Final-initial and initial-final dipoles for top quark decay kinematics

Refs.: [CS] : Catani, Seymour; hep-ph/9605323

[CDST]: Catani, Dittmaier, Seymour, Trocsanyi; hep-ph/0201036

[D] : Dittmaier; hep-ph/9904440

Momenta mapping

N+1 phase space :

$$\boxed{p_a = \underset{\substack{\uparrow \\ \text{emitter} \\ \text{OR spectator}}}{p_i} + \underset{\substack{\uparrow \\ \text{emitted}}}{p_j} + \underset{\substack{\uparrow \\ \text{sum of} \\ \text{all other final} \\ \text{state momenta}}}{K}} \qquad (1.1)$$

with $p_a^2 = m_a^2$, $p_i^2 = m_i^2$, $p_j^2 = 0$ (later $m_i = 0$, $m_a = m_{top}$)

$\tilde{N}$ phase space:

$$\boxed{\tilde{p}_a = \tilde{p}_i + K}$$

with $\tilde{p}_a^2 = m_a^2$, $\tilde{p}_i^2 = m_i^2$.

Note that K is not transformed, therefore

$$p_a - p_i - p_j = \tilde{p}_a - \tilde{p}_i =: P_{ia} . \qquad (1.2)$$

Also, $\tilde{p}_a \neq f(x, P_{ia})\, p_a$ since $m_a \neq 0$.

$\tilde{p}_a$ and $\tilde{p}_i$ are defined according to eq. (4.17) in [D],

$$\tilde{p}_i = \frac{\sqrt{\lambda_{ia}}}{\sqrt{\lambda((p_i+p_j)^2, p_{ia}^2, m_a^2)}} \left(p_a - \frac{p_{ia}\cdot p_a}{p_{ia}^2} p_{ia} \right) + \frac{p_{ia}^2 - m_a^2 + m_i^2}{2 p_{ia}^2} p_{ia} \,,$$

$$\tilde{p}_a = \tilde{p}_i - p_{ia} \,, \qquad (2.1)$$

with $\lambda_{ia} = \lambda(p_{ia}^2, m_a^2, m_i^2)$.

Subtraction terms

For the process $1_t \to 2_b + 3_w + 4_g + 5_g$ we find the following f-i and i-f dipoles,

$$\mathcal{D}_{24,1}\,,\quad \mathcal{D}_{25,1}\,,\quad \mathcal{D}_{45,1} \qquad (2.3)$$

$$\mathcal{D}_{14,2}\,,\quad \mathcal{D}_{15,2}\,,\quad \mathcal{D}_{14,5}\,,\quad \mathcal{D}_{15,4}\,. \qquad (2.4)$$

They are given by [CS],

$$\mathcal{D}_{ij,a} = \frac{-1}{2p_i\cdot p_j}\frac{1}{x_{ia}} \langle 1,\dots,\widetilde{ij},\dots,N+1;\tilde{a}|\, T_a\cdot T_{ij}\, V_{ij}^a \,|1,\dots,\widetilde{ij},\dots,N+1;\tilde{a}\rangle, \qquad (2.5)$$

$$\mathcal{D}_{aj,i} = \frac{-1}{2p_a\cdot p_j}\frac{1}{x_{ia}} \langle 1,\dots,\tilde{i},\dots,N+1;\widetilde{aj}|\, T_{aj}\cdot T_i\, V_{aj}^i \,|1,\dots,\tilde{i},\dots,N+1;\widetilde{aj}\rangle \qquad (2.6)$$

The dipole splitting functions V_{ij}^{a} in (2.5) are given by

eq.(4.16) in [D]:

$$\langle V_{qg}^{a} \rangle = 8\pi \mu^{2\varepsilon} \alpha_s \left\{ \frac{2}{2-x_{ia}-z_{ia}} - 1 - z_{ia} \right.$$

$$\left. - \frac{m_i^2}{p_i \cdot p_j} - \frac{m_i^2\, x_{ia}}{2\, p_i \cdot p_j} \cdot \frac{(1-z_{ia})^2}{z_{ia}} \cdot \frac{r_{ia}}{x_{ia}} \right\}$$

The second line does not contribute, $m_i = m_{bottom} = 0$.

