Final-initial and initial-final dipoles for top quark decay kinematics

Refs.: [CS] : Catani, Seymour; hep-ph/9605323
[CDST]: Catani, Dittmaier, Seymour, Trocsanyi; hep-ph/0201036
[D] : Dittmaier; hep-ph/9904440

Momenta mapping

N+1 phase space :

$$\boxed{p_a = \underbrace{p_i}_{\text{emitter OR spectator}} + \underbrace{p_j}_{\text{emitted}} + \underbrace{K}_{\text{sum of all other final state momenta}}} \qquad (1.1)$$

with $p_a^2 = m_a^2$, $p_i^2 = m_i^2$, $p_j^2 = 0$ (later $m_i = 0$, $m_a = m_{top}$)

$\tilde{N}$ phase space:

$$\boxed{\tilde{p}_a = \tilde{p}_i + K}$$

with $\tilde{p}_a^2 = m_a^2$, $\tilde{p}_i^2 = m_i^2$.

Note that K is not transformed, therefore

$$p_a - p_i - p_j = \tilde{p}_a - \tilde{p}_i =: P_{ia}. \qquad (1.2)$$

Also, $\tilde{p}_a \neq f(x, P_{ia})\, p_a$ since $m_a \neq 0$.

$\tilde{p}_a$ and $\tilde{p}_i$ are defined according to eq. (4.17) in [D],

$$\tilde{p}_i = \frac{\sqrt{\lambda_{ia}}}{\sqrt{\lambda((p_i+p_j)^2, P_{ia}^2, m_a^2)}} \left(p_a - \frac{P_{ia}\cdot p_a}{P_{ia}^2} P_{ia} \right) + \frac{P_{ia}^2 - m_a^2 + m_i^2}{2P_{ia}^2} P_{ia} ,$$

$$\tilde{p}_a = \tilde{p}_i - P_{ia} , \tag{2.1}$$

with $\lambda_{ia} = \lambda(P_{ia}^2, m_a^2, m_i^2)$.

Subtraction terms

For the process $1_t \rightarrow 2_b + 3_w + 4_g + 5_g$ we find the following f-i and i-f dipoles,

$$\mathcal{D}_{24,1} , \quad \mathcal{D}_{25,1} , \quad \mathcal{D}_{45,1} \tag{2.3}$$

$$\mathcal{D}_{14,2} , \quad \mathcal{D}_{15,2} , \quad \mathcal{D}_{14,5} , \quad \mathcal{D}_{15,4} . \tag{2.4}$$

They are given by [CS],

$$\mathcal{D}_{ij,a} = \frac{-1}{2p_i\cdot p_j} \frac{1}{x_{ia}} \langle 1,\dots,\tilde{ij},\dots,N+1; \tilde{a} | T_a\cdot T_{ij}\, V_{ij}^a | 1,\dots,\tilde{ij},\dots,N+1; \tilde{a} \rangle , \tag{2.5}$$

$$\mathcal{D}_{aj,i} = \frac{-1}{2p_a\cdot p_j} \frac{1}{x_{ia}} \langle 1,\dots,\tilde{i},\dots,N+1; \widetilde{aj} | T_{aj}\cdot T_i\, V_{aj}^i | 1,\dots,\tilde{i},\dots,N+1; \widetilde{aj} \rangle \tag{2.6}$$

The dipole splitting functions V_{ij}^{a} in (2.5) are given by

eq. (4.16) in [D]:

$$\langle V_{qg}^{a} \rangle = 8\pi \mu^{2\varepsilon} \alpha_s \left\{ \frac{2}{2 - x_{ia} - z_{ia}} - 1 - z_{ia} \right.$$
$$\left. - \frac{m_i^2}{p_i \cdot p_j} - \frac{m_i^2 x_{ia}}{2 p_i \cdot p_j} \cdot \frac{(1-z_{ia})^2}{z_{ia}} \cdot \frac{r_{ia}}{x_{ia}} \right\}$$

The second line does not contribute, $m_i = m_{bottom} = 0$.

eq. (5.40) in [CS]:

$$\langle \mu | V_{gg}^{a} | \nu \rangle = 16\pi \mu^{2\varepsilon} \alpha_s \left\{ -g^{\mu\nu} \left(\frac{1}{2 - x - z} + \frac{1}{2 - x - (1-z)} - 2 \right) \right.$$
$$+ (1-\varepsilon) \frac{1}{p_i \cdot p_j} \left(z_{ia} p_i^{\mu} - (1-z_{ia}) p_j^{\mu} \right) \times$$
$$\left. \left(z_{ia} p_i^{\nu} - (1-z_{ia}) p_j^{\nu} \right) \right\}$$

where

$$x_{ia} = 1 - \frac{p_i \cdot p_j}{p_i \cdot p_a + p_j \cdot p_a}, \qquad \text{eq. (3.10) in [D]}$$

$$z_{ia} = \frac{p_i \cdot p_a}{p_i \cdot p_a + p_j \cdot p_a},$$

$$r_{ia} = 1 + \frac{\overline{P}_{ia}^2 \left(\overline{P}_{ia}^2 + 2 m_a^2 \right)}{\lambda_{ia}} \cdot \frac{1 - x_{ia}}{x_{ia}}, \qquad \text{eq. (4.15) in [D]}$$

$$\overline{P}_{ia}^2 = P_{ia}^2 - m_a^2 - m_i^2. \qquad \text{eq. (4.12) in [D]}$$

The dipole splitting functions V_{aj}^{i} in (2.6) are given by eq. (4.32) in [D]:

$$\langle V_{qg}^{i} \rangle = 8\pi \mu^{2\varepsilon} \alpha_s \left\{ \frac{2}{2 - x_{ia} - z_{ia}} - (1 + x_{ia}) R_{ia} - \frac{x_{ia} m_a^2}{p_a \cdot p_j} - \frac{m_a^2}{2 p_a \cdot p_j} \cdot (1 - x_{ia})^2 \right\}$$

with the same x_{ia}, z_{ia} as above and

$$R_{ia} = \frac{\sqrt{(P_{ia}^2 + 2 m_a^2 x_{ia})^2 - 4 m_a^2 P_{ia}^2 x_{ia}^2}}{\sqrt{\lambda_{ia}}} \qquad \text{eq. (4.15) in [D]}$$

