeds. In the following argument however, we will assume that *any* obstructing cube along E is a candidate for being obstructing. This simplification is justified by Proposition 3.1, stating that the width of the partially ordered set modelling visibility for *all* cubes along E is at most a constant factor times larger than the partially ordered set modelling visibility along just the upper half of E.

3.2 Constructing a partially ordered set to model visibility along E

Consider the family of parallelepipeds related to P, with characteristic vertices $(p-i, j)$, $(i, p-j)$, and $(p-i, p-j)$. In this subsection, we construct a partially ordered set whose width corresponds to a set of cubes intersecting E that can be simultaneously seen. In calculating the lower bound of such a set, we arrive at the lower bound of $f_3(n)$. Consider two square faces at heights k_1 and k_2 intersecting the edge of $\mathcal{P}$ containing (i, j, p) and the origin, respectively. These corners are taken under the projection to the points $(\{\frac{i \cdot k_1}{p}\}, \{\frac{j \cdot k_1}{p}\}, 0)$ and $(\{\frac{i \cdot k_2}{p}\}, \{\frac{j \cdot k_2}{p}\}, 0)$, respectively. It follows that the face at height k_1 is visible from the observer face if either one of the following conditions are true.

1. $k_1 < k_2$, (face k_1 is lower than face k_2)

2. $\frac{i \cdot k_1}{p} > \frac{i \cdot k_2}{p}$ (the corner face k_1 "sticks out" from behind face k_2 with respect to x coordinate)

3. $\frac{j \cdot k_1}{p} > \frac{j \cdot k_2}{p}$ (the corner face k_1 "sticks out" from behind face k_2 with respect to y coordinate)

Thus, the face at height k_2 can only block the observer face's view of the face at height k_1 if every coordinate of $(\{\frac{i \cdot k_2}{p}\}, \{\frac{j \cdot k_2}{p}\}, p - k_2)$ is larger than the corresponding coordinate in $(\{\frac{i \cdot k_1}{p}\}, \{\frac{j \cdot k_1}{p}\}, p - k_1)$. As a result, the set of obstructing faces all intersecting the lateral edge of $\mathcal{P}$ containing (i, j, p), all of whom are visible from the observer face, corresponds to an antichain of the partially ordered set $(\{\frac{i \cdot k_2}{p}\}, \{\frac{j \cdot k_2}{p}\}, p - k) \mid 0 \leq k < p)$, where $(a_1, a_2, a_3) \leq (b_1, b_2, b_3)$ if and only if $a_i < b_i$ for all $1 \leq i \leq 3$. Multiplying the first two coordinates by p, and subtracting p from the third coordinate yields the simpler, yet structurally identical, poset $(ik \pmod p), jk \pmod p), -k) \mid 0 \leq k < p)$, equipped with the standard partial order relation on tuples. The width of this partially ordered set is the maximum number of obstructing cubes intersecting E that can possible be simultaneously visible from the observing face with respect to the restricted sight lines.

Now consider the three parallelepipeds with characteristic vertices $(p-i, j)$, $(i, p-j)$, and $(p-i, p-j)$ respectively. These parallelepipeds have partially ordered sets analogous to the one modelling visibility along E, with elements of the form $(-ik \pmod p), jk \pmod p), -k)$, $(ik \pmod p), -jk \pmod p), -k)$, and $(-ik \pmod p), -jk \pmod p), -k)$, respectively. It follows that we may sacrifice a constant factor of four and for each parallelepiped, consider its contribution of visible obstructing cubes to be the maximum width of one of the following four partially ordered sets taken under product order: $\{(\pm ik \pmod p), \pm jk \pmod p), k) \mid 0 \leq k < p\}$.

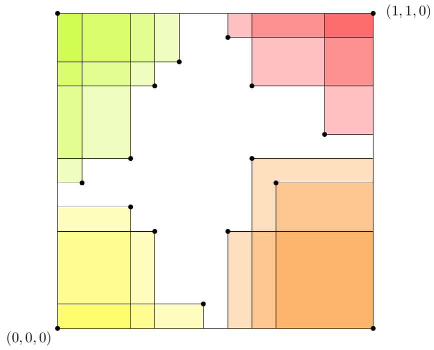

Figure 5: Obstructing cubes projected onto the observer cube. The images of projections along E are shown in yellow.

Proposition 3.1. *Let S be one of the four partially ordered sets of the form $\{(\pm ik \pmod p), \pm jk \pmod p), k) \mid 0 \leq k < p\}$, and let w be the width of such a poset. Denote S_+ and S_- to be the subsets of S for which $\frac{p-1}{2} < k < p$ and $0 \leq k < \frac{p-1}{2}$ respectively, and let w_+ and w_- the the two posets' respective widths. Then*

(i) $w_1 = w_2$

(ii) $w_1, w_2 \geq \frac{w-1}{2}$

Proof. For $a \in S$, we shall refer to $-a$ as the reflection of a over the point $(\frac{p-1}{2}, \frac{p-1}{2}, \frac{p-1}{2})$. Observe that $a \in S_+$ if and only if $-a \in S_-$. Furthermore, notice that if $b, c \in S$ are incomparable under product wise ordering, then so too are $-b$ and $-c$. It follows that negating any antichain of S_+ yields an antichain of S_- of the same size, and vice versa, proving (i).

To see that (ii) is true, note that any antichain A of S, except when A consists solely of the point corresponding to $k = \frac{p-1}{2}$, can be decomposed into two antichains, A_+ and A_-, at least one of which is of size at least $\frac{|A|-1}{2}$. The claim then immediately follows from (i), after taking A to be a maximum antichain of S. □

Remark 3.1. The proof of Proposition 3.1 can easily be extended to hold true for any partially ordered set of the form $\{(\pm t_1 k \pmod p), \cdots, \pm t_m k \pmod p), k) \mid 0 \leq k < p\}$ taken under product order, where m is some integer greater than two.

3.3 Bounding the Width

Our objective now is to find the lower bound on the maximum width among the four posets. To bound the width, we show that there exists an antichain of one of the four previously mentioned partially ordered sets of sufficiently large size, and do so in a manner motivated by the following observation. Consider the partially ordered set $S = \{(ai \pmod p), aj \pmod p), a) \mid 1 \leq a \leq p-1\}$, viewing each element as a point within $[0, p]^3$. It follows that any maximal antichain of S corresponds to a lattice spanned by two elements of the partially ordered set and nearly perpendicular to to the $(1, 1, 1)$ vector. As we have the ability to change the sign of any coordinate, it suffices to consider only the absolute values of the elements. In the spirit of this, we construct linearly independent vectors in S, with all coordinates as small as possible absolute value wise, and use the lattice spanned by the two vectors to make a statement about the width of one of the four partially ordered sets.

Lemma 3.1. *There exists a $v \in S$ for which the absolute value of all of v's coordinates are at most $p^{\frac{2}{3}}$.*

Proof. We use a pigeonhole argument. Imagine each element of S as a point within a cube of side length p. Divide the cube into an $m \times m \times m$ grid of cubes, with each cube having dimensions $\frac{p}{m} \times \frac{p}{m} \times \frac{p}{m}$. If m satisfies that $m^3 < p$, then at least one of these cubes must have two points inside it. Take one of these cubes with two points inside of it, call the two points v_1 and v_2. Taking $v_3 = v_1 - v_2$, we see that all of v_3's coordinates are at most the side length of the box. We want this box side length, $\frac{p}{m}$, to be as small as possible to get the tightest bound on the size of v_3's coordinates. So we want m to be as large as possible. Doing so yields $m^3 < p \to m < p^{\frac{1}{3}}$, hence we can take $m = p^{\frac{1}{3}}$, which tells us that the dimension of the box is $\frac{p}{m} = p^{\frac{2}{3}}$. □

Lemma 3.2. *The width of S is $\Omega(p^{\frac{8}{3}})$.*

Proof. We use a similar pigeonhole argument to generate a suitable second vector with small coordinates to build our antichain grid from. We first divide the $p \times p \times p$ cube into an $m \times m \times m$ grid of cubes, with each cube having dimensions $\frac{p}{m} \times \frac{p}{m} \times \frac{p}{m}$. By setting m so that $km^3 < p$, we have that there is a cube C with at least $k + 1$ points from S inside it. Simplifying, we have that $m < \left(\frac{p}{k}\right)^{\frac{1}{3}}$, so we take $m = \left(\frac{p}{k}\right)^{\frac{1}{3}}$.

