Final-initial and initial-final dipoles for top quark decay kinematics

Refs.: [CS] : Catani, Seymour; hep-ph/9605323

[CDST]: Catani, Dittmaier, Seymour, Trocsanyi; hep-ph/0201036

[D] : Dittmaier; hep-ph/9904440

Momenta mapping

N+1 phase space :

$$\boxed{p_a = \underbrace{p_i}_{\substack{\text{emitter}\\ \text{OR spectator}}} + \underbrace{p_j}_{\text{emitted}} + \underbrace{K}_{\substack{\text{sum of}\\ \text{all other final}\\ \text{state momenta}}}} \qquad (1.1)$$

with $\quad p_a^2 = m_a^2, \quad p_i^2 = m_i^2, \quad p_j^2 = 0 \qquad$ (later $m_i = 0$, $m_a = m_{top}$)

$\tilde{N}$ phase space:

$$\boxed{\tilde{p}_a = \tilde{p}_i + K}$$

with $\quad \tilde{p}_a^2 = m_a^2, \quad \tilde{p}_i^2 = m_i^2$.

Note that K is not transformed, therefore

$$p_a - p_i - p_j = \tilde{p}_a - \tilde{p}_i =: P_{ia} . \qquad (1.2)$$

Also, $\tilde{p}_a \neq f(x, P_{ia})\, p_a$ since $m_a \neq 0$.

$\tilde{p}_a$ and $\tilde{p}_i$ are defined according to eq. (4.17) in [D],

$$\tilde{p}_i = \frac{\sqrt{\lambda_{ia}}}{\sqrt{\lambda((p_i+p_j)^2, P_{ia}^2, m_a^2)}}\left(p_a - \frac{P_{ia}\cdot p_a}{P_{ia}^2} P_{ia}\right) + \frac{P_{ia}^2 - m_a^2 + m_i^2}{2P_{ia}^2} P_{ia} ,$$

$$\tilde{p}_a = \tilde{p}_i - P_{ia} , \qquad (2.1)$$

with $\lambda_{ia} = \lambda(P_{ia}^2, m_a^2, m_i^2)$.

Subtraction terms

For the process $1_t \to 2_b + 3_w + 4_g + 5_g$ we find the following f-i and i-f dipoles,

$$\mathcal{D}_{24,1} , \quad \mathcal{D}_{25,1} , \quad \mathcal{D}_{45,1} \qquad (2.3)$$

$$\mathcal{D}_{14,2} , \quad \mathcal{D}_{15,2} , \quad \mathcal{D}_{14,5} , \quad \mathcal{D}_{15,4} . \qquad (2.4)$$

They are given by [CS],

$$\mathcal{D}_{ij,a} = \frac{-1}{2p_i\cdot p_j} \frac{1}{x_{ia}} \langle 1,\dots,\tilde{ij},\dots,N+1;\tilde{a} | T_a\cdot T_{ij}\, V_{ij}^a | 1,\dots,\tilde{ij},\dots,N+1;\tilde{a}\rangle , \qquad (2.5)$$

$$\mathcal{D}_{aj,i} = \frac{-1}{2p_a\cdot p_j} \frac{1}{x_{ia}} \langle 1,\dots,\tilde{i},\dots,N+1;\tilde{aj} | T_{aj}\cdot T_i\, V_{aj}^i | 1,\dots,\tilde{i},\dots,N+1;\tilde{aj}\rangle \qquad (2.6)$$

The dipole splitting functions V_{ij}^{a} in (2.5) are given by

eq. (4.16) in [D]:

$$\langle V_{qg}^{a} \rangle = 8\pi \mu^{2\varepsilon} \alpha_s \left\{ \frac{2}{2 - x_{ia} - z_{ia}} - 1 - z_{ia} \right.$$
$$\left. - \frac{m_i^2}{p_i \cdot p_j} - \frac{m_i^2 \, x_{ia}}{2\, p_i \cdot p_j} \cdot \frac{(1-z_{ia})^2}{z_{ia}} \cdot \frac{r_{ia}}{x_{ia}} \right\}$$

The second line does not contribute, $m_i = m_{bottom} = 0$.

eq. (5.40) in [CS]:

$$\langle \mu | V_{gg}^{a} | \nu \rangle = 16 \pi \mu^{2\varepsilon} \alpha_s \left\{ -g^{\mu\nu} \left(\frac{1}{2 - x - z} + \frac{1}{2 - x - (1-z)} - 2 \right) \right.$$
$$+ (1-\varepsilon) \frac{1}{p_i \cdot p_j} \left(z_{ia} p_i^{\mu} - (1-z_{ia}) p_j^{\mu} \right) \times$$
$$\left. \left(z_{ia} p_i^{\nu} - (1-z_{ia}) p_j^{\nu} \right) \right\}$$

where

$$x_{ia} = 1 - \frac{p_i \cdot p_j}{p_i \cdot p_a + p_j \cdot p_a}, \qquad \text{eq. (3.10) in [D]}$$

$$z_{ia} = \frac{p_i \cdot p_a}{p_i \cdot p_a + p_j \cdot p_a},$$

$$r_{ia} = 1 + \frac{\bar{P}_{ia}^2 \left(\bar{P}_{ia}^2 + 2 m_a^2 \right)}{\lambda_{ia}} \cdot \frac{1 - x_{ia}}{x_{ia}}, \qquad \text{eq. (4.15) in [D]}$$

$$\bar{P}_{ia}^2 = P_{ia}^2 - m_a^2 - m_i^2. \qquad \text{eq. (4.12) in [D]}$$

The dipole splitting functions V_{aj}^{i} in (2.6) are given by eq. (4.32) in [D]:

$$\langle V_{qg}^{i} \rangle = 8\pi \mu^{2\varepsilon} \alpha_s \left\{ \frac{2}{2-x_{ia}-z_{ia}} - (1+x_{ia}) R_{ia} - \frac{x_{ia} m_a^2}{p_a \cdot p_j} - \frac{m_a^2}{2 p_a \cdot p_j} \cdot (1-x_{ia})^2 \right\}$$

with the same x_{ia}, z_{ia} as above and

$$R_{ia} = \cancel{\frac{\sqrt{(\bar{P}_{ia}^2 + 2m_a^2 x_{ia})^2 - 4m_a^2 \bar{P}_{ia}^2 x_{ia}^2}}{\sqrt{\lambda_{ia}}}} \qquad \text{eq. (4.15) in [D]}$$

$\langle V_{qg}^{i} \rangle$ can be simplified because there is no collinear singularity.

$$\langle V_{qg}^{i} \rangle = 8\pi \mu^{2\varepsilon} \alpha_s \left\{ \frac{2}{2-x_{ia}-z_{ia}} - \frac{m_t^2}{p_a \cdot p_j} \right\}$$

Simplification of i-f dipole

$$\mathcal{D}_{qg,i} = \frac{-1}{2p_a \cdot p_j} \cdot \frac{1}{x_{ia}} \langle \ldots | T_{aj} \cdot T_i \, V_{aj}^i | \ldots \rangle$$

V_{aj}^i can be choosen to be

$$V_{aj}^i = 8\pi \mu^{2\varepsilon} \alpha_s \cdot \left\{ \frac{2}{2 - x_{ia} - z_{ia}} - \frac{m_t^2}{p_a \cdot p_j} \right\}.$$

because $\frac{2}{2-x_{ia}-z_{ia}} \xrightarrow{p_j \to 0} 2 \frac{p_i \cdot p_a}{(p_i + p_a)\cdot p_j}$.