$\langle V_{qg}^{i} \rangle$ can be simplified because there is no collinear singularity.

$$\langle V_{qg}^{i} \rangle = 8\pi \mu^{2\varepsilon} \alpha_s \left\{ \frac{2}{2 - x_{ia} - z_{ia}} - \frac{m_t^2}{p_a \cdot p_j} \right\}$$

Simplification of i-f dipole

$$\mathcal{D}_{qg,i} = \frac{-1}{2p_a\cdot p_j}\cdot\frac{1}{x_{ia}}\langle\ldots|T_{aj}\cdot T_i\, V_{aj}^i|\ldots\rangle$$

V_{aj}^i can be choosen to be

$$V_{aj}^i = 8\pi\mu^{2\varepsilon}\alpha_s\cdot\left\{\frac{2}{2-x_{ia}-z_{ia}} - \frac{m_t^2}{p_a\cdot p_j}\right\}.$$

because $\frac{2}{2-x_{ia}-z_{ia}} \xrightarrow{p_j\to 0} 2\,\frac{p_i\cdot p_a}{(p_i+p_a)\cdot p_j}$.

$$\boxed{\begin{aligned} 1-x &= \frac{p_i\cdot p_j}{(p_i+p_j)p_a} \\ 1-z &= \frac{p_a\cdot p_j}{(p_i+p_j)p_a}\end{aligned}}$$

i: quark

top

$$\Rightarrow \quad \mathcal{D}_{qg,i} = \frac{-1}{2p_i\cdot p_j}\,\frac{1}{x_{ia}}\cdot\langle\ldots|\left\{\frac{p_i\cdot p_j}{p_a\cdot p_j}\,8\pi\mu^{2\varepsilon}\alpha_s\left(\frac{2}{2-x_{ia}-z_{ia}} - \frac{m_t^2}{p_a\cdot p_j}\right)\right\}|\ldots\rangle$$

and add to $\mathcal{D}_{qg,a}$:

$$\Rightarrow \quad \langle V_{qg}^a\rangle^{new} = 8\pi\mu^{2\varepsilon}\alpha_s\left\{\frac{2}{2-x_{ia}-z_{ia}}\underbrace{\left(1+\frac{p_i\cdot p_j}{p_a\cdot p_j}\right)}_{\frac{2-x_{ia}-z_{ia}}{1-z_{ia}}} - 1 - z_{ia} - \frac{p_i\cdot p_j}{p_a\cdot p_j}\,\frac{m_t^2}{p_a\cdot p_j}\right\}$$

$$= 8\pi\mu^{2\varepsilon}\alpha_s\left\{\frac{2}{1-z_{ia}} - (1+z_{ia}) - \frac{1-x_{ia}}{1-z_{ia}}\,\frac{m_t^2}{p_a\cdot p_j}\right\}$$

$$= 8\pi\mu^{2\varepsilon}\alpha_s\left\{\frac{1+z_{ia}^2}{1-z_{ia}} - \frac{1-x_{ia}}{1-z_{ia}}\,\frac{m_t^2}{p_a\cdot p_j}\right\}$$

$\Rightarrow \quad \mathcal{D}_{qg,i}$ (i: gluon; top) there are always two contributions:

$$V_{qg_1,g_2} + V_{qg_2,g_1} = -\frac{1}{2p_i\cdot p_j}\,\frac{1}{x_{ia}}\,8\pi\mu^{2\varepsilon}\alpha_s \cdot \Bigg\{ \frac{p_i\cdot p_j}{p_a\cdot p_j}\left(\frac{2}{2-x_{ia}-z_{ia}} - \frac{m_t^2}{p_a\cdot p_j}\right) + \frac{p_i\cdot p_j}{p_a\cdot p_i}\left(\frac{2}{2-x_{ia}-(1-z_{ia})} - \frac{m_t^2}{p_a\cdot p_i}\right)\Bigg\}$$

x_{ia} is symm. in $i \leftrightarrow j$; $i \leftrightarrow j$ is equiv to $z_{ia} \leftrightarrow 1-z_{ia}$

$$= \frac{-1}{2p_i\cdot p_j}\,\frac{1}{x_{ia}}\Bigg\{\frac{1-x}{1-z}\left(\frac{2}{2-x_{ia}-z_{ia}} - \frac{m_t^2}{p_a\cdot p_j}\right) + \frac{1-x}{z}\left(\frac{2}{2-x_{ia}-(1-z_{ia})} - \frac{m_t^2}{p_a\cdot p_i}\right)\Bigg\}\,8\pi\mu^{2\varepsilon}\alpha_s$$

add this to $\mathcal{D}_{g_1g_2,a}$:

$$\langle\mu|V^a_{gg}|\nu\rangle^{new} = 16\pi\mu^{2\varepsilon}\alpha_s\Bigg\{-g^{\mu\nu}\Bigg[\frac{1}{2-x_{ia}-z_{ia}}\overbrace{\left(1+\frac{1-x_{ia}}{1-z_{ia}}\right)}^{\frac{2-z_{ia}-x_{ia}}{1-z_{ia}}} - 2 + \frac{1}{2-x_{ia}-(1-z_{ia})}\overbrace{\left(1+\frac{1-x}{z}\right)}^{(2-x-(1-z))/z} - \frac{1-x}{1-z}\,\frac{m_t^2}{2p_a\cdot p_j} - \frac{1-x}{z}\,\frac{m_t^2}{2p_a\cdot p_i}\Bigg] + (1-\varepsilon)\ \ldots \text{as usual}^{\mu\nu}\ldots\Bigg\}$$

$$= 16\pi\mu^{2\varepsilon}\alpha_s\Bigg\{-g^{\mu\nu}\Bigg[\frac{z}{1-z} + \frac{1-z}{z} - \frac{m_t^2}{2}(1-x)\left(\frac{1}{(1-z)\,p_a\cdot p_j} + \frac{1}{z\,p_a\cdot p_i}\right)\Bigg] + (1-\varepsilon)\ \ldots \text{as usual}^{\mu\nu}\ldots\Bigg\}$$