When trying to find two vectors, we need to make sure that these two vectors are linearly independent. To make sure that this is the case, we must find a k for which $k+1$ is greater than the longest possible arithmetic sequence of vectors in S that could be contained within C. Note that the side length of C is $\frac{p}{m} = k^{\frac{1}{3}} p^{\frac{2}{3}}$. Let s_1 be the maximal coordinate of the vector in S with the smallest maximal coordinate, absolute value wise (i.e the worst possible case for generating an arithmetic sequence of points within C). Let s_2 be the second smallest coordinate absolute value wise. We then see that k must satisfy $\frac{k^{\frac{1}{3}} p^{\frac{2}{3}}}{s_1} < k + 1$. We can effectively ignore this extra one, giving that $\frac{k^{\frac{1}{3}} p^{\frac{2}{3}}}{s_1} < k \longrightarrow \frac{p}{s_1^{\frac{3}{2}}} < k$. To obtain our lower bound, we take k to be as small as possible: $k = \frac{p}{s_1^{\frac{3}{2}}}$. Plugging this in, we see that the side length of our square is $k^{\frac{1}{3}} p^{\frac{2}{3}} = \frac{p}{\sqrt{s_1}}$.

It follows that $s_2 \leq \frac{p}{\sqrt{s_1}} \longrightarrow s_2\sqrt{s_1} \leq p \to s_2 s_1 \leq p\sqrt{s_1}$. By Lemma 3.1 $s_1 \leq p^{\frac{2}{3}}$, so we see that $s_2 s_1 \leq p^{\frac{4}{3}}$. Finally, our lattice spanned by these two vectors has size at least $\frac{p^2}{2 * s_1 s_2} \geq \frac{p^2}{2 * p^{\frac{4}{3}}} = \frac{1}{2} * p^{\frac{2}{3}}$, which is a constant factor off of $p^{\frac{2}{3}}$.

However, in order to prove the existence of our two dimensional antichain using the above vectors, we need a starting point in the middle of the grid that is a factor of p away from the sides of the grid. More concretely, we wish to find the greatest value of an integer $0 < M < \frac{p}{2}$ for which the following statement holds for all $i, j \in (1, 2, \dots p-1)$:

$$\text{There exists an } \epsilon \in (1, 2, \dots p-1) \text{ such that } M \leq \epsilon, \epsilon \cdot i\%p, \epsilon \cdot j\%p \leq p - M. \qquad (\dagger)$$

To find a suitable M, we use a union bound approach. To do so, note that each coordinate in S ranges from 1 to p-1, thus making the spread of our points uniform. Let I be the interval $(0, \frac{p}{6}) \cap (\frac{5p}{6}, p)$. Given a random point in S, the probability that the x coordinate is not within I is $\frac{1}{3}$. Similarly, the probabilities for the y and z coordinates not being in I is $\frac{1}{3}$. We now see that the probability that at least one of the coordinates is in I is at most $3 \cdot \frac{1}{3} = 1$. Thus, it follows that there exists at least one point for all i and j within I, hence obtaining $M = \frac{p}{6}$. This reduces the size of our lattice by a constant factor of $\frac{1}{36}$.

$\square$

Remark 3.2. Although we have no proof, computational evidence indicates that M (see $(\dagger)$) can be improved to $M = \lfloor \frac{p}{4} - 1 \rfloor$. Smaller M do not work.

Theorem 3.1. *The maximum number of unit cubes visible from an observing unit cube within a cube of side length p is $\Omega(p^{\frac{8}{3}})$.*

Proof. By Lemma 3.2, a single parallelepiped can contain $\Omega(p^{\frac{2}{3}})$ obstructions simultaneously visible from the observer face. Multiplying over all p^2 such parallelepipeds yields a lower bound of $\Omega(p^{\frac{8}{3}})$. $\square$

4 A lower bound for $f_d(n)$

4.1 Generalization of Geometric Setup and a Representative Partially Ordered Set

In this section, we generalize the bounds found in 3 to $d > 2$ numbers of dimensions. Where in three dimensions, we asked how many obstructing cubes could possibly be seen from an observer cube, we now ask how many *d-hypercubes*, which we will abbreviate to just hypercube, can be seen from the observer *d-hypercube*. The geometric approach however, is essentially the same. We take the time now to familiarize the reader with the basic approach.

As in Section 3, we assume that $n = p$ is a prime, an assumption that by Bertrand's postulate throws our bound off by at most a constant factor exponential in d. We then consider a d-hypercube $\mathcal{C}$ with side length p, and assume that the observer is the d-hypercube adjacent one of whose vertices is the origin, in doing so losing again a constant factor exponential in dimension.

Consider the point $(t_1, t_2, \cdots, t_{d-1}, p)$ on the upper $d-1$ hyperface of $\mathcal{C}$. This is the d-dimensional analogue of the point (i, j, p) (see Section 3). We may then construct the unique d-paralleletope $\mathcal{P}$ whose lower base is the bottom $d-1$-hyperface of the observer, and which contains the edge with endpoints at the origin and $(t_1, t_2, \cdots, t_{d-1}, p)$. As in Section 3, we refer to the point $(t_1, t_2, \cdots, t_{d-1}, p)$ as the characteristic vertex of $\mathcal{P}$. Additionally we will refer to the segment connecting the characteristic vertice of $\mathcal{P}$ to the origin as E. In each such choice of $\mathcal{P}$, we will try and maximize the number of obstructing hypercubes intersecting $\mathcal{E}$ that are simultaneously visible from the observer.

Edge E as the special property that any hypercube obstructing E is visible to the observer only by it's bottom hyperface, and as a result, any set of obstructing hypercubes intersecting E is visible if and only if the corresponding set of bottom hyperfaces are also all simultaneously visible. The task of computing large sets of simultaneously visible obstructing hyperfaces is naturally a question of partially ordered sets. In a near identical manner to 3.2, it can be seen that the largest number of simultaneously visible obstructing hypercubes along E is the width of the set

$$\{(at_1, at_2, \cdots, at_{d-1}, -a) \mid 0 \leq a \leq p\} \tag{†}$$

taken under product order. By considering the family of d-paralleletopes whose characteristic vertices can be obtained by switching some of the t_i's in $(t_1, t_2, \cdots, t_{d-1}, p)$ to $p - t_i$'s, we see that, at a loss of constant factor exponential in d, we may for $\mathcal{P}$ consider not just the width of (†), but the maximum of the widths of all partially ordered sets of the form

$$\{(\pm at_1, \pm at_2, \cdots, \pm at_{d-1}, -a) \mid 0 \leq a \leq p\} \tag{‡}$$

taken under product order.