$$\boxed{\begin{aligned} 1 - x &= \frac{p_i \cdot p_j}{(p_i + p_j) p_a} \\ 1 - z &= \frac{p_a \cdot p_j}{(p_i + p_j) p_a} \end{aligned}}$$

i: quark

top

$$\Rightarrow \quad \mathcal{D}_{qg,i} = \frac{-1}{2p_i \cdot p_j} \frac{1}{x_{ia}} \cdot \langle \ldots | \left\{ \frac{p_i \cdot p_j}{p_a \cdot p_j} 8\pi \mu^{2\varepsilon} \alpha_s \left(\frac{2}{2 - x_{ia} - z_{ia}} - \frac{m_t^2}{p_a \cdot p_j} \right) \right\} | \ldots \rangle$$

and add to $\mathcal{D}_{qg,a}$:

$$\Rightarrow \quad \langle V_{qg}^a \rangle^{new} = 8\pi \mu^{2\varepsilon} \alpha_s \left\{ \frac{2}{2 - x_{ia} - z_{ia}} \underbrace{\left(1 + \frac{p_i \cdot p_j}{p_a \cdot p_j} \right)}_{\frac{2 - x_{ia} - z_{ia}}{1 - z_{ia}}} - 1 - z_{ia} - \frac{p_i \cdot p_j}{p_a \cdot p_j} \frac{m_t^2}{p_a \cdot p_j} \right\}$$

$$= 8\pi \mu^{2\varepsilon} \alpha_s \left\{ \frac{2}{1 - z_{ia}} - (1 + z_{ia}) - \frac{1 - x_{ia}}{1 - z_{ia}} \frac{m_t^2}{p_a \cdot p_j} \right\}$$

$$= 8\pi \mu^{2\varepsilon} \alpha_s \left\{ \frac{1 + z_{ia}^2}{1 - z_{ia}} - \frac{1 - x_{ia}}{1 - z_{ia}} \frac{m_t^2}{p_a \cdot p_j} \right\}$$

$\Rightarrow \mathcal{D}_{qg,i}$ (q: top, i: gluon) there are always two contributions:

$$V_{qg_1,g_2} + V_{qg_2,g_1} = -\frac{1}{2p_i \cdot p_j}\frac{1}{x_{ia}}\, 8\pi\mu^{2\varepsilon}\alpha_s \cdot \left\{ \frac{p_i\cdot p_j}{p_a\cdot p_j}\left(\frac{2}{2-x_{ia}-z_{ia}} - \frac{m_t^2}{p_a\cdot p_j}\right) + \frac{p_i\cdot p_j}{p_a\cdot p_i}\left(\frac{2}{2-\underset{\substack{x_{ia}\text{ is symm.}\\ \text{in } i\leftrightarrow j}}{x_{ia}}-\underset{\substack{i\leftrightarrow j \text{ is equiv. to}\\ z_{ia}\leftrightarrow 1-z_{ia}}}{(1-z_{ia})}} - \frac{m_t^2}{p_a\cdot p_i}\right)\right\}$$

$$= \frac{-1}{2p_i\cdot p_j}\frac{1}{x_{ia}}\left\{\frac{1-x}{1-z}\left(\frac{2}{2-x_{ia}-z_{ia}} - \frac{m_t^2}{p_a\cdot p_j}\right) + \frac{1-x}{z}\left(\frac{2}{2-x_{ia}-(1-z_{ia})} - \frac{m_t^2}{p_a\cdot p_i}\right)\right\} 8\pi\mu^{2\varepsilon}\alpha_s$$

add this to $\mathcal{D}_{g_1g_2,a}$:

$$\langle\mu|V_{gg}^a|\nu\rangle^{\text{new}} = 16\pi\mu^{2\varepsilon}\alpha_s\left\{-g^{\mu\nu}\left[\frac{1}{2-x_{ia}-z_{ia}}\overbrace{\left(1+\frac{1-x_{ia}}{1-z_{ia}}\right)}^{\frac{2-z_i-x_{ia}}{1-z_{ia}}} - 2 + \frac{1}{2-x_{ia}-(1-z_{ia})}\overbrace{\left(1+\frac{1-x}{z}\right)}^{(2-x-(1-z))/z} - \frac{1-x}{1-z}\frac{m_t^2}{2p_a\cdot p_j} - \frac{1-x}{z}\frac{m_t^2}{2p_a\cdot p_i}\right] + (1-\varepsilon)\ \ldots\text{as usual}^{\mu\nu}\ldots\right\}$$

$$= 16\pi\mu^{2\varepsilon}\alpha_s\left\{-g^{\mu\nu}\left[\frac{z}{1-z}+\frac{1-z}{z} - \frac{m_t^2}{2}(1-x)\left(\frac{1}{(1-z)\,p_a\cdot p_j} + \frac{1}{z\,p_a\cdot p_i}\right)\right] + (1-\varepsilon)\ \ldots\text{ as usual}^{\mu\nu}\ldots\right\}$$

Results of page 4a, 4b can be used to simplify eq. (2.3), (2.4):

$\mathcal{D}^{\text{new}}_{24,1}$ includes soft limit of $\mathcal{D}_{14,2}$,

$\mathcal{D}^{\text{new}}_{25,1}$ includes soft limit of $\mathcal{D}_{15,2}$,

$\mathcal{D}^{\text{new}}_{45,1}$ includes soft limit of $(\mathcal{D}_{14,5} + \mathcal{D}_{15,4})$.

Therefore, the new rule for writing down all dipoles is:

$$\text{Dipoles} = \sum_{\substack{i,j \\ i<j}} \sum_{k \neq i,j} \mathcal{D}_{ij,k} + \sum_{i,j} \mathcal{D}^{\text{new}}_{ij,a} + \text{no i-f contribution}$$

Phase space factorization in D=4

We have to examine

$$PS = \int d\phi^{(3)}\left(p_a \,|\, p_i, p_j, K \right) \qquad (5.1)$$

and want to split it into the phase space for p_j and the rest. Since the particle j recoils against inital state particle a, the <u>whole</u> phase space depends on the energy fraction $\tilde{x}$ taken away. This x-dependence cannot be integrated analytically. Thus, we want to derive

$$PS = \int dx \cdot \int d\phi^{(2)}\left(\tilde{p}_a(x) \,|\, \tilde{p}_i(x), K \right) \times \int d\phi^{(1)}(p_j) . \qquad (5.2)$$

To extract an explicit parametrization of $\int d\phi^{(1)}(p_j)$ in (5.2) we take the following steps.

1.) we factorize (5.1) into

$$PS = \int \frac{dq^2}{2\pi}\, d\phi^{(2)}(p_a | q, K) \times d\phi^{(2)}(q \,|\, p_i, p_j) \qquad (5.3)$$

2.) We choose a frame and parametrization such that $d\phi^{(2)}(p_a|q,K)$ in (5.3) and $d\phi^{(2)}(\tilde{p}_a|\tilde{p}_i,K)$ in (5.2) can be identified (up to Lorentz invariant factors).

3.) This allows us to read off $d\phi^{(1)}(p_j)$ in (5.2) by comparing with the remaining pieces in (5.3). In particular, $\int dq^2$ will turn into $\int dx$ and $d\phi^{(2)}(q|p_i,p_j)$ will result into an integral over z_{ia} and an azimuth angle of particle j.