Results of page 4a, 4b can be used to simplify eq. (2.3), (2.4):

$$\begin{cases} \mathcal{D}^{\text{New}}_{24,1} & \text{includes soft limit of } \mathcal{D}_{14,2}, \\ \mathcal{D}^{\text{New}}_{25,1} & \text{includes soft limit of } \mathcal{D}_{15,2}, \\ \mathcal{D}^{\text{New}}_{45,1} & \text{includes soft limit of } (\mathcal{D}_{14,5} + \mathcal{D}_{15,4}). \end{cases}$$

Therefore, the new rule for writing down all dipoles is:

$$\text{Dipoles} = \sum_{\substack{i,j \\ i<j}} \sum_{k \neq i,j} \mathcal{D}_{ij,k} + \sum_{i,j} \mathcal{D}^{\text{New}}_{ij,a} + \text{no i-f contribution}$$

Phase space factorization in D=4

We have to examine

$$PS = \int d\phi^{(3)}(p_a \mid p_i, p_j, K) \tag{5.1}$$

and want to split it into the phase space for p_j and the rest. Since the particle j recoils against inital state particle a, the <u>whole</u> phase space depends on the energy fraction $\tilde{x}$ taken away. This x-dependence cannot be integrated analytically. Thus, we want to derive

$$PS = \int dx \cdot \int d\phi^{(2)}(\tilde{p}_a(x) \mid \tilde{p}_i(x), K) \times \int d\phi^{(1)}(p_j). \tag{5.2}$$

To extract an explicit parametrization of $\int d\phi^{(1)}(p_j)$ in (5.2) we take the following steps.

1.) we factorize (5.1) into

$$PS = \int \frac{dq^2}{2\pi} d\phi^{(2)}(p_a \mid q, K) \times d\phi^{(2)}(q \mid p_i, p_j) \tag{5.3}$$

2.) We choose a frame and parametrization such that $d\phi^{(2)}(p_a|q,K)$ in (5.3) and $d\phi^{(2)}(\tilde{p}_a|\tilde{p}_i,K)$ in (5.2) can be identified (up to Lorentz invariant factors).

3.) This allows us to read off $d\phi^{(1)}(p_j)$ in (5.2) by comparing with the remaining pieces in (5.3).
In particular, $\int dq^2$ will turn into $\int dx$ and $d\phi^{(2)}(q|p_i,p_j)$ will result into an integral over z_{ia} and an azimuth angle of particle j.

Step 1:

$$\int d\phi^{(3)}(p_a|p_i,p_j,K) = \int \frac{d^D p_i}{(2\pi)^D} \cdot \frac{d^D p_j}{(2\pi)^D} \cdot \frac{d^D K}{(2\pi)^D} \cdot (2\pi)^D \delta^{(D)}(p_a - p_i - p_j - K)$$
$$\times\, 2\pi\,\delta^+(p_i^2 - m_i^2) \cdot 2\pi\,\delta^+(p_j^2 - m_j^2) \cdot 2\pi\,\delta^+(K^2 - m_K^2)$$

insert $1 = \int d^D q\;\delta^{(D)}(q - p_i - p_j)$,

$1 = \int dq^2\;\delta^+(q^2 - (p_i+p_j)^2)$

$$\boxed{\delta^+(p^2) = \delta(p^2)\,\theta(p^0)}$$

$$\ldots = (2\pi)^{3-2D} \int dq^2 \cdot d^D q \cdot d^D K\;\delta^+(q^2 - (p_i+p_j)^2)\,\delta^+(K^2 - m_K^2) \cdot \delta^{(D)}(p_a - q - K)$$
$$\times\, d^D p_i \cdot d^D p_j\;\delta^+(p_i^2 - m_i^2) \cdot \delta^+(p_j^2 - m_j^2) \cdot \delta^{(D)}(q - p_i - p_j)$$

$$= \int \frac{dq^2}{2\pi}\, d\phi^{(2)}(p_a|q,K) \times d\phi^{(2)}(q|p_i,p_j) \qquad (6.1)$$

<u>Step 2:</u> We choose a frame where $\vec{p}_a + \vec{\tilde{p}}_a = \vec{0}$.

- Let's parametrize $d\phi^{(2)}(p_a | q, K)$ in (6.1),

$$\int d\phi^{(2)}(p_a | q, K) = (2\pi)^{2-D} \int d^D q \cdot d^D K \cdot \delta^{(D)}(p_a - q - K)\, \delta^+(q^2 - m_q^2)\, \delta^+(K^2 - m_K^2)$$

$$= (2\pi)^{2-D} \int \frac{d^{D-1} q}{2 q^0}\, \delta\left(m_a^2 + m_q^2 - m_K^2 - 2 p_a \cdot q\right)$$

$\boxed{D=4}$

$$= (2\pi)^{2-D} \int dq^0\, \frac{|\vec{q}|}{2}\, d\Omega_q\, \delta\left(m_a^2 + m_q^2 - m_K^2 - 2 p_a^0 q^0 + 2 \vec{p}_a \vec{q}\right)$$

Integrate over δ-fct., then introduce $t = (p_a - q)^2$.

$$\Rightarrow \quad d\Omega_q = d\cos\theta_q \cdot d\varphi_q = \frac{dt}{2 |\vec{p}_a| \cdot |\vec{q}|} \cdot d\varphi_q$$

$$\ldots = (2\pi)^{2-D} \frac{1}{8 |\vec{p}_a|\, p_a^0} \int dt \cdot d\varphi_q . \qquad (7.1)$$