eq.(5.40) in [CS]:

$$\langle \mu | V_{gg}^{a} | \nu \rangle = 16\pi \mu^{2\varepsilon} \alpha_s \left\{ -g^{\mu\nu} \left(\frac{1}{2-x-z} + \frac{1}{2-x-(1-z)} - 2 \right) \right.$$

$$+ (1-\varepsilon) \frac{1}{p_i \cdot p_j} \left(z_{ia} p_i^{\mu} - (1-z_{ia}) p_j^{\mu} \right) \times$$

$$\left. \left(z_{ia} p_i^{\nu} - (1-z_{ia}) p_j^{\nu} \right) \right\}$$

where

$$x_{ia} = 1 - \frac{p_i \cdot p_j}{p_i \cdot p_a + p_j \cdot p_a} ,$$

$$z_{ia} = \frac{p_i \cdot p_a}{p_i \cdot p_a + p_j \cdot p_a} ,$$

eq. (3.10) in [D]

$$r_{ia} = 1 + \frac{\bar{P}_{ia}^{2} \left(\bar{P}_{ia}^{2} + 2 m_a^2 \right)}{\lambda_{ia}} \cdot \frac{1-x_{ia}}{x_{ia}} ,$$

eq. (4.15) in [D]

$$\bar{P}_{ia}^{2} = P_{ia}^{2} - m_a^2 - m_i^2 .$$

eq. (4.12) in [D]

The dipole splitting functions V_{aj}^{i} in (2.6) are given by eq. (4.32) in [D]:

$$\langle V_{qg}^{i} \rangle = 8\pi \mu^{2\varepsilon} \alpha_s \left\{ \frac{2}{2-x_{ia}-z_{ia}} - (1+x_{ia}) R_{ia} - \frac{x_{ia} m_a^2}{p_a . p_j} - \frac{m_a^2}{2 p_a . p_j} \cdot (1-x_{ia})^2 \right\}$$

with the same x_{ia}, z_{ia} as above and

$$R_{ia} = \frac{\sqrt{(P_{ia}^2 + 2m_a^2 x_{ia})^2 - 4m_a^2 P_{ia}^2 x_{ia}^2}}{\sqrt{\lambda_{ia}}} \qquad \text{eq. (4.15) in [D]}$$

Phase space factorization in $D=4$

We have to examine

$$PS = \int d\phi^{(3)}\left(p_a \,|\, p_i, p_j, k \right) \tag{5.1}$$

and want to split it into the phase space for p_j and the rest. Since the particle j recoils against initial state particle a, the <u>whole</u> phase space depends on the energy fraction $\tilde{x}$ taken away. This x-dependence cannot be integrated analytically. Thus, we want to derive

$$PS = \int dx \cdot \int d\phi^{(2)}\left(\tilde{p}_a(x) \,|\, \tilde{p}_i(x), k \right) \times \int d\phi^{(1)}(p_j) . \tag{5.2}$$

To extract an explicit parametrization of $\int d\phi^{(1)}(p_j)$ in (5.2) we take the following steps.

1.) we factorize (5.1) into

$$PS = \int \frac{dq^2}{2\pi} \, d\phi^{(2)}(p_a \,|\, q, k) \times d\phi^{(2)}(q \,|\, p_i, p_j) \tag{5.3}$$

2.) We choose a frame and parametrization such that $d\phi^{(2)}(p_a|q,k)$ in (5.3) and $d\phi^{(2)}(\tilde{p}_a|\tilde{p}_i,k)$ in (5.2) can be identified (up to Lorentz invariant factors).

3.) This allows us to read off $d\phi^{(1)}(p_j)$ in (5.2) by comparing with the remaining pieces in (5.3). In particular, $\int dq^2$ will turn into $\int dx$ and $d\phi^{(2)}(q|p_i,p_j)$ will result into an integral over z_{ia} and an azimuth angle of particle j.