<u>Step 1:</u>

$$\int d\phi^{(3)}(p_a|p_i,p_j,K) = \int \frac{d^Dp_i}{(2\pi)^D} \cdot \frac{d^Dp_j}{(2\pi)^D} \cdot \frac{d^DK}{(2\pi)^D} \cdot (2\pi)^D \delta^{(D)}(p_a-p_i-p_j-K)$$
$$\times\, 2\pi\delta^+(p_i^2-m_i^2)\cdot 2\pi\delta^+(p_j^2-m_j^2)\cdot 2\pi\delta^+(K^2-m_K^2)$$

insert $1=\int d^Dq\ \delta^{(D)}(q-p_i-p_j)$,

$1=\int dq^2\ \delta^+(q^2-(p_i+p_j)^2)$

$$\boxed{\delta^+(p^2)=\delta(p^2)\,\theta(p^0)}$$

$$\ldots = (2\pi)^{3-2D}\int dq^2\cdot d^Dq\cdot d^DK\ \delta^+(q^2-(p_i+p_j)^2)\ \delta^+(K^2-m_K^2)\cdot\ \delta^{(D)}(p_a-q-K)$$
$$\times\ d^Dp_i\cdot d^Dp_j\ \delta^+(p_i^2-m_i^2)\cdot\delta^+(p_j^2-m_j^2)\cdot\delta^{(D)}(q-p_i-p_j)$$

$$= \int\frac{dq^2}{2\pi}\ d\phi^{(2)}(p_a|q,K)\times d\phi^{(2)}(q|p_i,p_j) \tag{6.1}$$

Step 2: We choose a frame where $\vec{p}_a + \vec{\tilde{p}}_a = \vec{0}$.

- Let's parametrize $d\phi^{(2)}(p_a|q,K)$ in (6.1),

$$\int d\phi^{(2)}(p_a|q,K) = (2\pi)^{2-D}\int d^Dq \cdot d^DK \cdot \delta^{(D)}(p_a-q-K)\,\delta^+(q^2-m_q^2)\,\delta^+(K^2-m_K^2)$$

$$= (2\pi)^{2-D}\int \frac{d^{D-1}q}{2q^0}\,\delta\left(m_a^2+m_q^2-m_K^2-2p_a\cdot q\right)$$

$\boxed{D=4}$

$$= (2\pi)^{2-D}\int dq^0\,\frac{|\vec{q}|}{2}\,d\Omega_q\,\delta\left(m_a^2+m_q^2-m_K^2-2p_a^0q^0+2\vec{p}_a\cdot\vec{q}\right)$$

Integrate over δ-fct., then introduce $t=(p_a-q)^2$.

$$\Rightarrow\ d\Omega_q = d\cos\theta_q\cdot d\varphi_q = \frac{dt}{2|\vec{p}_a|\cdot|\vec{q}|}\cdot d\varphi_q$$

$$\ldots = (2\pi)^{2-D}\frac{1}{8|\vec{p}_a|\,p_a^0}\int dt\cdot d\varphi_q. \qquad (7.1)$$

- Let's parametrize $d\phi^{(2)}(\tilde{p}_a|\tilde{p}_i,K)$ in (5.2),

$$\int d\phi^{(2)}(\tilde{p}_a|\tilde{p}_i,K) = (2\pi)^{2-D}\int d^D\tilde{p}_i\cdot d^DK\ \delta^{(D)}(\tilde{p}_a-\tilde{p}_i-K)\,\delta^+(\tilde{p}_i^2-m_i^2)\,\delta^+(K^2-m_K^2)$$

$\boxed{D=4}$

$$= (2\pi)^{2-D}\int d\tilde{p}_i^0\,\frac{|\vec{\tilde{p}}_i|}{2}\,d\tilde{\Omega}_i\,\delta\left(m_a^2+m_i^2-m_K^2-2\tilde{p}_a^0\tilde{p}_i^0+2\vec{\tilde{p}}_a\cdot\vec{\tilde{p}}_i\right)$$

Integrate over δ-fct, then introduce $\tilde{t}=(\tilde{p}_a-\tilde{p}_i)^2$

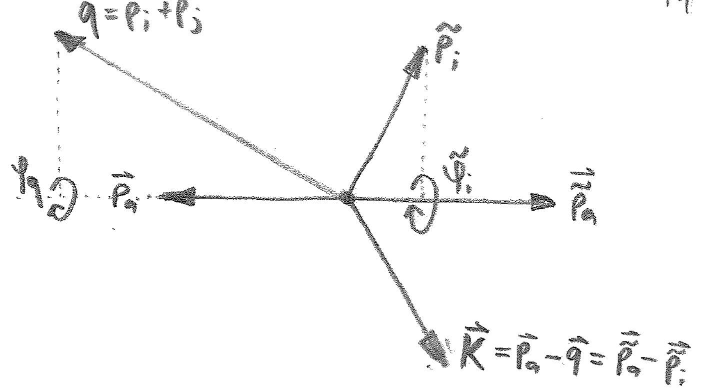
$$\Rightarrow\ d\tilde{\Omega}_i = d\cos\tilde{\theta}_i\,d\tilde{\varphi}_i = \frac{d\tilde{t}}{2|\vec{\tilde{p}}_a|\cdot|\vec{\tilde{p}}_i|}\cdot d\tilde{\varphi}_i$$

$$\ldots = (2\pi)^{2-D}\frac{1}{8|\vec{\tilde{p}}_a|\cdot\tilde{p}_a^0}\int d\tilde{t}\cdot d\tilde{\varphi}_i\ . \qquad (7.2)$$

- Now, we argue that (7.1) and (7.2) can be identified.

Since we are in a frame where $\vec{p}_a + \vec{\tilde{p}}_a = \vec{0}$ and $p_a^2 = \tilde{p}_a^2 = m_a^2$, $p_a^0 = \tilde{p}_a^0$ and $|\vec{p}_a| = |\vec{\tilde{p}}_a|$. Furthermore, $t = \tilde{t}$ because $p_a - p_i - p_j = \tilde{p}_a - \tilde{p}_i$, cf. (1.2).

Also the integrations over φ_q and $\tilde{\varphi}_i$ can be identified if we choose $\varphi_q = \tilde{\varphi}_i$ for $\varphi_q = 0$.



<u>Step 3:</u> Let's parametrize $\int d\phi^{(2)}(q|p_i,p_j)$ in (5.3).
Since this is a separately Lorentz invariant phase space, we choose a different frame, where $\vec{q} = \vec{0}$.

$$\Rightarrow \quad \int d\phi^{(2)}_{D=4}(q|p_i,p_j) = (2\pi)^{2-D} \int dp_i^0 \, \frac{|\vec{p}_i|}{2} \, d\cos\theta_i \, d\varphi_i \, \delta(m_q^2 + m_i^2 - m_j^2 - 2p_i^0 q^0)$$

The polar axis is oriented to $\vec{p}_a$, so that for $v = (p_a - p_i)^2$, $dv = 2|\vec{p}_i| \cdot |\vec{p}_a| d\cos\theta_i$.

$$\ldots = (2\pi)^{2-D} \frac{1}{8|\vec{p}_a| q^0} \int dv \cdot d\varphi_i$$

In the frame where $\vec{q}=0$, we have

$$|\vec{P}_a| = \frac{m_a^2+q^2-K^2}{2q^2} \quad \text{and} \quad q^0 = q\cdot q .$$

Furthermore,
$$v = (P_a-P_i)^2 = m_a^2+m_i^2 - z_{ia}\left(m_a^2-P_{ia}^2+(P_i+P_j)^2\right)$$
$$dv = \left(P_{ia}^2-m_a^2-(P_i+P_j)^2\right)\cdot dz_{ia} .$$

The first integral in (6.1) can be written as

$$\int\frac{dq^2}{2\pi} = \int\frac{dx_{ia}}{x^2}\,\frac{\bar{P}_{ia}^2}{2\pi}, \quad \text{since} \quad P_{ia}^2-m_a^2-q^2 = \frac{\bar{P}_{ia}^2}{x_{ia}} .$$

