Bayesian Analysis II, Fall 2020

Lab Exercise 4:

1.

a)

$$X_j \mid \lambda_j \sim \text{Poisson}(t_j \lambda_j) \qquad t_j \equiv \text{known}$$
$$\lambda_j \mid \beta \overset{iid}{\sim} \text{Gamma}(a, 1/\beta) \qquad a \equiv \text{known}$$
$$\beta \sim \text{Gamma}(\gamma, 1/\delta) \qquad \gamma \equiv \text{known},\ \delta \equiv \text{known}$$

a.) Derive the posterior density $f(\underline{\lambda}, \beta \mid \underline{X}) \equiv p(\underline{\lambda}, \beta \mid \underline{X})$

Lkhd:
$$p(\underline{X} \mid \underline{\lambda}) = \prod^{p} p(X_j \mid \lambda_j) = \prod^{p} (\lambda_j t_j)^{X_j} e^{-\lambda_j t_j} \cdot \frac{1}{X_j!} = e^{-\sum^{p} \lambda_j t_j} \prod^{p} (\lambda_j t_j)^{X_j} \cdot \frac{1}{X_j!}$$

Prior $\underline{\lambda}$:
$$p(\underline{\lambda} \mid \beta) = \prod^{p} p(\lambda_j \mid \beta) = \prod^{p} \frac{1}{\beta^{-a} \Gamma(a)} \lambda_j^{a-1} e^{-\lambda_j \beta} = \left(\frac{\beta^a}{\Gamma(a)}\right)^{p} e^{-\beta \sum^{p} \lambda_j} \prod^{p} \lambda_j^{a-1}$$

Prior β:
$$p(\beta) = \frac{1}{\delta^{-\gamma} \Gamma(\gamma)} \beta^{\gamma-1} e^{-\beta\delta}$$

Joint dist. $p(\underline{\lambda}, \beta, \underline{X}) = p(\underline{X} \mid \underline{\lambda})\, p(\underline{\lambda} \mid \beta)\, p(\beta)$

$$p(\underline{\lambda}, \beta, \underline{X}) = \left(\frac{\beta^a}{\Gamma(a)}\right)^{p} \exp\left(-\beta \sum^{p} \lambda_j\right) \left[\prod^{p} \lambda_j^{a-1}\right] \exp\left(-\sum^{p} \lambda_j t_j\right) \underbrace{\left[\prod^{p} (\lambda_j t_j)^{X_j} \cdot \frac{1}{X_j!}\right]}_{\prod^{p} \lambda_j^{X_j} t_j^{X_j} \frac{1}{X_j!}}$$
$$\frac{1}{\delta^{-\gamma} \Gamma(\gamma)} \beta^{\gamma-1} \exp(-\beta\delta)$$

$$\propto \beta^{ap} \exp\left(-\beta \sum^{p} \lambda_j\right) \left[\prod^{p} \lambda_j^{a-1}\right] \exp\left(-\sum^{p} \lambda_j t_j\right) \left[\prod^{p} (\lambda_j t_j)^{X_j} \frac{1}{X_j!}\right] \beta^{\gamma-1} \exp(-\beta\delta)$$

Joint posterior $p(\underline{\lambda}\, \beta \mid \underline{X})$

$$p(\underline{\lambda}\, \beta \mid \underline{X}) \propto p(\underline{\lambda}\, \beta\, \underline{X})$$
$$\propto \beta^{ap} \exp\left(-\beta \sum^{p} \lambda_j\right) \left[\prod^{p} \lambda_j^{a-1}\right] \exp\left(-\sum^{p} \lambda_j t_j\right) \left[\prod^{p} \lambda_j^{X_j}\right] \beta^{\gamma-1} \exp(-\beta\delta)$$

b)

Joint posterior:

$$p(\underline{\lambda}, \beta | \underline{x}) \propto \beta^{ap} \exp\left(-\beta \sum^{p} \lambda_j\right) \left[\prod^{p} \lambda_j^{a-1}\right] \exp\left(-\sum^{p} \lambda_j t_j\right) \left[\prod^{p} \lambda_j^{x_j}\right] \beta^{\gamma-1} \exp(-\beta \delta)$$