- Let's parametrize $d\phi^{(2)}(\tilde{p}_a | \tilde{p}_i, K)$ in (5.2),

$$\int d\phi^{(2)}(\tilde{p}_a | \tilde{p}_i, K) = (2\pi)^{2-D} \int d^D \tilde{p}_i \cdot d^D K\, \delta^{(D)}(\tilde{p}_a - \tilde{p}_i - K)\, \delta^+(\tilde{p}_i^2 - m_i^2)\, \delta^+(K^2 - m_K^2)$$

$\boxed{D=4}$

$$= (2\pi)^{2-D} \int d\tilde{p}_i^0\, \frac{|\vec{\tilde{p}}_i|}{2}\, d\tilde{\Omega}_i\, \delta\left(m_a^2 + m_i^2 - m_K^2 - 2 \tilde{p}_a^0 \tilde{p}_i^0 + 2 \vec{\tilde{p}}_a \cdot \vec{\tilde{p}}_i\right)$$

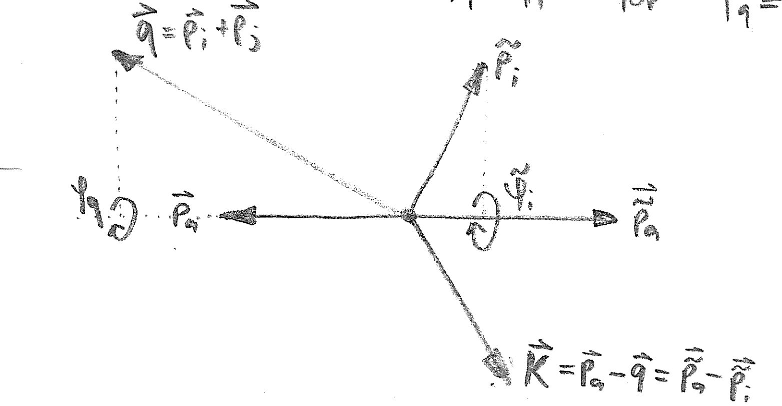
Integrate over δ-fct, then introduce $\tilde{t} = (\tilde{p}_a - \tilde{p}_i)^2$

$$\Rightarrow \quad d\tilde{\Omega}_i = d\cos\tilde{\theta}_i\, d\tilde{\varphi}_i = \frac{d\tilde{t}}{2 |\vec{\tilde{p}}_a| \cdot |\vec{\tilde{p}}_i|} \cdot d\tilde{\varphi}_i$$

$$\ldots = (2\pi)^{2-D} \frac{1}{8 |\vec{\tilde{p}}_a| \cdot \tilde{p}_a^0} \int d\tilde{t} \cdot d\tilde{\varphi}_i . \qquad (7.2)$$

- Now, we argue that (7.1) and (7.2) can be identified.

Since we are in a frame where $\vec{p}_a + \vec{\tilde{p}}_a = \vec{0}$ and $p_a^2 = \tilde{p}_a^2 = m_a^2$, $p_a^0 = \tilde{p}_a^0$ and $|\vec{p}_a| = |\vec{\tilde{p}}_a|$. Furthermore, $t = \tilde{t}$ because $p_a - p_i - p_j = \tilde{p}_a - \tilde{p}_i$, cf. (1.2).
Also the integrations over φ_q and $\tilde{\varphi}_i$ can be identified if we choose $\varphi_q = \tilde{\varphi}_i$ for $\varphi_q = 0$.



<u>Step 3:</u> Let's parametrize $\int d\phi^{(2)}(q|p_i,p_j)$ in (5.3).
Since this is a separately Lorentz invariant phase space, we choose a different frame, where $\vec{q} = \vec{0}$.

$$\Rightarrow \quad \int d\phi^{(2)}_{D=4}(q|p_i,p_j) = (2\pi)^{2-D} \int dp_i^0 \, \frac{|\vec{p}_i|}{2} \, d\cos\theta_i \, d\varphi_i \; \delta\left(m_q^2 + m_i^2 - m_j^2 - 2p_i^0 q^0\right)$$

The polar axis is oriented to $\vec{p}_a$, so that for $v = (p_a - p_i)^2$, $dv = 2|\vec{p}_i| \cdot |\vec{p}_a| d\cos\theta_i$.

$$\ldots = (2\pi)^{2-D} \frac{1}{8|\vec{p}_a| q^0} \int dv \cdot d\varphi_i$$

In the frame where $\vec{q}=0$, we have

$$|\vec{P}_a| = \frac{m_a^2+q^2-K^2}{2q^2} \quad \text{and} \quad q^0 = q.q.$$

Furthermore,
$$v = (P_a - P_i)^2 = m_a^2 + m_i^2 - z_{ia}\left(m_a^2 - P_{ia}^2 + (P_i+P_j)^2\right)$$
$$dv = \left(P_{ia}^2 - m_a^2 - (P_i+P_j)^2\right) \cdot dz_{ia}.$$

The first integral in (6.1) can be written as

$$\int \frac{dq^2}{2\pi} = \int \frac{dx_{ia}}{x^2}\frac{\bar{P}_{ia}^2}{2\pi}, \quad \text{since} \quad P_{ia}^2 - m_a^2 - q^2 = \frac{\bar{P}_{ia}^2}{x_{ia}}.$$

Now, we can write (5.3)

$x_{ia}=x$

$$PS = \int \frac{dx_{ia}}{x_{ia}^2} \cdot \frac{\bar{P}_{ia}^2}{2\pi} \cdot d\phi^{(2)}\left(\tilde{P}_a(x) \,|\, \tilde{P}_i(x), K\right) \cdot \frac{(2\pi)^{2-D}}{8}$$
$$\times \frac{\bar{P}_{ia}^2}{2\left(\bar{P}_{ia}^2 + x m_a^2 - x P_{ia}^2\right)} \cdot \frac{\bar{P}_{ia}^2}{x} \int dz_{ia} \cdot d\varphi_i .$$

NOTE: INTEGRATION LIMITS ON dx AND dz HAVE TO BE DETERMINED SOMEHOW.