Now, we can write (5.3)

$x_{ia}=x$

$$PS = \int\frac{dx_{ia}}{x_{ia}^2}\cdot\frac{\bar{P}_{ia}^2}{2\pi}\cdot d\phi^{(2)}\left(\tilde{P}_a(x)\,\middle|\,\tilde{P}_i(x),K\right)\cdot\frac{(2\pi)^{2-D}}{8}$$
$$\times\frac{\bar{P}_{ia}^2}{2\left(\bar{P}_{ia}^2+xm_a^2-xP_{ia}^2\right)}\cdot\frac{\bar{P}_{ia}^2}{x}\int dz_{ia}\cdot d\varphi_i .$$

determine integration limits for $\int dz_{ia}$:

original integral is: $\int_{-1}^{+1} d\cos\theta$

$$v = (p_a - p_i)^2 = m_a^2 + m_i^2 - 2E_a E_i + 2|\vec{p}_a|\cdot|\vec{p}_i|\cdot\cos\theta$$

$$\Rightarrow \quad \sim \int_{v_-}^{v_+} dv \dots \quad \text{with}$$

$$v_\pm = m_a^2 + m_i^2 - 2E_a E_i \pm 2|\vec{p}_a|\cdot|\vec{p}_i|$$

in the frame where $\vec{q} = 0$:

$$E_a \cdot E_i = \frac{(m_a^2 + q^2 - P_{ia}^2)(q^2 + m_i^2)}{4q^2}$$

$$|\vec{p}_a|\cdot|\vec{p}_i| = \frac{|q^2 - m_i^2|\,\lambda^{1/2}(q^2, m_a^2, P_{ia}^2)}{4q^2}$$

$$\Rightarrow \quad \sim \dots \int_{z_-}^{z_+} dz \dots \quad \text{with}$$

$$z_\pm = \frac{m_a^2 + m_i^2 - v_\pm}{m_a^2 - P_{ia}^2 + q^2} = \frac{2E_a E_i \mp 2|\vec{p}_a|\cdot|\vec{p}_i|}{m_a^2 - P_{ia}^2 + q^2}$$

$$= \frac{1}{m_a^2 - P_{ia}^2 + q^2} \cdot \frac{1}{2q^2} \Big\{ (m_a^2 + q^2 - P_{ia}^2)(q^2 + m_i^2) \mp (q^2 - m_i^2)\,\lambda^{1/2}(q^2, m_a^2, P_{ia}^2) \Big\}$$

$$z_\pm \overset{N6}{=} \frac{x^2}{2\bar{p}_{ia}^2\left(\bar{p}_{ia}^2 - x\left(p_{ia}^2 - m_a^2\right)\right)} \cdot \left\{ \frac{\bar{p}_{ia}^2}{x^2}\left(\bar{p}_{ia}^2 - x\left(\bar{p}_{ia}^2 + 2m_i^2\right)\right) \mp \bar{p}_{ia}^2\left(\frac{x-1}{x}\right)\frac{1}{x}\left[\ldots \text{see N6} \ldots\right]^{1/2} \right\}$$

$$= \frac{1}{2\cancel{\bar{p}_{ia}^2}\left(\bar{p}_{ia}^2 - x\left(p_{ia}^2 - m_a^2\right)\right)} \left\{ \cancel{\bar{p}_{ia}^2}\left(\bar{p}_{ia}^2 - x\left(\bar{p}_{ia}^2 + 2m_i^2\right)\right) \mp \cancel{\bar{p}_{ia}^2}(x-1)\left[\left(\bar{p}_{ia}^2 + 2m_a^2 x\right)^2 - 4m_a^2 p_{ia}^2 x^2\right]^{1/2} \right\}$$

in agreement with eqn (4.22) in [D].

note: $\bar{p}_{ia}^2(x-1) \geq 0$ since $\bar{p}_{ia}^2 < 0$.

<u>useful</u>:

$$z_- + z_+ = \frac{\bar{p}_{ia}^2 - x\left(\bar{p}_{ia}^2 + 2m_i^2\right)}{\bar{p}_{ia}^2 - x\left(p_{ia}^2 - m_a^2\right)},$$

$$z_- - z_+ = \frac{(x-1)\left[\ldots \text{see above} \ldots\right]^{1/2}}{\bar{p}_{ia}^2 - x\left(p_{ia}^2 - m_a^2\right)},$$

$$1 - z_\pm = \frac{(x-1)\left(-\bar{p}_{ia}^2 \pm \left[\ldots \text{see above} \ldots\right]^{1/2}\right)}{2\left(\bar{p}_{ia}^2 - x\left(p_{ia}^2 - m_a^2\right)\right)},$$

$$2 + z_- + z_+ = \frac{-3\bar{p}_{ia}^2(x-1) - 4m_i^2 x}{\bar{p}_{ia}^2 - x\left(p_{ia}^2 - m_a^2\right)}$$

$$\overset{m_i=0}{\downarrow}$$

$$= \frac{1}{2(p_{ia}^2-m_a^2)(1-x)}\Big\{(p_{ia}^2-m_a^2)(1-x) \mp (1-x)\Big[(p_{ia}^2-m_a^2)^2 + 4(p_{ia}^2-m_a^2)m_a^2 x + \underbrace{4m_a^4x^2 - 4m_a^2p_{ia}^2x^2}_{-4m_a^2x^2(p_{ia}^2-m_a^2)}\Big]^{1/2}\Big\}$$

$$= \frac{1}{2(p_{ia}^2-m_a^2)}\Big\{(p_{ia}^2-m_a^2) \mp \Big[(p_{ia}^2-m_a^2)^2 - (p_{ia}^2-m_a^2)4m_a^2(x^2-x)\Big]^{1/2}\Big\}$$

$$= \frac{1}{2}\Big\{1 \mp \Big[1 - \frac{4m_a^2}{p_{ia}^2-m_a^2}(x^2-x)\Big]^{1/2}\Big\}$$

$$= \frac{1}{2}\Big\{1 \mp \Big[1 + \underbrace{\bar{\mu}_a^2}_{>0}\,\underbrace{\underbrace{(x^2-x)}_{x(x-1)}}_{<0}\Big]^{1/2}\Big\}$$

case $m_i = 0$:

$$z_\pm \overset{N7}{=} \frac{1}{2\bar{p}_{ia}^2(1-x)} \left\{ \bar{p}_{ia}^2 (1-x) \mp (1-x)\, \bar{p}_{ia}^2 \bar{\mu}_a \left((x-\tfrac{1}{2})^2 - \mu_a^{-2} \right)^{1/2} \right\}$$

$$= \frac{1}{2}\left\{ 1 \mp \bar{\mu}_a \left((x-\tfrac{1}{2})^2 - \mu_a^{-2} \right)^{1/2} \right\} = \frac{1}{2}\left\{ 1 \mp \left(1 + \bar{\mu}_a^2 (x^2 - x) \right)^{1/2} \right\}$$

useful:

$$z_- + z_+ = 1$$

$$z_- - z_+ = \bar{\mu}_a \left((x-\tfrac{1}{2})^2 - \mu_a^{-2} \right)^{1/2} \longrightarrow z_- > z_+$$

$$= 1 - 2z_+$$

$$1 - z_\pm = z_\mp$$

$$2 + z_+ + z_- = 3$$

examine the sqrt in 9c :

$$\left[\left(x - \tfrac{1}{2}\right)^2 - \mu_a^{-2} \right]^{1/2}$$

$$\Rightarrow \quad \left(x - \tfrac{1}{2}\right)^2 - \mu_a^{-2} \overset{!}{\geq} 0$$

$$\Leftrightarrow \quad \left(x - \tfrac{1}{2} + \mu_a^{-1}\right)\left(x - \tfrac{1}{2} - \mu_a^{-1}\right) \geq 0$$

$$\Rightarrow \quad x \overset{!}{\geq} \frac{1}{2} + \frac{\sqrt{p_{ia}^2}}{2 m_a} \qquad \text{OR} \qquad x \overset{!}{<} \frac{1}{2} - \frac{\sqrt{p_{ia}^2}}{2 m_a}$$

not possible, since we have to allow $x \to 1$.