As we iterate over all such paralleletopes $\mathcal{P}$, we run the risk of counting a given obstructing hypercube in arbitrarily many such paralleletopes. To avoid this, we employ the same technique used in Section 3. Specifically, we restrict the possible obstructing cubes in each paralleletope to the obstructing hypercubes in the upper half of the paralleletope. In other words, we would only consider elements of (‡) corresponding to $\frac{p}{2} \leq a < p$. However by the generalization to $d > 2$ dimensions of Proposition 3.1 , the width of the ensuing partially ordered set differs from that of (‡) from at most a factor of two, so we may ignore this range restriction on a and consider the entirety of the set. We shall henceforth refer to this family of partially ordered sets as $S_{\mathcal{P}}$.

Recall that the argument used in three dimensions (see Section 3), visualized the elements of the given partially ordered set as points within a cube of side length p, from which perspective an antichain could be viewed as a plane perpendicular to a vector all of whose coordinates were positive. In generalizing this argument to higher dimensions, we investigate the lattice whose elements are intimately related to those of (‡).

4.2 Lattices and the LLL Lattice Basis Reduction Algorithm

We first recall several important properties of lattices. Suppose we have some basis B of the lattice $\mathcal{L}$ of rank d. We shall refer to the d dimensional-volume of the fundamental region given by B as the *covolume* of $\mathcal{L}$, and equivalently (when the dimension of $\mathcal{L}$ coincides with d), as the absolute value of the determinant of the matrix obtained by taking the elements of B as rows. In light of this, when it is contextually appropriate, we use the terms covolume and determinant interchangeably.

Proposition 4.1. *The covolume of a lattice $\mathcal{L}$ is independent of basis.*

Let $B = (b_1, b_2, \cdots, b_d)$ be a basis for a d dimensional lattice L. Let $B^* = (b_1^*, b_2^*, \cdots, b_d^*)$ be the result when the Gram-Schmidt orthogonalization procedure is applied to B, and let $\mu_{i,j} = \frac{b_i b_j^*}{b_j^* \cdot b_j^*}$ be the orthogonal projection coefficient of b_i onto b_j. We now define an LLL reduced basis [4]

Definition 4.1. The basis $B = (b_1, b_2, \cdots, b_d)$ is said to be *LLL reduced* if the following two conditions are met

(i) $\mid \mu_{i,j} \mid \leq \frac{1}{2}$ for all $1 \leq j < i \leq d$

(ii) $\mid b_i^* + \mu_{i,i-1} b_{i-1}^* \mid^2 \geq \frac{3}{4} \mid b_{i-1}^* \mid^2$ for all $1 \leq i \leq d$

One can think of such a basis as being a "good approximation" of an orthogonal basis, as demonstrated by the following proposition, taken from [4] (Proposition 1.6).

Proposition 4.2 (Lenstra, Lenstra, Lovász)**.** *Let $B = (b_1, \cdots, b_d)$ be a reduced basis for a lattice $\mathcal{L}$ in $\mathbb{R}^d$. Then we have*

$$d(\mathcal{L}) \leq \prod_{i=1}^{n} \mid b_i \mid \leq 2^{\frac{d(d-1)}{4}} d(\mathcal{L})$$

where $d(\mathcal{L})$ is covolume of the $\mathcal{L}$.

Proposition 4.2 as actually a stronger version of the more general result relating the product of the magnitudes of a collection of vectors to the covolume of the fundamental region that they determine.

Proposition 4.3. *Let $B = (b_1, \cdots, b_n)$ be a reduced basis for a lattice $\mathcal{L}$ in $\mathbb{R}^m$. Then*

$$d(\mathcal{L}) \leq \prod_{i=1}^{n} \mid b_i \mid$$

where $d(\mathcal{L})$ is the covolume of $\mathcal{L}$. Equality holds if and only if the elements of B are orthogonal.

Proof. Rotate B so that the collection of vectors lies entirely within the n-hyperplane determined by the first n coordinate axes. As the vectors' respective magnitudes are preserved, as is the covolume, the result then follows from Hadamard's Inequality on the rotated basis. □

4.3 An LLL Reduced Basis of a Familiar Lattice

Let p be a prime, and take $(t_1, t_2, \cdots, t_{d-1}, 1)$ the the characteristic vertex of the paralleletope $\mathcal{P}$.

Definition 4.2. Let $\mathcal{L}_{\mathcal{P}}$ be the d-dimensional lattice $\mathcal{L}$ whose basis consists of the vector $b_1 = (t_1, t_2, \cdots, t_{d-1}, 1)$, and $d-1$ vectors of the form $b_k = (0, \cdots, 0, p, 0, \cdots, 0)$, where the k-th such vector has all zeros except for a p in the k-th coordinate starting from the left for $2 \leq k \leq d$.

Lemma 4.1. *The covolume of $\mathcal{L}_{\mathcal{P}}$ is p^{d-1}.*

Proof. Upon expansion of the determinant of the matrix given by taking the basis given in Definition 4.2 as rows

$$\begin{vmatrix} t_1 & t_2 & t_3 & \dots & t_{d-1} & 1 \\ p & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p & 0 \end{vmatrix}$$

all terms equal zero with the exception of one times the $(d-1)$ by $(d-1)$ diagonal matrix whose diagonal entries are all p, and whose determinant is $\pm p^{d-1}$, with sign dependent on the parity of d. The lemma then follows by taking absolute value. □

Furthermore, observe that the intersection of $\mathcal{L}$ with $[0, p)^d$ is precisely the multiples of the characteristic vertex $(t_1, t_2, \cdots, t_{d-1}, 1)$ taken modulo p and as points in $\mathbb{R}^d$.

Theorem 4.1. *Their exists an element of $S_{\mathcal{P}}$ whose width is $\Omega(p^{1-\frac{1}{d}})$.*

Proof. By Lemma 4.1, the covolume of $\mathcal{L}_{\mathcal{P}}$ is p^{d-1}. Let $B^* = (b_1^*, b_2^*, \cdots, b_d^*)$ be an LLL reduced basis of $\mathcal{L}_{\mathcal{P}}$. Without loss of generality, assume $\mid b_i \mid \leq \mid b_{i+1} \mid$ for all $1 \leq i \leq d-1$. By propositions 4.2 and 4.3, we have that

$$p^{d-1} \leq \prod_{i=1}^{d} |b_i| \leq 2^{\frac{d(d-1)}{4}} p^{d-1} \qquad (\S)$$

It follows then that $\prod_{i=1}^{d-1} \mid b_i \mid \leq (2^{\frac{d(d-1)}{4}} p^{d-1})^{\frac{d-1}{d}} = 2^{\frac{(d-1)^2}{4}} p^{\frac{(d-1)^2}{d}}$. It follows from Proposition 4.3 that the covolume of the fundamental region determined by the first $d-1$ vectors of B^* is at most $2^{\frac{(d-1)^2}{4}} p^{\frac{(d-1)^2}{d}}$.

