(3) $x|n,\theta \sim Bin(n,\theta)$ (x = single sample)

$n \sim Poisson(\lambda)$

$\theta \sim Beta(\alpha,\beta)$

a) derive $p(\theta|n,x)$, $p(n|\theta,x)$

Joint $p(\theta,n,x) = p(x|\theta,n)\,p(n)\,p(\theta)$

$$= \underbrace{\binom{n}{x}\theta^x(1-\theta)^{n-x}}\ \underbrace{\lambda^n \exp(-\lambda)\frac{1}{n!}}\ \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

Full-cond. for θ:

$$p(\theta|n,x) \propto \theta^x(1-\theta)^{n-x}\theta^{\alpha-1}(1-\theta)^{\beta-1} = \theta^{x+\alpha-1}(1-\theta)^{n-x+\beta-1}$$

$\therefore\ \theta|n,x \sim Beta(x+\alpha,\ n-x+\beta)$ $\square$

Full-cond. for n:

$$p(n|\theta,x) \propto \binom{n}{x}(1-\theta)^{n-x}\lambda^n\frac{1}{n!} = \frac{n!}{x!(n-x)!}(1-\theta)^{n-x}\lambda^n\frac{1}{n!}$$

$$\propto (1-\theta)^{n-x}\lambda^n\frac{1}{(n-x)!} = \lambda^x(1-\theta)^{n-x}\lambda^{n-x}\frac{1}{(n-x)!}$$

$$\propto [(1-\theta)\lambda]^{n-x}\frac{1}{(n-x)!}$$

$\therefore$ pdf of $n|\theta,x$: $\underbrace{[(1-\theta)\lambda]^{n-x}\frac{1}{(n-x)!}}_{Poisson((1-\theta)\lambda)}$