Step 1:

$$\int d\phi^{(3)}(p_a|p_i,p_j,K) = \int \frac{d^D p_i}{(2\pi)^D} \cdot \frac{d^D p_j}{(2\pi)^D} \cdot \frac{d^D K}{(2\pi)^D} \cdot (2\pi)^D \delta^{(D)}(p_a - p_i - p_j - K)$$
$$\times 2\pi \delta^+(p_i^2 - m_i^2) \cdot 2\pi \delta^+(p_j^2 - m_j^2) \cdot 2\pi \delta^+(K^2 - m_K^2)$$

insert
$$1 = \int d^D q \; \delta^{(D)}(q - p_i - p_j),$$
$$1 = \int dq^2 \; \delta^+(q^2 - (p_i + p_j)^2)$$

$$\boxed{\delta^+(p^2) = \delta(p^2)\,\theta(p^0)}$$

$$\ldots = (2\pi)^{3-2D} \int dq^2 \cdot d^D q \cdot d^D K \; \delta^+(q^2 - (p_i+p_j)^2)\, \delta^+(K^2 - m_K^2) \cdot \delta^{(D)}(p_a - q - K)$$
$$\times d^D p_i \cdot d^D p_j \; \delta^+(p_i^2 - m_i^2) \cdot \delta^+(p_j^2 - m_j^2) \cdot \delta^{(D)}(q - p_i - p_j)$$

$$= \int \frac{dq^2}{2\pi} \, d\phi^{(2)}(p_a|q,K) \times d\phi^{(2)}(q|p_i,p_j) \qquad (6.1)$$

Step 2: We choose a frame where $\vec{p}_a + \vec{\tilde{p}}_a = \vec{0}$.

- Let's parametrize $d\phi^{(2)}(p_a|q,K)$ in (6.1),

$$\int d\phi^{(2)}(p_a|q,K) = (2\pi)^{2-D}\int d^Dq \cdot d^DK \cdot \delta^{(D)}(p_a-q-K)\,\delta^+(q^2-m_q^2)\,\delta^+(K^2-m_K^2)$$

$$= (2\pi)^{2-D}\int \frac{d^{D-1}q}{2q^0}\,\delta\left(m_a^2+m_q^2-m_K^2-2p_a\cdot q\right)$$

$\boxed{D=4}$

$$= (2\pi)^{2-D}\int dq^0\,\frac{|\vec{q}|}{2}\,d\Omega_q\,\delta\left(m_a^2+m_q^2-m_K^2-2p_a^0q^0+2\vec{p}_a\vec{q}\right)$$

Integrate over δ-fct., then introduce $t=(p_a-q)^2$.

$$\Rightarrow d\Omega_q = d\cos\theta_q\cdot d\varphi_q = \frac{dt}{2|\vec{p}_a|\cdot|\vec{q}|}\cdot d\varphi_q$$

$$\ldots = (2\pi)^{2-D}\frac{1}{8|\vec{p}_a|\,p_a^0}\int dt\cdot d\varphi_q. \qquad (7.1)$$

- Let's parametrize $d\phi^{(2)}(\tilde{p}_a|\tilde{p}_i,K)$ in (5.2),

$$\int d\phi^{(2)}(\tilde{p}_a|\tilde{p}_i,K) = (2\pi)^{2-D}\int d^D\tilde{p}_i\cdot d^DK\;\delta^{(D)}(\tilde{p}_a-\tilde{p}_i-K)\,\delta^+(\tilde{p}_i^2-m_i^2)\,\delta^+(K^2-m_K^2)$$

$\boxed{D=4}$

$$= (2\pi)^{2-D}\int d\tilde{p}_i^0\,\frac{|\vec{\tilde{p}}_i|}{2}\,d\tilde{\Omega}_i\,\delta\left(m_a^2+m_i^2-m_K^2-2\tilde{p}_a^0\tilde{p}_i^0+2\vec{\tilde{p}}_a\cdot\vec{\tilde{p}}_i\right)$$

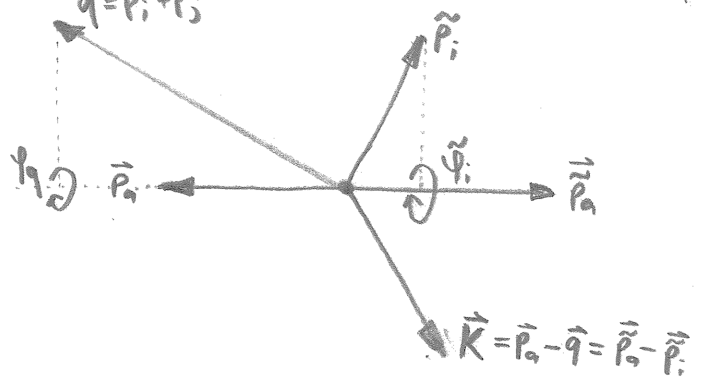
Integrate over δ-fct, then introduce $\tilde{t}=(\tilde{p}_a-\tilde{p}_i)^2$