Full-cond. $p(\beta | \underline{\lambda}, \underline{x})$

$$p(\beta | \underline{\lambda}, \underline{x}) \propto \beta^{ap} \exp\left(-\beta \sum^{p} \lambda_j\right) \beta^{\gamma-1} \exp(-\beta \delta)$$

$$= \beta^{ap+\gamma-1} \exp\left(-\left(\beta \sum^{p} \lambda_j + \beta \delta\right)\right)$$

$$= \beta^{ap+\gamma-1} \exp\left(-\beta\left(\delta + \sum^{p} \lambda_j\right)\right)$$

b

$$\therefore \beta | \underline{\lambda}, \underline{x} \sim Gamma\left(ap+\gamma, \left(\delta + \sum^{p} \lambda_j\right)^{-1}\right)$$

Full-cond. $p(\lambda_j | \beta, \underline{\lambda}_{j-1}, \underline{x})$,

$$p(\lambda_j | \beta, \underline{\lambda}_{j-1}, \underline{x}) \propto \exp\left(-\beta \sum^{p} \lambda_j\right) \left[\prod^{p} \lambda_j^{a-1}\right] \exp\left(-\sum^{p} \lambda_j t_j\right) \left[\prod^{p} \lambda_j^{x_j}\right]$$

$$\propto \exp(-\beta \lambda_j) \exp(-\lambda_j t_j) \prod^{p} \lambda_j^{a-1} \lambda_j^{x_j}$$

$$= \exp(-\beta \lambda_j) \exp(-\lambda_j t_j) \prod^{p} \lambda_j^{x_j+a-1}$$

$$= \exp(-\beta \lambda_j - \lambda_j t_j) \prod^{p} \lambda_j^{x_j+a-1}$$

$$\propto \exp(-\lambda_j(\beta + t_j)) \lambda_j^{x_j+a-1}$$

$$\therefore \lambda_j | \beta, \underline{\lambda}_{j-1}, \underline{x} \sim Gamma\left(x_j + a, \frac{1}{\beta + t_j}\right)$$

c)

c.) Gibbs sampler

1. Initialize $\beta^{(0)}$ & $\underline{\lambda}^{(0)}$

2. Draw $\beta^{(1)}$ from $Gamma(ap+\gamma, (\delta + \sum \lambda_j^{(0)})^{-1})$
   Draw $\lambda_1^{(1)}$ from $Gamma(x_1 + a, (\beta^{(1)} + t_1)^{-1})$
   ⋮
   Draw $\lambda_p^{(1)}$ from $Gamma(x_p + a, (\beta^{(1)} + t_p)^{-1})$

3. Draw $\beta^{(2)}$ from $Gamma(ap+\gamma, (\delta + \sum^{p} \lambda_j^{(1)})^{-1})$
   —"— $\lambda_1^{(2)}$ from $G(x_1 + a, (\beta^{(2)} + t_1)^{-1})$
   ⋮
   —"— $\lambda_p^{(2)}$ from $G(x_p + a, (\beta^{(2)} + t_p)^{-1})$

4 Repeat until enough samples

2.

a)

$x_1, \dots, x_n | \mu, \lambda \sim N(\mu, \frac{1}{\lambda})$

$\mu \sim N(\mu_0, \frac{1}{\tau})$, μ_0 = known, τ = known

$\lambda \sim Gamma(\alpha, \frac{1}{\beta})$, α = known, β = known

2

Lkhd:

$$p(\underline{x}|\mu,\lambda) = \prod^n p(x_i|\mu,\lambda) = \prod^n (2\pi\lambda^{-1})^{-\frac{1}{2}} \exp\left(-\frac{1}{2\lambda^{-1}}(x_i-\mu)^2\right)$$

$$= (2\pi\lambda^{-1})^{-\frac{n}{2}} \exp\left(-\frac{1}{2\lambda^{-1}}\sum^n (x_i-\mu)^2\right)$$

Prior for μ:

$$p(\mu) = (2\pi\tau^{-1})^{-\frac{1}{2}} \exp\left(-\frac{1}{2\tau^{-1}}(\mu-\mu_0)^2\right)$$

Prior for λ:

$$p(\lambda) = \frac{1}{\beta^{-\alpha}\Gamma(\alpha)} \lambda^{\alpha-1} \exp(-\lambda\beta)$$