Note that by the union bound, there exists an element $\ell \in \mathcal{L}_{\mathcal{P}}$ within the region $(\frac{p}{2d}, \frac{p(2d-1)}{2d})^d$. Suppose that all of the b_i contain a coordinate whose absolute value it at least $\frac{p}{2d}$, for $1 \leq i \leq d-1$. It follows that each such b_i has magnitude at least $\frac{p}{2d}$. Referring back to (§), this would imply that

$$\left(\frac{p}{2d}\right)^{d-1} \leq 2^{\frac{(d-1)^2}{4}} p^{\frac{(d-1)^2}{d}} \to \frac{p}{2d} \leq 2^{\frac{d-1}{4}} p^{\frac{d-1}{d}}$$

However d is fixed, and so for sufficiently large p, the inequality fails, implying that for sufficiently large p, there exists a b_i whose magnitude, and therefore largest coordinate absolute value wise is at most $\frac{p}{2d}$. As our b_i are sorted by increasing magnitude, suppose that for all $1 \leq i \leq k$, $\mid b_i \mid \leq \frac{p}{2d}$. It follows that the sublattice $\mathcal{L}_\ell$ of $\mathcal{L}_{\mathcal{P}}$ obtained by adding ℓ to every element of the sublattice spanned by the first k elements of B^* after translation by contains points within the region $(\frac{p}{2d}, \frac{p(2d-1)}{2d})^d$. Now observe that

$$\prod_{i=1}^{k} \mid b_i \mid \leq 2^{\frac{(d-1)^2}{4}} p^{\frac{(d-1)^2}{d} - (d-1-k)}$$

and so by Proposition 4.3, we see that the covolume of the fundamental region spanned by the first k basis vectors is at most $2^{\frac{(d-1)^2}{4}} p^{\frac{(d-1)^2}{d} - (d-1-k)}$. It follows that the intersection of $\mathcal{L}_\ell$ with $(\frac{p}{2d}, \frac{p(2d-1)}{2d})^d$ contains, up to a constant factor, at least

$$\frac{p^k}{2^{\frac{(d-1)^2}{4}} p^{\frac{(d-1)^2}{d} - (d-1-k)}} = 2^{-\frac{(d-1)^2}{4}} p^{\frac{d-1}{d}}$$

and so the number of points in this intersection is $\Omega(p^{\frac{(d-1)}{d}})$.

Now imagine the basis $(b_1, \cdots, b_k)$ of $\mathcal{L}_\ell$ as vectors in $\mathbb{R}^d$. Let v be some element of the orthogonal complement of this collection of vectors. Recall the set $S_{\mathcal{P}}$ of partially ordered sets, related to each other by the negation of coordinates. Now observe that choosing any partially ordered set in $S_{\mathcal{P}}$ in which the t_i coordinate is negated, has the effect of changing the sign of the corresponding coordinate in v. In this manner, by choosing the right element of $S_{\mathcal{P}}$, call it s, we can ensure that all of the coordinates of v are positive. It follows, after adjusting the signs of the coordinates of $\mathcal{L}_\ell$ to account for the choice of s, that the vector corresponding to any two elements in $\mathcal{L}_\ell$ is orthogonal to v, and thus has neither all positive or all negative coordinates. Combining this with the fact that each element in the intersection of $\mathcal{L}_\ell$ and $(\frac{p}{2d}, \frac{p(2d-1)}{2d})^d$ corresponds to a unique element of s, we see that the elements of this intersection are in bijection with an antichain of s, and so we see that the width of s is $\Omega(p^{\frac{(d-1)}{d}})$, as desired. □

4.4 Conclusion

In the prior subsection, we analyzed the set $S_{\mathcal{P}}$ of partially ordered sets representing visibility within the family of d-paralleletopes related to $\mathcal{P}$ by negating (modulo p) some subset of the coordinates of $\mathcal{P}$'s characteristic vertex. We found the maximum number of simultaneously visible obstructing hypercubes contained within this family of paralleletopes to be $\Omega(p^{\frac{(d-1)}{d}})$. Summing over all such families yields a lower bound on $f_d(n)$.

Theorem 4.2. *The maximum number of unit d-hypercubes visible from an observing unit d-hypercube within a d-hypercube of side length p is* $\Omega(p^{d-\frac{1}{d}})$.

Proof. Each such paralleletope as described in Section 4.1 belongs to a family of 2^{d-1} paralleletopes inside of which a number of d-hypercubes not less than the maximum width among the elements of the set $S_{\mathcal{P}}$ can be seen. By Theorem 4.1, the maximum of these widths is $\Omega(p^{\frac{(d-1)}{d}})$. Multiplying by $\frac{p^{d-1}}{2^{d-1}}$ such families yields a lower bound of $\Omega(p^{d-\frac{1}{d}})$. □

5 A Toy Upper Bound

In the following three sections, we work within the frame of the parallelepiped model we used for the lower bound. Whereas in the lower bound we bounded from below the largest number of cubes that could possibly be seen from within the confines of each parallelepiped, we now bound this value from above. We do this by making use of the same partially ordered set S_d as used in the lower bound, this time obtaining an upper bound of the width by constructing chain covers of S_d. By Dilworth's Theorem, these chain covers will then serve as upper bounds on the width of the partially ordered set.

In addition, the following three sections will treat visibility as it applies in the general case of $d > 2$ dimensions. As we are working within the confines of a particular d-paralleletope, we let $(t_1, t_2, \ldots, t_{d-1}, p)$ serve as the $d > 3$ dimensional analogue of the characteristic vertex, which in three dimensions was $(1, i, j)$.

5.1 Preliminaries

Recall that we are working with S_d (see definition **??**).

Definition 5.1. Let $m_{(p,d)}$ denote $\min(\max(at_1, at_2, \cdots, at_{d-1}, a))$ as a ranges from 0 to $p-1$.

Lemma 5.1. *The width of* S_d *is at most a constant factor greater than* $m_{(p,d)}$.

Proof. We generate a chain cover of S_d from the first $p-1$ multiples of $\vec{v} = (v_1, v_2, \cdots, v_{k-1}, v_d)$, the element (or one of) of S_d whose maximum coordinate is $m_{(p,d)}$. Writing down all the multiples in the order they appear, we traverse this list from its start. On step one, we create a chain and add the first multiple, namely v, into it. On step k, we examine the tuple $k \cdot v$. If each of the coordinates of $k \cdot v$ is greater modulo p than its corresponding coordinate in $(k-1) \cdot v$, then we append $k \cdot v$ onto the end of the current chain. Otherwise, we terminate the current chain and cast it aside, adding $k \cdot v$ to a new chain. As p is prime, the multiples of v will take on every value in S_d. It follows that after step $p-1$, the collection of chains formed by the process forms a chain cover on S_d ($0 \cdot v$ can be appended to the beginning of any of the antichains). Note that an existing chain is completed and a new chain is started at step k if and only if one of the coordinates in the transition from $(k-1) \cdot v$ to $k \cdot v$ exceeds $p \pmod p$ and "loops back" to a smaller modulo p value. The number of steps where this occurs in at least one coordinate is at most the sum of the number of times it occurs in each coordinate, which is equal to $v_1 + v_2 + \cdots + v_d \leq km_{(p,d)}$. As the size of any chain cover of S_d is greater than the width of S_d, we are done. ☐

As a result of Lemma 5.1, it suffices to place an upper bound on $m_{(p,d)}$.

6 The Discrete Fourier Transform

Before we proceed, we take the time to familiarize the reader with several important notions and analytic techniques that will play crucial roles in the calculations of Section 7. In the proof of the upper bound of our reduced visibility problem, we use techniques from Fourier analysis. Below, we provide statements and proofs of the theorems that we will use.