$$\Rightarrow \quad \boxed{x \overset{!}{\geq} \frac{m_a + \sqrt{p_{ia}^2}}{2 m_a}}$$

in accordance with [D] eq. (4,14)

$$\cancel{\left[\left(x - \tfrac{1}{2}\right)^2 - \mu_a^{-2} \right]^{1/2} < \left[\tfrac{1}{4} \right]^{1/2} = \tfrac{1}{2}}$$

$$\cancel{\mu_a [\ldots]^{1/2} < \mu_a \cdot \tfrac{1}{2} = \left[\frac{4 m_a^2}{-p_{ia}^2} \cdot \frac{1}{4} \right]^{1/2} = \left[\frac{m_a^2}{m_a^2 - p_{ia}^2} \right]^{1/2}}$$

Also, $\mu_a \left[\left(x - \tfrac{1}{2}\right)^2 - \mu_a^{-2} \right]^{1/2} = 1$ for $x = 1$

i.e. $z_+(x=1) = 0$

$z_-(x=1) = 1$

repeat step 3 in D dimensions:

$$\int d\phi_D^{(2)}(q|p_i,p_j) = (2\pi)^{2-D} \int \frac{d^{D-1}p_i}{2E_i}\, \delta^+\left((q-p_i)^2 - m_j^2 \right)$$

$\boxed{\text{choose frame where } \vec{q}=\vec{0} \text{ and set } m_j = 0}$

$$= (2\pi)^{2-D} \int \frac{d|\vec{p}_i|}{2E_i} |\vec{p}_i|^{D-2} \cdot d\Omega_i^{D-1}\, \delta^+(m_q^2 + m_i^2 - 2E_q E_i)$$

$$\boxed{|\vec{p}_i| = \sqrt{E_i^2 - m_i^2}, \quad d|\vec{p}_i| = \frac{E_i\, dE_i}{|\vec{p}_i|}}$$

$$= (2\pi)^{2-D} \int dE_i\, \frac{1}{2} |\vec{p}_i|^{D-3}\, d\Omega_i^{D-1}\, \frac{1}{2E_q}\, \delta^+\left(E_i - \frac{m_q^2 + m_i^2}{2E_q}\right)$$

$$= (2\pi)^{2-D} \frac{1}{4E_q} |\vec{p}_i|^{D-3} \int d\theta\, \sin^{D-3}\theta \cdot d\Omega_i^{D-2}$$

choose $v = (p_a - p_i)^2 = m_a^2 + m_i^2 - 2E_a E_i + 2|\vec{p}_a|\cdot|\vec{p}_i|\cdot\cos\theta$

$dv = -2|\vec{p}_a|\cdot|\vec{p}_i|\sin\theta\cdot d\theta$

$\cos\theta = \sqrt{1-\sin^2\theta} = \dfrac{v - m_a^2 - m_i^2 + 2E_a E_i}{2|\vec{p}_a|\cdot|\vec{p}_i|}$

$$\boxed{\begin{aligned} E_a &= \frac{m_a^2 + m_q^2 - m_k^2}{2m_q}, \\ E_i &= \frac{m_q^2 + m_i^2}{2m_q}, \\ E_q &= m_q \end{aligned}}$$

$|\vec{p}_i| = \sqrt{\lambda(m_q^2, m_i^2, 0)}\, \dfrac{1}{2m_q} = \dfrac{1}{2m_q} |m_q^2 - m_i^2|,$

$|\vec{p}_a| = \sqrt{\lambda(m_q^2, m_a^2, m_k^2)}\, \dfrac{1}{2m_q}$

$$= (2\pi)^{2-D} \frac{1}{4m_q} \frac{|\vec{p}_i|^{D-4}}{2|\vec{p}_a|} (-1) \int dv\, \sin^{D-4}\theta \cdot d\Omega_i^{D-2} \qquad (10.1)$$

Now, the full phase space integral (5.3) becomes

$$PS = \int \frac{dq^2}{2\pi} \, d\phi^{(2)}(p_a | q, K) \times d\phi^{(2)}(q | p_i, K)$$

see NS:

$$= \int \overbrace{\frac{dx}{2\pi} \cdot \frac{K^2 - m_i^2 - m_a^2}{x^2}}^{dq^2/2\pi} \cdot d\phi^{(2)}(\tilde{p}_a | \tilde{p}_i, K) \cdot \overbrace{\frac{1}{4} \lambda^{-1/2}(q^2, m_a^2, K^2)}^{\frac{1}{8\sqrt{q^2} \cdot |\vec{p}_a|}}$$

$$(2\pi)^{2-D} \cdot \underbrace{\frac{(q^2 - m_i^2)^{-2\varepsilon}}{2^{-2\varepsilon} \sqrt{q^2}^{\,-2\varepsilon}}}_{|\vec{p}_i|^{D-4}} \cdot (-1) \cdot \underbrace{\int \frac{dz}{x} (K^2 - m_i^2 - m_a^2)}_{dv} \cdot \sin^{-2\varepsilon}\theta \cdot d\Omega_i^{D-2}$$

$$= -(2\pi)^{1-D} \int \frac{dx}{x^3} \, d\phi^{(2)}(\tilde{p}_a | \tilde{p}_i, -p_{ia}) \cdot (p_{ia}^2 - m_i^2 - m_a^2)^2 \cdot \frac{1}{4} \lambda^{-1/2}(q^2, m_a^2, p_{ia}^2)$$

$$\times \, 2^{2\varepsilon} (q^2)^{\varepsilon} (q^2 - m_i^2)^{-2\varepsilon} \int dz \cdot \sin^{-2\varepsilon}\theta \cdot d\Omega_i^{D-2} \qquad (11.1)$$

$$\boxed{m_i = 0}$$

$$\overset{\downarrow}{=} -(2\pi)^{1-D} \int \frac{dx}{x^3} \, d\phi^{(2)}(\tilde{p}_a | \tilde{p}_i, -p_{ia}) \cdot \frac{\overline{p_{ia}}^{\,4}}{\lambda^{1/2}(q^2, m_a^2, p_{ia}^2)}$$

$$\times \, 4^{\varepsilon - 1} (q^2)^{-\varepsilon} \int dz \, \sin^{-2\varepsilon}\theta \, d\Omega_i^{D-2} \qquad (11.2)$$

re-write $\sin^{-2\varepsilon}$ term in (11.2):

$$\sin^{-2\varepsilon}\theta = \left(\sin^2\theta\right)^{-\varepsilon} = \left(1-\cos^2\theta\right)^{-\varepsilon}$$

$$= \left(1-\cos\theta\right)^{-\varepsilon}\cdot\left(1+\cos\theta\right)^{-\varepsilon}$$

$$= \left(\cos\theta_+ - \cos\theta\right)^{-\varepsilon}\cdot\left(\cos\theta - \cos\theta_-\right)^{-\varepsilon}$$