determine integration limits for $\int dz_{ia}$:

original integral is: $\int_{-1}^{+1} d\cos\theta$

$$v = (p_a - p_i)^2 = m_a^2 + m_i^2 - 2E_aE_i + 2|\vec{p}_a|\cdot|\vec{p}_i|\cdot\cos\theta$$

$$\Rightarrow \quad \sim \int_{v_-}^{v_+} dv \ldots \quad \text{with}$$

$$v_\pm = m_a^2 + m_i^2 - 2E_aE_i \pm 2|\vec{p}_a|\cdot|\vec{p}_i|$$

in the frame where $\vec{q} = 0$:

$$E_a \cdot E_i = \frac{(m_a^2 + q^2 - P_{ia}^2)(q^2 + m_i^2)}{4q^2}$$

$$|\vec{p}_a|\cdot|\vec{p}_i| = \frac{|q^2 - m_i^2|\,\lambda^{1/2}(q^2, m_a^2, P_{ia}^2)}{4q^2}$$

$$\Rightarrow \quad \sim \ldots \int_{z_-}^{z_+} dz \ldots \quad \text{with}$$

$$z_\pm = \frac{m_a^2 + m_i^2 - v_\pm}{m_a^2 - P_{ia}^2 + q^2} = \frac{2E_aE_i \mp 2|\vec{p}_a|\cdot|\vec{p}_i|}{m_a^2 - P_{ia}^2 + q^2}$$

$$= \frac{1}{m_a^2 - P_{ia}^2 + q^2} \cdot \frac{1}{2q^2}\Big\{(m_a^2 + q^2 - P_{ia}^2)(q^2 + m_i^2) \mp (q^2 - m_i^2)\,\lambda^{1/2}(q^2, m_a^2, P_{ia}^2)\Big\}$$

N6

$$z_\pm = \frac{x^2}{2\,\bar{p}_{1a}^2\left(\bar{p}_{1a}^2 - x\left(p_{1a}^2 - m_a^2\right)\right)} \cdot \left\{ \frac{\bar{p}_{1a}^2}{x^2}\left(\bar{p}_{1a}^2 - x\left(\bar{p}_{1a}^2 + 2m_i^2\right)\right) \mp \bar{p}_{1a}^2\left(\frac{x-1}{x}\right)\frac{1}{x}\left[\ldots \text{see N6} \ldots\right]^{1/2} \right\}$$

$$= \frac{1}{2\cancel{\bar{p}_{1a}^2}\left(\bar{p}_{1a}^2 - x\left(p_{1a}^2 - m_a^2\right)\right)} \left\{ \cancel{\bar{p}_{1a}^2}\left(\bar{p}_{1a}^2 - x\left(\bar{p}_{1a}^2 + 2m_i^2\right)\right) \mp \cancel{\bar{p}_{1a}^2}(x-1)\left[\left(\bar{p}_{1a}^2 + 2m_a^2 x\right)^2 - 4m_a^2 p_{1a}^2 x^2\right]^{1/2} \right\}$$

in agreement with eqn (4.22) in [D].

note: $\bar{p}_{1a}^2(x-1) \geq 0$ since $\bar{p}_{1a}^2 < 0$.

<u>useful</u>:

$$z_- + z_+ = \frac{\bar{p}_{1a}^2 - x\left(\bar{p}_{1a}^2 + 2m_i^2\right)}{\left(\bar{p}_{1a}^2 - x\left(p_{1a}^2 - m_a^2\right)\right)},$$

$$z_- - z_+ = \frac{(x-1)\left[\ldots \text{see above} \ldots\right]^{1/2}}{\bar{p}_{1a}^2 - x\left(p_{1a}^2 - m_a^2\right)},$$

$$1 - z_\pm = \frac{(x-1)\left(-\bar{p}_{1a}^2 \pm \left[\ldots \text{see above} \ldots\right]^{1/2}\right)}{2\left(\bar{p}_{1a}^2 - x\left(p_{1a}^2 - m_a^2\right)\right)},$$

$$2 + z_- + z_+ = \frac{-3\bar{p}_{1a}^2(x-1) - 4m_i^2 x}{\bar{p}_{1a}^2 - x\left(p_{1a}^2 - m_a^2\right)}$$

repeat step 3 in D dimensions:

$$\int d\phi_D^{(2)}(q|p_i,p_j) = (2\pi)^{2-D}\int \frac{d^{D-1}p_i}{2E_i}\,\delta^+\left((q-p_i)^2-m_j^2\right)$$

choose frame where $\vec{q}=\vec{0}$ and set $m_j=0$

$$= (2\pi)^{2-D}\int \frac{d|\vec{p}_i|}{2E_i}\,|\vec{p}_i|^{D-2}\cdot d\Omega_i^{D-1}\,\delta^+(m_q^2+m_i^2-2E_qE_i)$$

$|\vec{p}_i| = \sqrt{E_i^2-m_i^2}$, $\quad d|\vec{p}_i| = \frac{E_i\,dE_i}{|\vec{p}_i|}$

$$= (2\pi)^{2-D}\int dE_i\,\frac{1}{2}|\vec{p}_i|^{D-3}\,d\Omega_i^{D-1}\,\frac{1}{2E_q}\,\delta^+\left(E_i-\frac{m_q^2+m_i^2}{2E_q}\right)$$

$$= (2\pi)^{2-D}\,\frac{1}{4E_q}\,|\vec{p}_i|^{D-3}\int d\theta\,\sin^{D-3}\theta\cdot d\Omega_i^{D-2}$$

choose $\quad v=(p_a-p_i)^2 = m_a^2+m_i^2-2E_aE_i+2|\vec{p}_a|\cdot|\vec{p}_i|\cdot\cos\theta$

$dv = -2|\vec{p}_a|\cdot|\vec{p}_i|\sin\theta\cdot d\theta$

$\cos\theta = \sqrt{1-\sin^2\theta} = \frac{v-m_a^2-m_i^2+2E_aE_i}{2|\vec{p}_a|\cdot|\vec{p}_i|}$