Definition 6.1. Let $e_p(x) = e^{\frac{2\pi i x}{p}}$. Consider some function $f : \mathbb{Z}_p^k \longrightarrow \mathbb{C}$. We define the discrete Fourier transform of f, denoted as $\hat{f}$ as follows

$$\hat{f}(\vec{w}) := \sum_{\vec{x} \in \mathbb{Z}_p^k} e_p(\vec{w} \cdot \vec{x}) f(\vec{x})$$

Theorem 6.1 (Classical). *For any two functions $f, g : \mathbb{Z}_p^k \longrightarrow \mathbb{C}$, the following holds*

$$\sum_{\vec{x} \in \mathbb{Z}_p^k} f(\vec{x}) \overline{g(\vec{x})} = \frac{1}{p^k} \sum_{\vec{w} \in \mathbb{Z}_p^k} \hat{f}(\vec{w}) \overline{\hat{g}(\vec{w})}$$

Proof. To prove that this is true, we reduce the right hand side of the equation. Expanding out and rearranging the summation, we have that

$$\frac{1}{p^k} \sum_{\vec{w} \in \mathbb{Z}_p^k} \hat{f}(\vec{w}) \overline{\hat{g}(\vec{w})} = \frac{1}{p^k} \sum_{\vec{w}, \vec{x}_1, \vec{x}_2 \in \mathbb{Z}_p^k} e_p(\vec{w} \cdot \vec{x}_1) f(\vec{x}_1) \overline{e_p(\vec{w} \cdot x_2) g(\vec{x}_2)}$$

$$= \frac{1}{p^k} \sum_{\vec{w},\vec{x}_1,\vec{x}_2 \in \mathbb{Z}_p^k} e_p(\vec{w} \cdot (\vec{x}_1 - \vec{x}_2)) f(\vec{x}_1) \overline{g(\vec{x}_2)}$$

We can switch our summation to obtain

$$\frac{1}{p^k} \sum_{\vec{x}_1,\vec{x}_2 \in \mathbb{Z}_p^k} \left(f(\vec{x}_1)\overline{g(\vec{x}_2)} \sum_{\vec{w} \in \mathbb{Z}_p^k} e_p(\vec{w} \cdot (\vec{x}_1 - \vec{x}_2) \right)$$

Now if $\vec{x}_1 = \vec{x}_2$, then

$$\sum_{\vec{w} \in \mathbb{Z}_p^k} e_p(\vec{w} \cdot (\vec{x}_1 - \vec{x}_2) = \sum_{\vec{w} \in \mathbb{Z}_p^k} 1 = p^k$$

If $\vec{x}_1 \neq \vec{x}_2$, then we can write $\vec{x}_1 - \vec{x}_2 = \vec{v} = (v_1, v_2, \cdots, v_k)$, from which it follows that

$$\sum_{\vec{w} \in \mathbb{Z}_p^k} e_p(\vec{w} \cdot \vec{v}) = \prod_{i=1}^{k} \sum_{j=0}^{p-1} e_p(jv_i) = \prod_{i=1}^{k} \sum_{j=0}^{p-1} e_p(v_i)^j = 0$$

And so we may write

$$\frac{1}{p^k} \sum_{\vec{x}_1,\vec{x}_2 \in \mathbb{Z}_p^k} \left(f(\vec{x}_1)\overline{g(\vec{x}_2)} \sum_{\vec{w} \in \mathbb{Z}_p^k} e_p(\vec{w} \cdot (\vec{x}_1 - \vec{x}_2) \right)$$

$$= \frac{1}{p^k} \left(\sum_{\vec{x} \in \mathbb{Z}_p^k} f(\vec{x})\overline{g(\vec{x})}(p^k) \right) + \frac{1}{p^k} \left(\sum_{\vec{x}_1,\vec{x}_2 \in \mathbb{Z}_p^k} f(\vec{x})\overline{g(\vec{x})}(0) \right)$$

Which equals

$$\sum_{\vec{x} \in \mathbb{Z}_p^k} f(\vec{x})\overline{g(\vec{x})}$$

as desired. □ □

Additionally, we introduce the following lemma to be used later.

Lemma 6.1. *Define $h(x)$ to be equal to one if $0 \leq x < n$ and zero otherwise. Then*

$$\left|\hat{h}(x)\right| \leq \min\left(\frac{p}{2|x|}, n\right)$$

Proof. We first expand the left hand side.

$$\left|\hat{h}(x)\right| = \left|\sum_{w \in \mathbb{F}_n} e_p(wx)\right| = \left|\frac{e_p(nx) - 1}{e_p(x) - 1}\right| = \frac{|e_p(nx) - 1|}{|e_p(x) - 1|} \leq \frac{2}{|e_p(x) - 1|}$$

Now note that the bottom is at most the length of the arc subtended by $\frac{2\pi|x|}{p} \ll \frac{|x|}{p}$ radians, and so we have that $\frac{2}{|e_p(x)-1|} \leq \frac{2}{\frac{|x|}{p}} = \frac{2p}{|x|}$. To finish, observe that

$$\left|\hat{h}(x)\right| = \left|\sum_{w \in \mathbb{F}_n} e_p(wx)\right| \leq \sum_{w \in \mathbb{F}_n} |e_p(wx)| = n$$

by the triangle inequality. It follows that $\left|\hat{h}(x)\right| \leq \min\left(\frac{p}{2|x|}, n\right)$ is desired. $\square$

We now generalize Lemma 6.1 to higher dimensions.

Lemma 6.2 (Generalization of Lemma 6.1)**.** *Let $h : \mathbb{F}_n^d \mapsto \mathbb{C}$ be such that $h(\vec{x}) = 1$ if for every coordinate x_i of $\vec{x}$, $0 \leq x_i \leq n-1$, and let $h(\vec{x})$ be zero otherwise. Then*

$$\left|\hat{h}(x_1, \cdots, x_d)\right| \leq \prod_{k=1}^{d} \min\left(\frac{p}{2|x_k|}, n\right)$$

Proof. It suffices to note that

$$\left|\hat{h}(x_1, \cdots, x_d)\right| = \left|\sum_{\vec{w} \in \mathbb{F}_n^d} e_p(\vec{w} \cdot \vec{x})\right| = \prod_{k=1}^{d} \left|\sum_{w \in \mathbb{F}_n} e_p(w_k x_k)\right| \leq \prod_{k=1}^{d} \min\left(\frac{p}{2|x_k|}, n\right)$$

where the last inequality is Lemma 6.1 applied to each coordinate of the x_i. $\square$

7 Proof of Toy Upper Bound

In this section we compute a bound on $m_{(p,d)}$ (see Section 5). For brevity we shall henceforth refer to this value as m_p.

Definition 7.1. Let $f : \mathbb{Z}_p^d \mapsto \{0, 1\}$ be such that $f(\vec{w})$ equals 1 if $\vec{w}$ is a scalar integer multiple of $\vec{t} = (t_1, \cdots, t_{d-1}, 1)$ modulo p and zero otherwise.

Lemma 7.1. *For any $\vec{w} \in \mathbb{Z}_p^d$, the following holds.*

$$\hat{f}(\vec{w}) = \begin{cases} p & \textit{if } \vec{w} \cdot \vec{t} = 0 \\ 0 & \textit{else} \end{cases}$$

Proof. We first show that if $\vec{w} \cdot \vec{t} = 0$, then $\hat{f}(\vec{w}) = p$. Now observe that in this case

$$\hat{f}(\vec{w}) = \sum_{\vec{x} \text{ is multiple of } \vec{t} \pmod{p}} e_p(\vec{w} \cdot \vec{x}) f(\vec{x}) = \sum_{\vec{x} \text{ is multiple of } \vec{t} \pmod{p}} e_p(0)(1) = p$$

as desired.

Now let $\vec{w}$ be such that $\vec{w} \cdot \vec{t}$ is nonzero modulo p, call it m. It then follows that

$$\hat{f}(\vec{w}) = \sum_{\vec{x} \text{ is multiple of } \vec{t} \pmod p} e_p(\vec{w} \cdot \vec{x})(1) = \sum_{k=0}^{p-1} e_p(km \pmod p) = 0$$

and so the lemma is proven. $\square$

Definition 7.2. We say that $1_S : \mathbb{Z}_p^d \mapsto \{0, 1\}$ is the *indicator function* for the set S if for all $\vec{w} \in \mathbb{Z}_p^d$, $1_S(\vec{w})$ equals one if every coordinate of w is in S, and zero otherwise.