[with $\cos\theta_\pm = \pm 1$

since integration limits $z_\pm$ correspond to $\cos\theta = \pm 1$

and $\underset{\boxed{m_i=0}}{\cos\theta =} \dfrac{\overline{P}_{ia}^{\,2}\,(2z-1)}{\left[\overline{P}_{ia}^{\,2}\left(\overline{P}_{ia}^{\,2} + 4m_a^2 x(1-x)\right)\right]^{1/2}}$

$\times 2$

$$\Rightarrow \sin^{-2\varepsilon}\theta \overset{\boxed{m_i=0}}{=} \left(\frac{2\,\overline{P}_{ia}^{\,2}}{\left(\overline{P}_{ia}^{\,2} + 4m_a^2 x(1-x)\right)}\right)^{-\varepsilon}\cdot\left(z_+ - z\right)^{-\varepsilon}\left(z - z_-\right)^{-\varepsilon} \qquad (12.1)$$

Thus, (11.2) becomes

$$PS = +(2\pi)^{1-D}\int dx\ \frac{\overline{P}_{ia}^{\,4}}{4\,\overline{p}_{ia}}\ d\phi^{(2)}\left(\tilde{p}_a \,|\, \tilde{p}_i, -P_{ia}\right)\cdot\frac{1}{x^2}\left(\left(x-\tfrac{1}{2}\right)^2 - \mu_a^{-2}\right)^{-1/2}$$

$$\cdot\ \frac{1}{(1-x)^{\varepsilon}}\cdot\int_{z_-}^{z_+} dz\ \left(\frac{2x\left(\overline{P}_{ia}^{\,2} + 4m_a^2 x(1-x)\right)}{\overline{P}_{ia}^{\,4}\left(z-z_+\right)\left(z-z_-\right)}\right)^{\varepsilon} d\Omega_i^{D-2} \qquad (12.2)$$

in (12.2)

$$\left(\bar{P}_{ia}^{\,2} + 4m_a^2 x(1-x)\right) \overset{\text{see N7}}{\doteq} \bar{P}_{ia}^{\,2}\,\bar{\mu}_a^{\,2}\left((x-\tfrac{1}{2})^2 - \mu_a^{-2}\right)$$

and

$$\frac{1}{(z-z_+)(z-z_-)} = \frac{1}{z_+-z_-}\left(\frac{1}{z-z_+} - \frac{1}{z-z_-}\right)$$

$$= \frac{-1}{\bar{\mu}_a\left((x-\frac{1}{2})^2-\mu_a^{-2}\right)^{1/2}}\left(\frac{1}{z-z_+} - \frac{1}{z-z_-}\right)$$

thus

$$PS = (2\pi)^{1-D}\int dx\ \frac{\bar{P}_{ia}^{\,4}}{4\bar{\mu}_a}\ d\phi^{(2)}(\tilde{p}_a|\tilde{p}_i, -p_{ia})\ \frac{1}{x^2}\left((x-\tfrac{1}{2})^2-\mu_a^{-2}\right)^{-1/2}$$

$$\cdot\ \frac{1}{(1-x)^\varepsilon}\left(\frac{-2x\bar{\mu}_a}{\bar{P}_{ia}^{\,2}}\left((x-\tfrac{1}{2})-\mu_a^{-2}\right)^{1/2}\right)^{\varepsilon}\cdot$$

$$\int_{z_-}^{z_+} dz\left(\frac{1}{z-z_+} - \frac{1}{z-z_-}\right)^{\varepsilon} d\Omega_i^{D-2} \qquad (12a.1)$$

$$d_{aj,i}(x) = \int_{z_-}^{z_+} dz \left(\frac{1}{(z-z_+)(z-z_-)} \right)^{\varepsilon} \cdot \mathcal{D}_{aj,i}(z,x)$$

I. case : $(aj,i) = (1_t, 5_g, 2_b)$

$$\mathcal{D}_{aj,i} = \overbrace{8\pi \mu^{2\varepsilon} \alpha_s}^{\mathcal{N}} \cdot \frac{-1}{2 p_i \cdot p_j} \frac{1}{x} \left\{ \frac{1+z^2}{1-z} - \frac{1-x}{1-z} \frac{m_t^2}{p_a \cdot p_j} \right\} \langle \ldots || \ldots \rangle$$

see N5

$$\stackrel{\downarrow}{=} \mathcal{N} \langle || \rangle \frac{1}{1-x} \frac{1}{\bar{p}_{ia}^2} \left\{ \frac{1+z^2}{1-z} + \frac{2x(1-x)}{1-z} \frac{m_t^2}{\bar{p}_{ia}^2} \right\}$$

$$= \mathcal{N} \langle || \rangle \left(A \frac{1+z^2}{1-z} + \frac{B}{1-z} \right)$$

with $\quad A = \frac{1}{\bar{p}_{ia}^2} \frac{1}{1-x} \quad , \quad B = \frac{m_t^2}{\bar{p}_{ia}^4} 2x$

expand to $\mathcal{O}(\varepsilon)$

$$d_{aj,i}(x) \stackrel{\downarrow}{=} \mathcal{N} \langle || \rangle \cdot \frac{1}{\bar{p}_{ia}^2} \left\{ \frac{3}{2} \frac{1}{1-x} \cdot \bar{\mu}_a \left((x-\tfrac{1}{2})^2 - \mu_a^{-2} \right)^{1/2} \right.$$

$$+ \left(\frac{2}{1-x} + \frac{2x m_t^2}{\bar{p}_{ia}^2} \right) \log\left(\frac{z_+}{z_-} \right)$$

$$\left. + \mathcal{O}(\varepsilon) \right\}$$

$\rightsquigarrow$ NOT WORKING BECAUSE $z_+(x=1) = 0$.

$$\hat{I}_1(z_0) = \int dz \, \frac{1}{(z-z_0)^\varepsilon} \frac{1}{1-z} = \left(\frac{z-z_0}{z-1}\right)^\varepsilon \left(\frac{1}{z-z_0}\right)^\varepsilon \frac{1}{\varepsilon} \, {}_2F_1(\ldots)$$

$$= \frac{1}{\varepsilon} \frac{1}{(z-1)^\varepsilon} \, {}_2F_1\left(\varepsilon, \varepsilon, 1+\varepsilon, \frac{z_0-1}{z-1}\right)$$

$$I_1(z_+) = \int_{z_-}^{z_+} dz \, \hat{I}_1(z_+) = \lim_{z \to z_+} \hat{I}_1(z_+) - \lim_{z \to z_-} \hat{I}_1(z_+)$$

$$= \underbrace{\frac{\pi}{\sin(\varepsilon\pi)} \left(\frac{1}{z_+-1}\right)^\varepsilon}_{\text{can be expanded}} - \left(\frac{1}{z_--1}\right)^\varepsilon {}_2F_1\left(\varepsilon, \varepsilon, 1+\varepsilon, \frac{z_+-1}{z_--1}\right)$$

$$I_1(z_-) = \int_{z_-}^{z_+} dz \, \hat{I}_1(z_-) = -\frac{\pi}{\sin(\varepsilon\pi)} \left(\frac{1}{z_--1}\right)^\varepsilon + \underbrace{\left(\frac{1}{z_+-1}\right)^\varepsilon}_{\text{can be expanded}} {}_2F_1\left(\varepsilon, \varepsilon, 1+\varepsilon, \frac{z_--1}{z_+-1}\right)$$

$${}_2F_1\left(\varepsilon, \varepsilon, 1+\varepsilon, \frac{z_+-1}{z_--1}\right) = \sum_{n=0}^{\infty} \frac{\Gamma(\varepsilon+n)^2}{\Gamma(\varepsilon)^2} \cdot \frac{\Gamma(1+\varepsilon)}{\Gamma(1+n+\varepsilon)} \frac{1}{n!} \left(\frac{z_-}{z_+}\right)^n$$