$E_a = \frac{m_a^2+m_q^2-m_k^2}{2m_q}$, $\quad |\vec{p}_i| = \sqrt{\lambda(m_q^2,m_i^2,0)}\,\frac{1}{2m_q} = \frac{1}{2m_q}|m_q^2-m_i^2|$,

$E_i = \frac{m_q^2+m_i^2}{2m_q}$

$E_q = m_q$, $\quad |\vec{p}_a| = \sqrt{\lambda(m_q^2,m_a^2,m_k^2)}\,\frac{1}{2m_q}$

$$= (2\pi)^{2-D}\,\frac{1}{4m_q}\,\frac{|\vec{p}_i|^{D-4}}{2|\vec{p}_a|}\,(-1)\int dv\,\sin^{D-4}\theta\cdot d\Omega_i^{D-2} \qquad (10.1)$$

Now, the full phase space integral (5.3) becomes

$$PS = \int \frac{dq^2}{2\pi} \; d\phi^{(2)}(p_a | q, K) \cdot d\phi^{(2)}(q | p_i, K)$$

$$= \int \overbrace{\frac{dx}{2\pi} \cdot \frac{K^2 - m_i^2 - m_a^2}{x^2}}^{\text{see NS: } dq^2/2\pi} \cdot d\phi^{(2)}(\tilde{p}_a | \tilde{p}_i, K) \cdot \overbrace{\frac{1}{4} \lambda^{-1/2}(q^2, m_a^2, K^2)}^{\frac{1}{8\sqrt{q^2} \cdot |\tilde{p}_a|}, \text{ see NS}}$$

$$(2\pi)^{2-D} \cdot \underbrace{\frac{(q^2 - m_i^2)^{-2\varepsilon}}{2^{-2\varepsilon} \sqrt{q^2}^{-2\varepsilon}}}_{|\vec{p}_i|^{D-4}} \cdot (-1) \cdot \underbrace{\int \frac{dz}{x} (K^2 - m_i^2 - m_a^2)}_{dv} \cdot \sin^{-2\varepsilon}\theta \cdot d\Omega_i^{D-2}$$

$$= -(2\pi)^{1-D} \int \frac{dx}{x^3} \; d\phi^{(2)}(\tilde{p}_a | \tilde{p}_i, -p_{ia}) \cdot (p_{ia}^2 - m_i^2 - m_a^2)^2 \cdot \frac{1}{4} \lambda^{-1/2}(q^2, m_a^2, p_{ia}^2)$$

(11.1) $$\times \; 2^{2\varepsilon} (q^2)^{\varepsilon} (q^2 - m_i^2)^{-2\varepsilon} \int dz \cdot \sin^{-2\varepsilon}\theta \cdot d\Omega_i^{D-2}$$

$$D_{aj,i}(x) := \int_{z_-}^{z_+} dz \; \sin^{-2\varepsilon}(\theta) \cdot \mathcal{D}_{aj,i}(z,x)$$

<u>case</u>: $(aj,i) = (1_t\, 5_g, 2_b)$

$$\mathcal{D}_{aj,i} = \overbrace{8\pi\mu^{2\varepsilon}\alpha_s}^{\mathcal{N}} \frac{-1}{2p_i \cdot p_j} \frac{1}{x} \left\{ \frac{1+z^2}{1-z} - \frac{1-x}{1-z} \frac{m_t^2}{p_a \cdot p_j} \right\} \langle \ldots || \ldots \rangle$$

$$\overset{NS}{\underset{\downarrow}{=}} \mathcal{N} \cdot \langle \ldots || \ldots \rangle \cdot \frac{1}{1-x} \frac{1}{\bar{P}_{ia}^2} \left\{ \frac{1+z^2}{1-z} + \frac{(1-x)2x}{1-z} \frac{m_t^2}{\bar{P}_{ia}^2} \right\}$$

$$= \left(A \cdot \frac{1+z^2}{1-z} + B \cdot \frac{1}{1-z} \right) \mathcal{N} \langle \ldots || \ldots \rangle$$

with $A = \frac{1}{\bar{P}_{ia}^2} \cdot \frac{1}{1-x}$, $B = \frac{m_t^2}{\bar{P}_{ia}^4} 2x$

first try without the $\sin^{-2\varepsilon}(\theta)$ terms:

$$D_{aj,i} = \frac{1}{2} \left\{ A(z_- - z_+)(2 + z_- + z_+) + 2(2A+B) \log\left(\frac{1-z_-}{1-z_+} \right) \right\}$$

$$= \frac{1}{2\bar{P}_{ia}^2} \left\{ \frac{\left[(\bar{P}_{ia}^2 + 2m_a^2 x)^2 - 4m_a^2 P_{ia}^2 x^2 \right]^{1/2} \cdot \left(3\bar{P}_{ia}^2 (x-1) + 4m_i^2 x \right)}{\left[\bar{P}_{ia}^2 - x(P_{ia}^2 - m_a^2) \right]^2} \right.$$

(12.1) $$\left. + 4\left(\frac{1}{1-x} + \frac{x m_t^2}{\bar{P}_{ia}^2} \right) \log\left(\frac{-\bar{P}_{ia}^2 - [\ldots \text{as above} \ldots]^{1/2}}{-\bar{P}_{ia}^2 + [\ldots \text{as above} \ldots]^{1/2}} \right) \right\}$$

setting $m_i = 0$:

$$\frac{1}{2\overline{P}_{ia}^2}\left\{\frac{[\ldots]^{1/2}\cdot 3\overline{P}_{ia}^2(x-1)}{\overline{P}_{ia}^4(1-x)^2} + 4\ldots\right\}$$

$$= \frac{1}{2\overline{P}_{ia}^2}\left\{\frac{+3[\ldots]^{1/2}(-1)}{\overline{P}_{ia}^2(1-x)} + 4\ldots\right\}$$