Definition 7.3. Let $g_{(d,k)} : \mathbb{Z}_p^d \mapsto \mathbb{R}$ be the indicator function $1_{[0, \frac{mp}{k})}$ convoluted with itself k times, where k is a fixed constant parameter. In the notation of Definition **??**, we write

$$g_{(d,k)} = 1_{[0, \frac{mp}{k})} * \cdots * 1_{[0, \frac{mp}{k})}$$

Lemma 7.2. $|\hat{g}_{(d,k)}(\vec{0})| = (\frac{mp}{k})^{kd}$.

Proof. Upon expansion, we have that

$$|\hat{g}_{(d,k)}(\vec{0})| = \sum_{\vec{x} \in \mathbb{Z}_p^d} g_{(d,k)}(\vec{x}) = \sum_{\vec{x} \in \mathbb{Z}_p^d} \sum_{\vec{v}_1 + \cdots + \vec{v}_k = \vec{x}} \prod_{i=1}^{k} 1_{[0, \frac{mp}{k})}(v_i) = \sum_{v_1, \cdots, v_k \in \mathbb{Z}_p^d} \prod_{i=1}^{k} 1_{[0, \frac{mp}{k})}(v_i)$$

Let S be the subset of $\mathbb{Z}_p^d$ for which $1_{[0, \frac{mp}{k})}$ evaluates to one. Then the right hand side of above equals $(\sum_{\vec{v} \in S} 1)^k$. The result then follows from the fact that $|S| = (\frac{mp}{k})^d$. $\square$

In general, we can make the more general statement as follows:

Lemma 7.3. *Let $\vec{x} = (x_1, \cdots, x_d)$. Then the following holds*

$$|\hat{g}_{(d,k)}(\vec{x})| \leq \prod_{i=1}^{d} \min\left(\frac{p}{2|x_i|}, \frac{mp}{k}\right)^k \leq \left(\frac{p}{2 \max |x_i|} \left(\frac{mp}{k}\right)^{d-1}\right)^k$$

Proof. We have that

$$\hat{g}_{(d,k)}(\vec{x}) = \sum_{\vec{w} \in \mathbb{Z}_p^d} e_p(\vec{w} \cdot \vec{x}) \sum_{\vec{v}_1 + \cdots + \vec{v}_k = \vec{w}} \prod_{i=1}^{k} 1_{[0, \frac{mp}{k})}(\vec{v}_i)$$

Using the same S as in Lemma 7.2, the right hand side of above equals $\sum_{\vec{v}_1, \cdots, \vec{v}_k \in S} e_p(\vec{x} \cdot \Sigma \vec{v}_i)$. Rewriting $e_p(\vec{x} \cdot \Sigma \vec{v}_i) = \prod_{i=1}^{k} e_p(\vec{x} \cdot \vec{v}_i)$ and exploiting symmetry, the expression reduces to

$$\sum_{\vec{v}_1, \cdots, \vec{v}_k \in S} \prod_{i=1}^{k} e_p(\vec{x} \cdot \vec{v}_i) = \left(\sum_{\vec{v} \in S} e_p(\vec{x} \cdot \vec{v})\right)^k = \left(\hat{1}_{[0, \frac{mp}{k})}(\vec{x})\right)^k$$

The left inequality follows immediately from Lemma 6.2 applied to $1_{[0,\frac{m_p}{k})}(\vec{x})$. The right inequality is true by the definition of minimum. $\square$

Lemma 7.4. *m_p is $O(p^{\frac{d-1}{d}}\log p)$.*

Proof. Observe that

$$\sum_{\vec{x}\in\mathbb{Z}_p^d} f(\vec{x})\overline{g_{(d,k)}(\vec{x})} = \sum_{\vec{x}\in\mathbb{Z}_p^d} f(\vec{x})g_{(d,k)}(\vec{x}) = f(0)g_{(d,k)}(0) = 1$$

However by Theorem 6.1, we also have that

$$p^d\sum_{\vec{x}\in\mathbb{Z}_p^d} f(\vec{x})\overline{g_{(d,k)}(\vec{x})} = \sum_{\vec{x}\in\mathbb{Z}_p^d} \hat{f}(\vec{x})\overline{\hat{g}_{(d,k)}(\vec{x})}$$

Note that $\hat{f}$ is only supported on α if $\vec{\alpha}\cdot(x_1,x_2,\ldots,x_d)\equiv 0(\text{mod } p)$, hence

$$p^d = \sum_{\vec{x}\in\mathbb{Z}_p^d} \hat{f}(\vec{x})\overline{\hat{g}_{(d,k)}(\vec{x})} = \sum_{\vec{\alpha}\cdot(x_1,x_2,\ldots,x_d)\equiv 0} \hat{f}(\vec{\alpha})\overline{\hat{g}_{(d,k)}(\vec{\alpha})}$$

$$= \sum_{\vec{\alpha}\cdot(x_1,x_2,\ldots,x_d)\equiv 0} p\overline{\hat{g}_{(d,k)}(\vec{\alpha})} \to p^{d-1} = \sum_{\vec{\alpha}\cdot(x_1,x_2,\ldots,x_d)\equiv 0} \overline{\hat{g}_{(d,k)}(\vec{\alpha})}$$

where the third equality is due to Lemma 7.1. Expanding on this we have

$$\hat{g}_{(d,k)}(\vec{0}) + \sum_{\substack{\vec{\alpha}\cdot(x_1,x_2,\ldots,x_d)\equiv 0\\ \alpha\neq\vec{0}}} \hat{g}_{(d,k)}(\vec{\alpha}) = p^{d-1} \to \hat{g}_{(d,k)}(\vec{0}) \leq p^{d-1} + \sum_{\substack{\vec{\alpha}\cdot(x_1,x_2,\ldots,x_d)\equiv 0\\ \alpha\neq\vec{0}}} |\hat{g}_{(d,k)}(\vec{\alpha})| \quad (\dagger_1)$$

due to the triangle inequality. For each α, let $\alpha = (\alpha_1,\cdots,\alpha_d)$. Plugging the results of lemmas 7.2 and 7.3 into ($\dagger_1$), we have

$$\left(\frac{m_p}{k}\right)^{dk} \leq p^{d-1} + \sum_{\substack{\vec{\alpha}\cdot(x_1,x_2,\ldots,x_d)\equiv 0\\ \alpha\neq\vec{0}}} |\hat{g}_{(d,k)}(\vec{\alpha})| \leq p^{d-1} + \sum_{\substack{\vec{\alpha}\cdot(x_1,x_2,\ldots,x_d)\equiv 0\\ \alpha\neq\vec{0}}} \left(\frac{p}{2\max|\alpha_i|}\left(\frac{m_p}{k}\right)^{d-1}\right)^k$$

Dividing both sides by $\left(\frac{m_p}{k}\right)^{(d-1)k}$ yields

$$\begin{aligned}\left(\frac{m_p}{k}\right)^k &\leq \frac{p^{d-1}}{(\frac{m_p}{k})^{(d-1)k}} + \sum_{\vec{\alpha}\cdot(x_1,x_2,\ldots,x_d)\equiv 0\alpha\neq\vec{0}} \left(\frac{p}{2\max|\alpha_i|}\right)^k \\ &\leq \frac{p^{d-1}}{(\frac{m_p}{k})^{(d-1)k}} + p^{d-1}\max_{\vec{\alpha}\cdot(x_1,x_2,\ldots,x_d)\equiv 0\alpha\neq\vec{0}} \left(\frac{p}{2\max|\alpha_i|}\right)^k\end{aligned}$$

We now let $a_p = \min_{\substack{\vec{\alpha}\cdot(x_1,x_2,\ldots,x_n)\equiv 0\\ \alpha\not\equiv\vec{0}}} \max|\alpha_i|$. Then

$$\left(\frac{m_p}{k}\right)^k \leq \frac{p^{d-1}}{(\frac{m_p}{k})^{(d-1)k}} + p^{d-1} \max_{\substack{\vec{\alpha}\cdot(x_1,x_2,\ldots,x_d)\equiv 0\\ \alpha\not\equiv\vec{0}}} \left(\frac{p}{2\max|\alpha_i|}\right)^k$$
$$\leq \left(p^{d-1}+1\right)\max\left(\frac{p^{d-1}}{(\frac{m_p}{k})^{(d-1)k}}, (\frac{p}{a_p})^k\right)$$

We consider two cases.