$$\frac{1}{z_- - 1} = \frac{-1}{z_+} = \frac{-2}{1-\sqrt{1+\bar{\mu}_a^2 x(x-1)}}$$

$$= \frac{-2\left(1+\sqrt{1+\bar{\mu}_a^2 x(x-1)}\right)}{1-|\underbrace{1+\bar{\mu}_a^2 x(x-1)}_{\geq 0 \text{ see 9d}}|}$$

$$\boxed{\begin{array}{l}(1-\sqrt{x})(1+\sqrt{x}) \\ = 1-|x|\end{array}}$$

$$= 2 \cdot \frac{1+\sqrt{1+\bar{\mu}_a^2 x(x-1)}}{\bar{\mu}_a^2 x(x-1)}$$

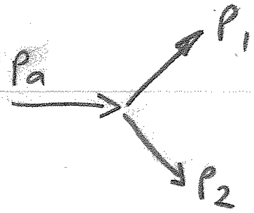
$$= \left(\frac{1}{x-1} - \frac{1}{x}\right) \cdot \frac{2}{\bar{\mu}_a^2}\left(1+\sqrt{1+\bar{\mu}_a^2 x(x-1)}\right)$$

$$\frac{z_+ - 1}{z_- - 1} = \frac{z_-}{z_+} = \frac{1+\sqrt{\dots}}{1-\sqrt{\dots}} \cdot \frac{1+\sqrt{\dots}}{1+\sqrt{\dots}} = \frac{(1+\sqrt{\dots})^2}{-\bar{\mu}_a^2 x(x-1)}$$

$$\frac{1}{z_+^{\varepsilon}}\left(\frac{z_+ - 1}{z_- - 1}\right)^n = \left(2\,\frac{1+\sqrt{\dots}}{-\bar{\mu}_a^2 x(x-1)}\right)^{\varepsilon}\left(\frac{(1+\sqrt{\dots})^{2}}{\left(-\bar{\mu}_a^2 x(x-1)\right)}\right)^n$$

$$= 2^{\varepsilon}\,\frac{(1+\sqrt{\dots})^{\varepsilon+2n}}{\left(-\bar{\mu}_a^2 x(x-1)\right)^{\varepsilon+n}}$$

N1



$$p_a = p_1 + p_2$$

$$p_a - p_1 = p_2$$

$$p_a^2 + p_1^2 - 2 p_a \cdot p_1 = p_2^2$$

goto rest frame of p_a : $p_a = \begin{pmatrix} E_a \\ \vec{o} \end{pmatrix} = \begin{pmatrix} m_a \\ \vec{o} \end{pmatrix}$

$$m_a^2 + m_1^2 - 2 m_a E_1 = m_2^2$$

$$\Rightarrow \quad \| \; E_1 = \frac{m_a^2 + m_1^2 - m_2^2}{2 m_a}$$

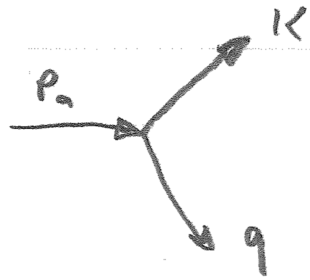
$$\| \; E_2 = m_a - E_1 = \frac{m_a^2 + m_2^2 - m_1^2}{2 m_a}$$

$$|\vec{p}_1|^2 = |\vec{p}_2|^2 = E_1^2 - m_1^2 = \frac{(m_a^2 + m_1^2 - m_2^2)^2 - 4 m_a^2 m_1^2}{4 m_a^2}$$

$$= \frac{m_a^4 + (m_1^2 - m_2^2)^2 + 2 m_a^2 (m_1^2 - m_2^2) - 4 m_a^2 m_1^2}{4 m_a^2}$$

$$= \frac{1}{4 m_a^2} \left((m_1 + m_2)^2 - m_a^2 \right) \cdot \left((m_1 - m_2)^2 - m_a^2 \right) = \lambda (m_a^2, m_1^2, m_2^2) \, \frac{1}{4 m_a^2}$$

$$\Rightarrow \quad \| \; |\vec{p}_1| = |\vec{p}_2| = \sqrt{\lambda(\underset{\uparrow \atop m_a^2}{s}, m_1^2, m_2^2)} \; \frac{1}{2 m_a}$$



in frame where $\vec{q} = 0$

$$\Rightarrow \qquad P_a^2 = (P_K + P_q)^2$$

$$\Leftrightarrow \qquad m_a^2 = m_K^2 + m_q^2 + 2 m_q E_K$$

$$\Rightarrow \qquad E_K = \frac{m_a^2 - m_q^2 - m_K^2}{2 m_q}$$

$$E_a = E_K + E_q = \frac{m_a^2 - m_q^2 - m_K^2}{2 m_q} + m_q$$

$$= \frac{m_a^2 + m_q^2 - m_K^2}{2 m_q}$$

$$|\vec{P}_a|^2 = E_a^2 - m_a^2 = \frac{(m_q^2 + m_a^2 - m_K^2) - 4 m_a^2 m_q^2}{4 m_q^2}$$

$$= \frac{1}{4 m_q^2} \lambda(m_q^2, m_a^2, m_K^2)$$

N3

$$z_{ia} = \frac{p_a \cdot p_i}{p_a p_i + p_a p_j} = \frac{1}{2} \frac{-(p_a - p_i)^2 + m_a^2 + m_i^2}{p_a p_i + p_a p_j}$$

$$p_i - p_a = P_{ia} - p_j \Rightarrow m_i^2 + m_a^2 - 2 p_i p_a = P_{ia}^2 + m_j^2 - 2 P_{ia} \cdot p_j$$

$$p_j - p_a = P_{ia} - p_i \Rightarrow m_j^2 + m_a^2 - 2 p_j p_a = P_{ia}^2 + m_i^2 - 2 P_{ia} \cdot p_i$$

$$\Rightarrow \quad 2(p_a p_i + p_a p_j) = 2 m_a^2 - 2 P_{ia}^2 + 2 \underbrace{P_{ia}}_{p_i + p_j - p_a} \cdot (p_i + p_j)$$

$$\Leftrightarrow \quad 4(p_a p_i + p_a p_j) = 2 m_a^2 - 2 P_{ia}^2 + 2 (p_i + p_j)^2$$

$$\Rightarrow \quad 2(p_a p_i + p_a p_j) = (p_i + p_j)^2 + m_a^2 - P_{ia}^2$$

$$\ldots = \frac{m_a^2 + m_i^2 - (p_a - p_i)^2}{(p_i + p_j)^2 - P_{ia}^2 + m_a^2}$$

$$\Rightarrow \quad \boxed{(p_a - p_i)^2 = - z_{ia} \left((p_i + p_j)^2 - P_{ia}^2 + m_a^2 \right) + m_a^2 + m_i^2}$$

$$x_{ia} = \frac{p_a(p_i+p_j) - p_i p_j}{p_a(p_i+p_j)}$$

$$P_{ia}^2 = (p_i + p_j - p_a)^2 = m_i^2 + m_j^2 + 2p_i p_j + m_a^2 - 2p_a(p_i+p_j)$$

$$\Rightarrow \quad p_a(p_i+p_j) - p_i p_j = \frac{1}{2}\left(-P_{ia}^2 + m_i^2 + m_j^2 + m_a^2\right)$$

N3

$$\Rightarrow \quad x_{ia} = \frac{-(P_{ia}^2 - m_i^2 - m_j^2 - m_a^2)}{(p_i+p_j)^2 - P_{ia}^2 + m_a^2}$$

