## $\boxed{\text{Final states dipoles with initial spectators}}$

Those have to satisfy the following conditions:

1) they must subtract soft singularity.;
2) they have to match on the soft limit;
3) we should write them using Ellis-Campbell maping;

To start, we define the collinear limit.

$$M^2 \to M_\mu P_{gg}^{\mu\nu} M_\nu \quad \text{, where } P_{gg}^{\mu\nu} \text{ is}$$

$$P_{g_i g_j}^{\mu\nu} \sim \left[ -g^{\mu\nu} \left( \frac{\xi}{1-\xi} + \frac{1-\xi}{\xi} \right) - 2(1-\varepsilon)\,\xi(1-\xi)\,\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} \right]$$

where $\xi$ & $k_\perp^\mu$ are defined as;

$$p_i^\mu = (1-\xi)\, p^\mu + k_\perp^\mu - \frac{k_\perp^2 n^\mu}{(1-\xi)(2pn)}$$

$$p_j^\mu = \xi\, p^\mu - k_\perp^\mu - \frac{k_\perp^2 n^\mu}{\xi\, 2pn}$$

& $p^\mu$, $p^2=0$ is the collinear directn.

We now use the fact that

$$\pi_{ij}^\mu\ \xi p_i^\mu - (1-\xi)\, p_j^\mu \Rightarrow k_\perp^\mu \quad \text{and} \quad 2p_i \cdot p_j = -\frac{k_\perp^2}{\xi(1-\xi)}$$

$$\Rightarrow P_{g_i g_j}^{\mu\nu} \sim \left[ -g^{\mu\nu} \left( \frac{\xi}{1-\xi} + \frac{1-\xi}{\xi} \right) + (1-\varepsilon)\, \frac{\pi_{ij}^\mu \pi_{ij}^\nu}{p_i p_j} \right].$$

To relate $\xi$ with Campbell-ELLIS

note that
$$p_i \cdot t = (1-\xi)\, p t \qquad p_j \cdot t = \xi\, p t \Rightarrow \frac{p_i t}{p_j t} = \frac{1-\xi}{\xi}\Big|_{\text{in the } p_i p_j \to 0 \text{ limit}}$$

We have $p_i t = \frac{m_t^2}{2}(1-r^2)(1-z)$

$$p_j t \div (t-p_i-p_j)^2 = m_t^2 r^2 \Rightarrow$$

$$0 = m_t^2 - m_t^2 r^2 - 2tp_i - 2tp_j + 2p_i p_j \Rightarrow$$

$$m_t^2(1-r^2) - m_t^2(1-r^2)(1-z) = 2tp_j + 2p_i p_j$$

$$\Rightarrow \quad 2tp_j = m_t^2(1-r^2)z + O(2p_i p_j) \Rightarrow$$

$$\frac{p_i t}{p_j t} = \frac{1-z}{z} \Rightarrow \quad \underline{z \leftrightarrow \xi}.$$

Now, we can also modify $\pi_{ij}^\mu$ by taking

$$\underline{\pi_{ij}^\mu = \xi p_i^\mu - (1-\xi)p_j^\mu \to z p_i^\mu - (1-z)p_j^\mu}$$

$$\to \frac{2tp_j}{m_t^2(1-r^2)} p_i^\mu - \frac{2tp_i}{m_t^2(1-r^2)} p_j^\mu \equiv$$

$$\boxed{\tilde{\pi}_{ij}^\mu \equiv \frac{2}{m_t^2(1-r^2)}\left(tp_j\, p_i^\mu - tp_i\, p_j^\mu\right)} \Rightarrow \tilde{\pi}_{ij}^\mu = (tp_j)p_i^\mu - tp_i p_j^\mu$$

$$D_{g_i;g_j} = \frac{-1}{2p_i p_j}\left\{-g^{\mu\nu}\frac{z}{1-z} + (1-\varepsilon)\frac{2}{m_t^4(1-r^2)^2}\frac{\tilde{\pi}_{ij}^\mu\, \tilde{\pi}_{ij}^\nu}{p_i p_j}\right\}$$