Case 1: First, suppose that

$$\max\left(\frac{p^{d-1}}{(\frac{m_p}{k})^{(d-1)k}}, (\frac{p}{a_p})^k\right) = \frac{p^{d-1}}{(\frac{m_p}{k})^{(d-1)k}}$$

Rearranging yields

$$\left(\frac{m_p}{k}\right)^k \leq (p^{d-1}+1)\max\left(\frac{p^{d-1}}{(\frac{m_p}{k})^{(d-1)k}}, (\frac{p}{a_p})^k\right) = (p^{d-1}+1)\frac{p^{d-1}}{(\frac{m_p}{k})^{(d-1)k}} \rightarrow$$

$$\left(\frac{m_p}{k}\right)^{dk} \leq (p^{d-1}+1)(p^{d-1}) \rightarrow m_p \ll k\cdot p^{\frac{2d-2}{d}\cdot\frac{1}{k}}$$

So as long as $k \geq 1$, this scenario contributes a complexity of at most $O(p^{\frac{d^2-1}{d}})$. This concludes the first case. ♣

Case 2: We suppose that

$$\max\left(\frac{p^{d-1}}{(\frac{m_p}{k})^{(d-1)k}}, (\frac{p}{a_p})^k\right) = \left(\frac{p}{a_p}\right)^k$$

Then rearranging the inequality gives:

$$\left(\frac{m_p}{k}\right)^k \leq (p^{d-1}+1)\max\left(\frac{p^{d-1}}{(\frac{m_p}{k})^{(d-1)k}}, (\frac{p}{a_p})^k\right) \leq (p^{n-1}+1)\cdot\left(\frac{p}{a_p}\right)^k \rightarrow$$

$$\left(\frac{m_p}{k}\right)^k \leq (p^{d-1}+1)\cdot\left(\frac{p}{a_p}\right)^k \rightarrow m_p\cdot a_p \leq k\cdot p\cdot(p^{d-1}+1)^{\frac{1}{k}}$$

Now we set $k=\log p$ (which is greater than three for large enough p) yielding

$$m_p\cdot a_p \leq \log p\cdot p\cdot(p^{d-1})^{\frac{1}{\log p}} = e^{d-1}\cdot p\cdot\log p \ll p\log p$$

From the fact that $m_p\cdot a_p \ll p\log p$, we rearrange the statement to a more workable form:

$$m_p \ll \frac{p\log p}{a_p}$$

We now introduce the parameter x, a bound on the size of a_p. Observe that

$$a_p \le x \to \frac{p \log p}{a_p} \ge \frac{p \log p}{x}$$

While

$$a_p \ge x \to \frac{p \log p}{a_p} \le \frac{p \log p}{x}$$

Let $\mathbb{E}(m_p)$ denote the expected value of m_p and let $\mathbb{E}(\frac{p \log p}{a_p})$ denote the expected value of $\frac{p \log p}{a_p}$ if $a_p < x$. Observe that for each value of a_p, there are p^{d-1} tuples $\vec{x}$ for which $\vec{x} \cdot \vec{\alpha} = 0$. It follows that

$$\begin{aligned}
\mathbb{E}(m_p) &\ll \max\left(\frac{p \log p}{x}, \mathbb{E}(\frac{p \log p}{a_p})\right) \\
&\ll \frac{p \log p}{x} + \frac{1}{p^d} \sum_{x \ge \alpha_i \ge 0} \frac{p \log p}{a_p} \cdot p^{d-1} \\
&= \frac{p \log p}{x} + \sum_{x \ge \alpha_i \ge 0} \frac{\log p}{a_p}.
\end{aligned}$$

Assuming that the coordinates of α are strictly descending, in doing so losing at most a constant degree of accuracy, we have

$$\begin{aligned}
\mathbb{E}(m_p) &\ll \frac{p \log p}{x} + \sum_{x \ge \alpha_1 \ge \alpha_2 \cdots \ge \alpha_d \ge 0} \frac{\log p}{\alpha_1} \\
&\ll \frac{p \log p}{x} + \log p \sum_{\alpha_1 = 0}^{x} \frac{1}{\alpha_1} (\alpha_1^{d-1}) \\
&\ll \frac{p \log p}{x} + x^{d-1}
\end{aligned}$$

We may now let x be $p^{\frac{1}{d}}$, yielding $\mathbb{E}(m_p) \ll p^{1-\frac{1}{d}} \log p$. This concludes the second case. ♣

It follows that we may set $k = \log p$, a choice in which both cases give that m_p is $O(p^{\frac{d-1}{d}} \log p)$. □

We are now ready to state the main result of this section.

Theorem 7.1. *The largest number of cubes visible in the d dimensional toy upper bound visibility environment is $O(p^{d-\frac{1}{d}} \log p)$.*

Proof. By Lemma 7.4, the largest number of visible obstructions within a d-paralleletope is $O(p^{\frac{d-1}{d}} \log p)$. Multiplying by p^{d-1} such paralleletopes yields the desired $O(p^{d-\frac{1}{d}} \log p)$. □

8 Upper Bound on Bottom Hyperfaces

Now that we have tackled the problem in a reduced visibility setting, we attempt to prove the general upper bound in which we relax the constraints on our lines of sight to account for all degrees of freedom. However, due to difficulties in modelling the precise nature in which d-dimensional obstructions can block each other, we reduce the obstructions to $d-1$-dimensional cubes by projecting the obstructions onto their bottom $d-1$-dimensional face, and instead consider the maximum number of these reduced obstructions that can be seen from a unit d-dimensional hypercube at the origin with respect to all possible lines of sight. We will prove that the upper bound of the maximum number of such obstructions that a unit d-dimensional hypercube cube can see in a grid of size n is $O(n^{\frac{d^2-1}{d}}\log(n))$.

We first characterize the aforementioned visibility. Using our terminology in sections 3 and 5, consider the parallelepiped $\mathcal{P}$ whose characteristic vertex is $(t_1, t_2, \ldots, t_{d-1}, p)$. Take the edge E connecting the origin and the characteristic vertex of $\mathcal{P}$. We only consider the set of obstructions that intersect E. The first crucial observation here is that if a d-hypercube intersects E, and V is its vertex inside $\mathcal{P}$, then if V is visible from a point of the unit $d-1$-hypersurface at the origin, V is visible from the coordinate $(1, 1, \ldots, 1, 0)$. Thus, in considering whether an obstruction is visible or not, we only need to consider its vertex V inside $\mathcal{P}$. Note that the coordinates of the vertices in $\mathcal{P}$ belonging to obstructing cubes that intersect E are of the form:

$$\left(\left\lceil\frac{t_1 \cdot a}{p}\right\rceil, \left\lceil\frac{t_2 \cdot a}{p}\right\rceil, \ldots, \left\lceil\frac{t_{d-1} \cdot a}{p}\right\rceil, a\right)$$

. This brings us to the following observation: Consider two obstructions intersecting E whose vertices inside $\mathcal{P}$ are X_1 and X_2, with X_1 having a smaller x_d-coordinate than X_2. If the x_i-coordinate slope connecting X_1 to $(1, 1, \ldots, 1, 0)$ is greater than the x_i-coordinate slope connecting X_2 to $(1, 1, \ldots, 1, 0)$ for every $1 \leq i \leq d-1$, then X_1 obstructs X_2.