$m_j = 0$

$$\Rightarrow \quad (p_i+p_j)^2 = \frac{-(P_{ia}^2 - m_i^2 - m_a^2)}{x_{ia}} + P_{ia}^2 - m_a^2$$

$$v \overset{N3}{=} (p_a - p_i)^2 = m_a^2 + m_i^2 - z_{ia}\Big(m_a^2 - P_{ia}^2 + \underbrace{(p_i+p_j)^2}_{q^2} \Big)$$

$$dv = \left(P_{ia}^2 - m_a^2 - q^2 \right) dz_{ia}$$

$$q^2 \overset{N4}{=} \frac{-(P_{ia}^2 - m_i^2 - m_a^2)}{x_{ia}} + P_{ia}^2 - m_a^2 \overset{m_i=0}{=} (P_{ia}^2 - m_a^2)\left(1 - \frac{1}{x}\right)$$

$$dq^2 = \frac{P_{ia}^2 - m_i^2 - m_a^2}{x_{ia}^2}\, dx$$

$$\Rightarrow \quad v = m_a^2 + m_i^2 + \frac{z_{ia}}{x_{ia}}\left(P_{ia}^2 - m_i^2 - m_a^2 \right)$$

$$dv = \frac{dz_{ia}}{x_{ia}}\left(P_{ia}^2 - m_i^2 - m_a^2 \right)$$

$$q^2 = (p_i + p_j)^2 = 2p_i\cdot p_j + m_i^2 \Rightarrow 2p_i\cdot p_j = (P_{ia}^2 - m_i^2 - m_a^2)\cdot\overbrace{\left(1 - \frac{1}{x}\right)}^{\frac{x-1}{x}}$$

$$v = (p_a - p_i)^2 = -2p_i\cdot p_a + m_i^2 + m_a^2 \Rightarrow 2p_a\cdot p_i = z\left(m_a^2 + m_i^2 - P_{ia}^2 + 2p_i\cdot p_j \right)$$

$$= -\frac{z}{x}\left(P_{ia}^2 - m_i^2 - m_a^2 \right)$$

$$p_a\cdot p_j = (p_i + p_j)p_a\,(1-z) \Rightarrow 2p_a\cdot p_j = -\frac{(P_{ia}^2 - m_i^2 - m_a^2)}{x}$$

$$\sin^{-2\varepsilon}\theta = (1 - \cos^2\theta)^{-\varepsilon} \underset{m_i^2=0}{=} \left(1 - \frac{(m_a^2 - P_{ia}^2)(1-2z)^2}{-P_{ia}^2 + m_a^2(1-2x)^2} \right)^{-\varepsilon}$$

$$= 1 - \varepsilon \log(1 - \%) + \frac{\varepsilon^2}{2}\log^2(1 - \%)$$

$$m_a^2 - p_{ia}^2 + q^2 = -\frac{\bar{p}_{ia}^{\,2}}{x} \,,$$

$$2q^2 = -\frac{2}{x}\left(\bar{p}_{ia}^{\,2} - x\left(p_{ia}^2 - m_a^2\right)\right) \,,$$

$$\left(m_a^2 + q^2 - p_{ia}^2\right)\left(q^2 + m_i^2\right) = \frac{\bar{p}_{ia}^{\,2}}{x^2}\left(\bar{p}_{ia}^{\,2} - x\left(\bar{p}_{ia}^{\,2} + 2m_i^2\right)\right) \,,$$

$$q^2 - m_i^2 = \bar{p}_{ia}^{\,2}\left(\frac{x-1}{x}\right) \,,$$

$$\lambda^{1/2}\left(q^2, m_a^2, p_{ia}^2\right) = \frac{1}{x}\left[\bar{p}_{ia}^{\,2}\left(\bar{p}_{ia}^{\,2} + 4x\,m_a^2\right) - 4x^2 m_a^2\left(m_i^2 + \bar{p}_{ia}^{\,2}\right)\right]^{1/2} .$$

$$\left[(\bar{P}_{ia}^2 + 2m_a^2 x)^2 - 4 m_a^2 P_{ia}^2 x^2 \right]$$

$$\boxed{m_i = 0} \downarrow$$

$$= - 4 m_a^2 \bar{P}_{ia}^2 \left(x^2 - x - \frac{\bar{P}_{ia}^2}{4 m_a^2} \right)$$

$$\checkmark = -4 m_a^2 \bar{P}_{ia}^2 \left(x - \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{\bar{P}_{ia}^2}{4m_a^2}} \right) \left(x - \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\bar{P}_{ia}^2}{4m_a}} \right)$$

introduce $\boxed{\mu_a^2 = \frac{4m_a^2}{\bar{P}_{ia}^2} \quad , \quad \bar{\mu}_a^2 = \frac{4 m_a^2}{-\bar{P}_{ia}^2}}$

$$= + \bar{P}_{ia}^4 \bar{\mu}_a^2 \left(x - \frac{1}{2} - \frac{1}{\mu_a} \right) \left(x - \frac{1}{2} + \frac{1}{\mu_a} \right)$$

$$= + \bar{P}_{ia}^4 \bar{\mu}_a^2 \left((x - \tfrac{1}{2})^2 - \frac{1}{\mu_a^2} \right)$$

$$-\bar{P}_{ia}^2 \pm [\ldots\%\ldots]^{1/2} = -\bar{P}_{ia}^2 \mp \bar{P}_{ia}^2 \bar{\mu}_a \left((x - \tfrac{1}{2})^2 - \frac{1}{\mu_a^2} \right)^{1/2}$$

$$= - \bar{P}_{ia}^2 \left[1 \pm \bar{\mu}_a \left((x - \tfrac{1}{2})^2 - \frac{1}{\mu_a^2} \right)^{1/2} \right]$$

$$\lambda(q^2, m_a^2, P_{ia}^2) = \frac{-\bar{P}_{ia}^2}{x^2} \left(4m_a^2 x^2 - 4m_a^2 x + m_a^2 - P_{ia}^2 \right)$$

$$= - \frac{4m_a^2 \bar{P}_{ia}^2}{x^2} \left(x^2 - x - \frac{\bar{P}_{ia}^2}{4m_a^2} \right)$$

$$= \frac{\bar{P}_{ia}^4 \bar{\mu}_a^2}{x^2} \left((x - \tfrac{1}{2})^2 - \mu_a^{-2} \right)$$

$$(q^2)^{-\varepsilon} = \left(-\bar{P}_{ia}^2 \cdot \frac{1-x}{x} \right)^{-\varepsilon} = \frac{1}{(1-x)^\varepsilon} \cdot \left(\frac{x}{-\bar{P}_{ia}^2} \right)^{\varepsilon}$$

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$$\int_{x_0}^{1} dx \, \frac{1}{(1-x)^{\varepsilon+1}} \cdot f(x) = \int_{x_0}^{1} dx \, \frac{1}{(1-x)^{1+\varepsilon}} \left[f(x) - f(1) \right]$$

$$+ \; f(1) \int_{x_0}^{1} dx \, \frac{1}{(1-x)^{1+\varepsilon}}$$

$$= \int_{x_0}^{1} dx \, \frac{1}{(1-x)^{1+\varepsilon}} \left[f(x) - f(1) \right] + f(1) \cdot \left[\frac{-1}{\varepsilon} + \log(1-x_0) + \mathcal{O}(\varepsilon) \right]$$

$$\Longrightarrow \quad \frac{1}{(1-x)^{1+\varepsilon}} \longrightarrow \delta(x-1) \cdot \left(\frac{-1}{\varepsilon} + \log(1-x_0) \right) + \left[\frac{1}{(1-x)^{1+\varepsilon}} \right]_+$$