$$\Rightarrow d\tilde{\Omega}_i = d\cos\tilde{\theta}_i\,d\tilde{\varphi}_i = \frac{d\tilde{t}}{2|\vec{\tilde{p}}_a|\cdot|\vec{\tilde{p}}_i|}\cdot d\tilde{\varphi}_i$$

$$\ldots = (2\pi)^{2-D}\frac{1}{8|\vec{\tilde{p}}_a|\cdot\tilde{p}_a^0}\int d\tilde{t}\cdot d\tilde{\varphi}_i\,. \qquad (7.2)$$

- Now, we argue that (7.1) and (7.2) can be identified.

Since we are in a frame where $\vec{p}_a + \vec{\tilde{p}}_a = \vec{0}$ and $p_a^2 = \tilde{p}_a^2 = m_a^2$, $p_a^0 = \tilde{p}_a^0$ and $|\vec{p}_a| = |\vec{\tilde{p}}_a|$. Furthermore, $t = \tilde{t}$ because $p_a - p_i - p_j = \tilde{p}_a - \tilde{p}_i$, cf. (1.2).
Also the integrations over φ_q and $\tilde{\varphi}_i$ can be identified if we choose $\varphi_q = \tilde{\varphi}_i$ for $\varphi_q = 0$.



<u>Step 3:</u> Let's parametrize $\int d\phi^{(2)}(q|p_i,p_j)$ in (5.3).
Since this is a separately Lorentz invariant phase space, we choose a different frame, where $\vec{q} = \vec{0}$.

$$\Rightarrow \quad \int d\phi^{(2)}(q|p_i,p_j) = (2\pi)^{2-D} \int dp_i^0 \, \frac{|\vec{p}_i|}{2} \, d\cos\theta_i \, d\varphi_i \, \delta(m_q^2 + m_i^2 - m_j^2 - 2p_i^0 q^0)$$

The polar axis is oriented to $\vec{p}_a$, so that for $v = (p_a - p_i)^2$, $dv = 2|\vec{p}_i| \cdot |\vec{p}_a| \cdot \cos\theta_i$.

$$\ldots = (2\pi)^{2-D} \frac{1}{8|\vec{p}_a| q^0} \int dv \cdot d\varphi_i$$

In the frame where $\vec{q}=0$, we have

$$|\vec{P}_a| = \frac{m_a^2+q^2-K^2}{2q^2} \quad \text{and} \quad q^0 = q \cdot q .$$

Furthermore,

$$v = (p_a-p_i)^2 = m_a^2 + m_i^2 - z_{ia}\left(m_a^2 - P_{ia}^2 + (p_i+p_j)^2\right)$$

$$dv = \left(P_{ia}^2 - m_a^2 - (p_i+p_j)^2\right) \cdot dz_{ia} .$$

The first integral in (6.1) can be written as

$$\int \frac{dq^2}{2\pi} = \int \frac{dx_{ia}}{x^2} \frac{\bar{P}_{ia}^2}{2\pi}, \quad \text{since} \quad P_{ia}^2 - m_a^2 - q^2 = \frac{\bar{P}_{ia}^2}{x_{ia}} .$$

Now, we can write (5.3)

$x_{ia} = x$

$$PS = \int \frac{dx_{ia}}{x_{ia}^2} \cdot \frac{\bar{P}_{ia}^2}{2\pi} \cdot d\phi^{(2)}\left(\tilde{P}_a(x) \,|\, \tilde{p}_i(x), K\right) \cdot \frac{(2\pi)^{2-D}}{8}$$

$$\times \frac{\bar{P}_{ia}^2}{2\left(\bar{P}_{ia}^2 + x m_a^2 - x P_{ia}^2\right)} \cdot \frac{\bar{P}_{ia}^2}{x} \int dz_{ia} \cdot d\varphi_i .$$

NOTE: INTEGRATION LIMITS ON dx AND dz HAVE TO BE DETERMINED SOMEHOW.