$$[\ldots] = -x^2\, 4m_a^2\overline{P}_{ia}^2 + x\, 4m_a^2\overline{P}_{ia}^2 + \overline{P}_{ia}^4$$

$$= -4m_a^2\overline{P}_{ia}^2\left[x^2 - x - \frac{\overline{P}_{ia}^2}{4m_a^2}\right]$$

$$x_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{\overline{P}_{ia}^2}{4m_a^2}}$$

$$= \frac{1}{2} \pm \sqrt{\frac{m_a^2 + \overline{P}_{ia}^2}{4m_a^2}}$$

$$= \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{\overline{P}_{ia}^2}{m_a^2}}$$

$$= -\overline{P}_{ia}^4\,\mu_a^2\left(x^2 - \frac{1}{\mu_a^2}\right)$$

$$\mu = \sqrt{\frac{m_a^2}{\overline{P}_{ia}^2}}$$

$$= -m_a^2\,\overline{P}_{ia}^2\left(x + \underbrace{\sqrt{\frac{\overline{P}_{ia}^2}{m_a^2}}}_{\mu}\right)\left(x - \sqrt{\frac{\overline{P}_{ia}^2}{m_a^2}}\right) =$$

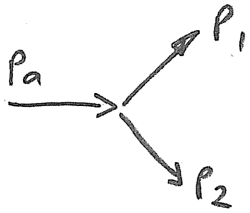
$$= \frac{1}{2\bar{p}_{ia}^2} \left\{ \frac{+3\cancel{\bar{p}_{ia}^2}\, \mu_a \left(x^2 - \frac{1}{\mu_a^2}\right)^{1/2}}{\cancel{\bar{p}_{ia}^2}\,(1-x)} \right.$$

$$\left. + 4\left(\frac{1}{1-x} + \frac{x m_e^2}{\bar{p}_{ia}^2}\right) \log\left(\frac{-\bar{p}_{ia}^2 + \bar{p}_{ia}^2 \mu_a \left(x^2 - \frac{1}{\mu^2}\right)^{1/2}}{-\bar{p}_{ia}^2 - \bar{p}_{ia}^2 \mu_a \left(x^2 - \frac{1}{\mu^2}\right)^{1/2}}\right) \right\}$$

$$= \frac{1}{2\bar{p}_{ia}^2} \left\{ \frac{3\mu_a \left(x^2 - \frac{1}{\mu_a^2}\right)^{1/2}}{(1-x)} \right.$$

$$\left. + 4\left(\frac{1}{1-x} + \frac{x m_e^2}{\bar{p}_{ia}^2}\right) \log\left(\frac{1 - \mu_a\left(x^2 - \frac{1}{\mu^2}\right)^{1/2}}{1 + \mu_a\left(x^2 - \frac{1}{\mu^2}\right)^{1/2}}\right) \right\}$$

N1



$$p_a = p_1 + p_2$$

$$p_a - p_1 = p_2$$

$$p_a^2 + p_1^2 - 2 p_a \cdot p_1 = p_2^2$$

goto rest frame of p_a: $p_a = \begin{pmatrix} E_a \\ \vec{0} \end{pmatrix} = \begin{pmatrix} m_a \\ \vec{0} \end{pmatrix}$

$$m_a^2 + m_1^2 - 2 m_a E_1 = m_2^2$$

$$\Rightarrow \quad E_1 = \frac{m_a^2 + m_1^2 - m_2^2}{2 m_a}$$

$$E_2 = m_a - E_1 = \frac{m_a^2 + m_2^2 - m_1^2}{2 m_a}$$

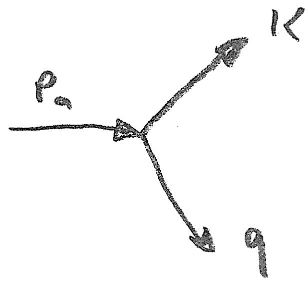
$$|\vec{p}_1|^2 = |\vec{p}_2|^2 = E_1^2 - m_1^2 = \frac{(m_a^2 + m_1^2 - m_2^2)^2 - 4 m_a^2 m_1^2}{4 m_a^2}$$

$$= \frac{m_a^4 + (m_1^2 - m_2^2)^2 + 2 m_a^2 (m_1^2 - m_2^2) - 4 m_a^2 m_1^2}{4 m_a^2}$$

$$= \frac{1}{4 m_a^2} \left((m_1 + m_2)^2 - m_a^2 \right) \cdot \left((m_1 - m_2)^2 - m_a^2 \right) = \lambda(m_a^2, m_1^2, m_2^2) \frac{1}{4 m_a^2}$$

$$\Rightarrow \quad |\vec{p}_1| = |\vec{p}_2| = \sqrt{\lambda(\underset{\uparrow\, m_a^2}{s}, m_1^2, m_2^2)} \; \frac{1}{2 m_a}$$

N2



in frame where $\vec{q} = 0$

$$\Rightarrow \quad p_a^2 = (p_K + p_q)^2$$

$$\Leftrightarrow \quad m_a^2 = m_K^2 + m_q^2 + 2 m_q E_K$$

$$\Rightarrow \quad E_K = \frac{m_a^2 - m_q^2 - m_K^2}{2 m_q}$$

$$E_a = E_K + E_q = \frac{m_a^2 - m_q^2 - m_K^2}{2 m_q} + m_q$$

$$= \frac{m_a^2 + m_q^2 - m_K^2}{2 m_q}$$

$$|\vec{p}_a|^2 = E_a^2 - m_a^2 = \frac{(m_q^2 + m_a^2 - m_K^2) - 4 m_a^2 m_q^2}{4 m_q^2}$$

$$= \frac{1}{4 m_q^2} \lambda(m_q^2, m_a^2, m_K^2)$$

N3

$$z_{ia} = \frac{p_a \cdot p_i}{p_a p_i + p_a p_j} = \frac{1}{2} \frac{-(p_a - p_i)^2 + m_a^2 + m_i^2}{p_a p_i + p_a p_j}$$