As a result, the true upper bound reduces to computing the width of the following partially ordered set, where the elements are taken under the order $X_1 \leq X_2$ if every coordinate in X_1 is greater than every coordinate in X_2:

$$\left\{\left(\frac{\lceil\frac{t_1 \cdot a}{p}\rceil - 1}{a}, \frac{\lceil\frac{t_2 \cdot a}{p}\rceil - 1}{a}, \ldots, \frac{\lceil\frac{t_{d-1} \cdot a}{p}\rceil - 1}{a}, p-a\right)\right\} \tag{†}$$

Since this condition alone is less restrictive than the poset of our reduced visibility problem, we add another constraint to arrive at the bounds we desire. To do so, note that obstructions near the origin will be overcounted many times since they belong to more parallelograms than the squares that are not as close to the origin. To combat this, we consider "primitive obstructions", in which we strive to count the obstructions along E that do not belong in a "lower parallelepiped" (one whose i^{th} coordinate is less than the i^{th} coordinate of the parallelepiped in consideration, for any $1 \leq i \leq d-1$.)

For that to happen, we do not want our obstructions to intersect any edge connecting $(0, 0, \ldots, 0, 1, 0, \ldots 0, 0)$ and the characteristic vertex, where the location of the 1 is taken over each of the first $d-1$ coordinates.

Lemma 8.1. *The poset* (†) *when strengthened to account for primitive obstructions yields the following poset:*

$$\left\{ \left(\frac{i \cdot k\%p}{k}, \frac{j \cdot k\%p}{k}, k \right) \mid i \cdot k\%p < k \ , \ j \cdot k\%p < k \right\}$$

Proof. For $1 \leq i \leq d-1$, we have that:

$$\left\lfloor \frac{t_i \cdot a}{p} \right\rfloor - 1 > a \cdot \frac{t_i - 1}{p}$$

This reduces to:

$$t_i \cdot a\%p < a$$

Our poset can then be written as:

$$\left\{ \left(\frac{i \cdot k\%p}{k}, \frac{j \cdot k\%p}{k}, k \right) \mid i \cdot k\%p < k \ , \ j \cdot k\%p < k \right\}$$

Where we are inverting the order of the prior poset. That is, $X_1 \leq X_2$ if every coordinate in X_1 is less than every coordinate in X_2. □

In order to find an upper bound on the width of this poset, we turn to Dilworth's theorem and instead find the upper bound on the size of the maximal chain cover of this poset.

To do so, consider l vectors of the form:

$$\left(\frac{t_1 \cdot l\%p}{l}, \frac{t_2 \cdot l\%p}{l}, \ldots, \frac{t_{d-1} \cdot l\%p}{l}, l \right)$$

that satisfy the conditions $(t_i \cdot l)\%p > l$ for $1 \leq i \leq d-1$. In doing so, we have that:

$$\left(\frac{t_1 \cdot a\%p}{a}, \frac{t_2 \cdot a\%p}{a}, \ldots, \frac{t_{d-1} \cdot a\%p}{a} \right) + \left(\frac{t_1 \cdot l\%p}{l}, \frac{t_2 \cdot l\%p}{l}, \ldots, \frac{t_{d-1} \cdot l\%p}{l} \right)$$

$$> \left(\frac{(t_1 \cdot a)\%p + (t_1 \cdot l)\%p}{l+a}, \frac{(t_2 \cdot a)\%p + (t_2 \cdot l)\%p}{l+a}, \ldots, \frac{(t_{d-1} \cdot a)\%p + (t_{d-1} \cdot l)\%p}{l+a} \right)$$

$$> \left(\frac{t_1(l+a)\%p}{l+a}, \frac{t_2(l+a)\%p}{l+a}, \ldots, \frac{t_{d-1}(l+a)\%p}{l+a} \right)$$

Where the last inequality fails for at most $\sum_{i=1}^{d-1}(t_i \cdot l)\%p$ cases. Thus, we can partition the poset into at most $l + \sum_{i=1}^{d-1}(t_i \cdot l)\%p$ chains. It now suffices to prove that the expected value of $l + \sum_{i=1}^{d-1}(t_i \cdot l)\%p$, with $l > t_i \cdot l\%p$ for all $1 \leq i \leq d-1$, is bounded above by a function of size $O(p^{\frac{d^2-1}{d}} \log p)$.

To do so, we transform an l vector into an m_v vector by noting that for all vectors $(t_1 \cdot a\%p, t_2 \cdot a\%p, \ldots, t_{d-1} \cdot a\%p, a)$ with maximal coordinate $m_p < \frac{p}{2}$, the vector $(t_1 \cdot (a+1)\%p, t_2 \cdot (a+1)\%p, \ldots, t_{d-1} \cdot (a+1), a+1$ satisfies $a+1 < (t_i(a+1))\%p$ for $1 \leq i \leq d-1$, and the maximal coordinate of this vector is less than or equal to $2m_p$.

To finish proving that $\mathbb{E}(m_p) << \mathbb{E}(l_p)$, it suffices to prove that $m_p < \frac{p}{2}$ for almost all t_i.

Lemma 8.2. *$m_p < \frac{p}{2}$ for enough vectors m_v such that the complexity of the upper bound of $\mathbb{E}(l_p)$ is the same as the complexity of the upper bound of $\mathbb{E}(m_p)$.*

Proof. This is a matter of manipulating the complexities of the lower bound. Since upper bound of the reduced visibility problem has a complexity of $O(p^{\frac{d^2-1}{d}} \log p)$, at most $O(p^{\frac{d^2-d-1}{d}} \log p)$ vectors can have a m_p value of greater than equal to $\frac{p}{2}$. Hence we can ignore the vectors with large m_p values since they contribute a complexity of less than $O(p^{\frac{d^2-1}{d}} \log p)$, and the remaining amount totals to our desired complexity. □

Which brings us to the main result of the upper-bound portion of this paper:

Theorem 8.1. *The largest number of reduced visible obstructions formed by projecting the unit d-hypercubes onto their bottom $d-1$-hyperface in the d dimensional true upper bound visibility environment is $O(n^{\frac{n^2-1}{n}} \log n)$.*

Proof. Summing over the cubes counted in each $\mathcal{P}$, whose characteristic vertex's largest coordinate is the x_d coordinate, gives us a count of at most $O(p^{d-1}) \cdot O(p^{\frac{d-1}{d}} \log p) = O(n^{\frac{d^2-1}{d}} \log n)$ unit hypercubes cubes that can be visible from the observer's hypercube at the origin among this set of $\frac{1}{d}p^d$ obstructing cubes. Therefore, the restricted true upper bound of the above set of obstructions has a complexity of $O(n^{\frac{d^2-1}{d}} \log n)$. □

9 Future Work

In this paper, a $\Omega(n^{\frac{d^2-1}{d}})$ lower bound on $f_d(n)$ was presented, as well as $O(n^{\frac{d^2-1}{d}} \log n)$ upper bounds in two environments, the first of which saw sight lines being restricted to only those parallel to the edges of the paralleletope in consideration, and second of which saw unit d-hypercubes replaced with their bottom hyperfaces. It remains to determine whether the aforementioned upper bound can be tightened to $O(n^{\frac{d^2-1}{d}})$. Furthermore, the question of placing a true upper bound on $f_d(n)$, in which neither sight nor shape is restricted lies unanswered.

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