Now, we discuss integration of the tensor $\dfrac{\tilde{\pi}^{\mu}_{ij} \cdot \tilde{\pi}^{\nu}_{ij}}{m_t^4(1-r^2)^2(p_i p_j)^2}$ We have

$$t^{\mu} \tilde{\pi}_{\mu, ij} = tp_j \, tp_i - tp_i \, tp_j = 0 \Rightarrow$$

$$\left\langle \frac{\tilde{\pi}^{\mu}_{ij} \cdot \tilde{\pi}^{\nu}_{ij}}{m_t^4(1-r^2)^2(p_i p_j)^2} \right\rangle \equiv T^{\mu\nu}(t, \hat{b}) \;\&\; T^{\mu\nu} t_{\mu} = 0$$

We write $T^{\mu\nu}(t, \hat{b}) = A\left(g^{\mu\nu} - \dfrac{t^{\mu}t^{\nu}}{t^2}\right) + B\, \tilde{b}_{\perp}^{\mu} \tilde{b}_{\perp}^{\nu}$,

where $\tilde{b}_{\perp}^{\mu} = \left(\hat{b}^{\mu} - \dfrac{(\hat{b}t)t^{\mu}}{t^2}\right) \Rightarrow$

Now, to find results for $A$ & $B$, we do:

$$g^{\mu\nu} T_{\mu\nu} = (d-1)A + B\, \tilde{b}_{\perp}^2$$

$$\tilde{b}_{\perp}^2 = \left(\hat{b} - \frac{(\hat{b}t)t}{t^2}\right)^2 = \frac{(\hat{b}t)^2}{t^2} - 2\frac{\hat{b}t \cdot \hat{b}t}{t^2} = -\frac{(\hat{b}t)^2}{t^2}$$

Now $(t-\hat{b})^2 = m_t^2 r^2 \Rightarrow m_t^2(1-r^2) = 2t\hat{b} \Rightarrow$

$$t\hat{b} = \frac{m_t^2(1-r^2)}{2} \Rightarrow$$

$$\boxed{\tilde{b}_{\perp}^2 = -\frac{m_t^2}{4}(1-r^2)^2} \Rightarrow$$

$$\boxed{g^{\mu\nu} T_{\mu\nu} = (d-1)A - \frac{B m_t^2}{4}(1-r^2)^2}$$

Also, contract with $\tilde{b}^\mu$ $\Rightarrow$

$$\tilde{b}^\mu \tilde{b}^\nu T_{\mu\nu} = -A\frac{(t\tilde{b})^2}{t^2} + B\frac{(\tilde{b}t)^4}{t^4} \equiv \frac{(\tilde{b}t)^2}{t^2}\left(B\frac{(\tilde{b}t)^2}{t^2} - A\right)$$

$LL(t-\tilde{w}^\mu)$

$$\boxed{\tilde{b}^\mu \tilde{b}^\nu T_{\mu\nu} \equiv \frac{m_t^2(1-z^2)^2}{4}\cdot\left(B\frac{m_t^2(1-z^2)^2}{4} - A\right).}$$

To an extent ~~that~~ I did this correctly, we have

$$\left\langle \frac{\tilde{\pi}^\mu_{ij}\tilde{\pi}^\nu_{ij}}{(p_i p_j)^2}\right\rangle \equiv m_t^2\left[\frac{(1+r)^2}{12\varepsilon} \mp \frac{1}{6}\frac{z^2(z^4-3r^2+4)\ln(z)}{(1-r^2)(1-z)^2} - \frac{1}{3}(1+z)^2\ln(1+z) + \frac{(3z^2-5)(3r^2-1)}{24(1-z)^2}\right]\left(g_{\mu\nu} - \frac{t_\mu t_\nu}{m_t^2}\right)$$

$$+\left(\frac{1}{3(1-r)^2\epsilon} \mp \frac{2}{3}\frac{z^4(z^2-3)\ln r}{(1-r^2)^3(1-z)^2} - \frac{4}{3}\frac{\ln(1+z)}{(1-z)^2} + \frac{(9z^4-14z^2+9)}{6(1-z^2)^2(1-r)^2}\right)\left(\tilde{b}^\mu - \frac{(\tilde{b}t)}{t^2}t_\mu\right)\left(\tilde{b}^\mu - \frac{(\tilde{b}t)}{t^2}t^\nu\right)$$

Now, $\langle\quad\rangle$ here stands for

$$\int_0^1 dz \int_0^{y_{max}} dy\; y^{-\varepsilon}(y_{max}-y)^{-\varepsilon}\times(z+r^2(1-z))^{-\varepsilon},$$ ~~but no other~~