$$\left[\begin{array}{l} p_i - p_a = P_{ia} - p_j \Rightarrow m_i^2 + m_a^2 - 2 p_i p_a = P_{ia}^2 + m_j^2 - 2 P_{ia} \cdot p_j \\ p_j - p_a = P_{ia} - p_i \Rightarrow m_j^2 + m_a^2 - 2 p_j p_a = P_{ia}^2 + m_i^2 - 2 P_{ia} \cdot p_i \\ \Rightarrow 2(p_a p_i + p_a p_j) = 2 m_a^2 - 2 P_{ia}^2 + 2 \underbrace{P_{ia}}_{p_i + p_j - p_a} \cdot (p_i + p_j) \\ \Leftrightarrow 4(p_a p_i + p_a p_j) = 2 m_a^2 - 2 P_{ia}^2 + 2 (p_i + p_j)^2 \\ \Rightarrow 2(p_a p_i + p_a p_j) = (p_i + p_j)^2 + m_a^2 - P_{ia}^2 \end{array}\right.$$

$$\ldots = \frac{m_a^2 + m_i^2 - (p_a - p_i)^2}{(p_i + p_j)^2 - P_{ia}^2 + m_a^2}$$

$$\Rightarrow \boxed{(p_a - p_i)^2 = - z_{ia} \left((p_i + p_j)^2 - P_{ia}^2 + m_a^2 \right) + m_a^2 + m_i^2}$$

N4

$$x_{ia} = \frac{p_a(p_i + p_j) - p_i p_j}{p_a(p_i + p_j)}$$

$$p_{ia}^2 = (p_i + p_j - p_a)^2 = m_i^2 + m_j^2 + 2p_i p_j + m_a^2 - 2p_a(p_i + p_j)$$

$$\Rightarrow \quad p_a(p_i + p_j) - p_i p_j = \frac{1}{2}\left(-p_{ia}^2 + m_i^2 + m_j^2 + m_a^2\right)$$

$$\overset{N3}{\Rightarrow} \quad x_{ia} = \frac{-\left(p_{ia}^2 - m_i^2 - m_j^2 - m_a^2\right)}{(p_i + p_j)^2 - p_{ia}^2 + m_a^2}$$

$$\overset{m_j = 0}{\Rightarrow} \quad (p_i + p_j)^2 = \frac{-\left(p_{ia}^2 - m_i^2 - m_a^2\right)}{x_{ia}} + p_{ia}^2 - m_a^2$$

$$v \overset{N3}{=} (p_a - p_i)^2 = m_a^2 + m_i^2 - z_{ia}\Big(m_a^2 - P_{ia}^2 + \underbrace{(p_i+p_j)^2}_{q^2}\Big)$$

$$dv = \left(P_{ia}^2 - m_a^2 - q^2\right) dz_{ia}$$

$$q^2 \overset{N4}{=} \frac{-(P_{ia}^2 - m_i^2 - m_a^2)}{x_{ia}} + P_{ia}^2 - m_a^2$$

$$dq^2 = \frac{P_{ia}^2 - m_i^2 - m_a^2}{x_{ia}^2}\, dx$$

$$\Rightarrow \quad v = m_a^2 + m_i^2 + \frac{z_{ia}}{x_{ia}}\left(P_{ia}^2 - m_i^2 - m_a^2\right)$$

$$dv = \frac{dz_{ia}}{x_{ia}}\left(P_{ia}^2 - m_i^2 - m_a^2\right)$$

$$q^2 = (p_i+p_j)^2 = 2p_i\cdot p_j + m_i^2 \Rightarrow 2p_i\cdot p_j = (P_{ia}^2 - m_i^2 - m_a^2)\cdot\overbrace{\left(1 - \frac{1}{x}\right)}^{\frac{x-1}{x}}$$

$$v = (p_a - p_i)^2 = -2p_i\cdot p_a + m_i^2 + m_a^2 \Rightarrow 2p_a\cdot p_i = z\left(m_a^2 + m_i^2 - P_{ia}^2 + 2p_i\cdot p_j\right)$$

$$= -\frac{z}{x}\left(P_{ia}^2 - m_i^2 - m_a^2\right)$$

$$p_a\cdot p_j = (p_i+p_j)p_a\,(1-z) \Rightarrow \quad 2p_a\cdot p_j = -\frac{(P_{ia}^2 - m_i^2 - m_a^2)}{x}$$

$$\sin^{-2\varepsilon}\theta = (1-\cos^2\theta)^{-\varepsilon} \underset{m_i^2=0}{\overset{\uparrow}{=}} \left(\left(1 - \frac{(m_a^2 - P_{ia}^2)(1-2z)^2}{-P_{ia}^2 + m_a^2(1-2x)^2}\right)^{-\varepsilon}\right)$$

$$= 1 - \varepsilon\log(1 - \%) + \frac{\varepsilon^2}{2}\log^2(1 - \%)$$

$$m_a^2 - p_{ia}^2 + q^2 = -\frac{\bar{p}_{ia}^2}{x} ,$$

$$2q^2 = -\frac{2}{x}\left(\bar{p}_{ia}^2 - x\left(p_{ia}^2 - m_a^2\right)\right) ,$$

$$\left(m_a^2 + q^2 - p_{ia}^2\right)\left(q^2 + m_i^2\right) = \frac{\bar{p}_{ia}^2}{x^2}\left(\bar{p}_{ia}^2 - x\left(\bar{p}_{ia}^2 + 2m_i^2\right)\right) ,$$

$$q^2 - m_i^2 = \bar{p}_{ia}^2 \left(\frac{x-1}{x}\right) ,$$

$$\lambda^{1/2}\left(q^2, m_a^2, p_{ia}^2\right) = \frac{1}{x}\left[\bar{p}_{ia}^2\left(\bar{p}_{ia}^2 + 4x m_a^2\right) - 4x^2 m_a^2\left(m_i^2 + \bar{p}_{ia}^2\right)\right]^{1/2} .$$