Joint dist: $p(\underline{x},\mu,\lambda) = p(\underline{x}|\mu,\lambda)\, p(\mu,\lambda) = p(\underline{x}|\mu,\lambda)\, p(\mu)\, p(\lambda)$

$$p(\underline{x},\mu,\lambda) = (2\pi\lambda^{-1})^{-\frac{n}{2}} \exp\left(-\frac{1}{2\lambda^{-1}}\sum(x_i-\mu)^2\right)(2\pi\tau^{-1})^{-\frac{1}{2}} \exp\left(-\frac{1}{2\tau^{-1}}(\mu-\mu_0)^2\right) \frac{1}{\beta^{-\alpha}\Gamma(\alpha)} \lambda^{\alpha-1} \exp(-\lambda\beta)$$

a

$p(\mu,\lambda|\underline{x})$

$$\propto \lambda^{\frac{n}{2}} \exp\left(-\frac{1}{2\lambda^{-1}}\sum^n (x_i-\mu)^2\right) \exp\left(-\frac{1}{2\tau^{-1}}(\mu-\mu_0)^2\right) \lambda^{\alpha-1} \exp(-\lambda\beta) \quad \square$$

b)

b) Full cond. $p(\lambda|\mu,\underline{x})$

b

$$p(\lambda|\mu,\underline{x}) \propto \lambda^{\frac{n}{2}} \lambda^{\alpha-1} \exp\left(-\frac{1}{2\lambda^{-1}}\sum^n (x_i-\mu)^2\right) \exp(-\lambda\beta)$$

$$= \lambda^{\frac{n}{2}+\alpha-1} \exp\left(-\frac{\lambda}{2}\sum (x_i-\mu)^2 - \lambda\beta\right)$$

$$= \lambda^{\frac{n}{2}+\alpha-1} \exp\left(-\lambda\left[\frac{1}{2}\sum^n (x_i-\mu)^2 + \beta\right]\right)$$

$$\therefore \lambda|\mu,\underline{x} \sim Gamma\left(\frac{n}{2}+\alpha, \left(\frac{1}{2}\sum^n (x_i-\mu)^2 + \beta\right)^{-1}\right)$$

b) Full cond $p(\mu|\lambda, \underline{X})$

$$p(\mu|\lambda,\underline{X}) \propto \exp\left(-\frac{1}{2\lambda^{-1}}\sum(x_i-\mu)^2\right)\exp\left(-\frac{1}{2\tau^{-1}}(\mu-\mu_0)^2\right)$$

$$= \exp\left(-\frac{1}{2}\left(\lambda\sum^n(x_i-\mu)^2 + \tau(\mu-\mu_0)^2\right)\right)$$

$$= \exp\left(-\frac{1}{2}\left[\lambda\sum^n(x_i^2-2x_i\mu+\mu^2) + \tau(\mu^2-2\mu_0\mu+\mu_0^2)\right]\right)$$

$$= \exp\left(-\frac{1}{2}\left[\lambda\sum^n x_i^2 - 2\lambda\mu\sum^n x_i + n\lambda\mu^2 + \tau\mu^2 - 2\mu_0\tau\mu + \tau\mu_0^2\right]\right)$$

$$\propto \exp\left(-\frac{1}{2}\left(n\lambda\mu^2+\tau\mu^2-2\lambda\mu\sum^n x_i - 2\mu_0\tau\mu\right)\right)$$

$$= \exp\left(-\frac{1}{2}\left[(n\lambda+\tau)\mu^2 - 2\left(\lambda\sum^n x_i + \mu_0\tau\right)\mu\right]\right)$$

$$\propto \exp\left(-\frac{1}{2}(n\lambda+\tau)\left(\mu-\frac{\lambda\sum^n x_i+\mu_0\tau}{n\lambda+\tau}\right)^2\right)$$

$$= \exp\left(-\frac{1}{2(n\lambda+\tau)^{-1}}\left(\mu-\frac{\lambda\sum^n x_i+\mu_0\tau}{n\lambda+\tau}\right)^2\right)$$

$$\therefore \mu|\lambda,\underline{X} \sim N\left(\frac{\lambda\sum^n x_i+\mu_0\tau}{n\lambda+\tau}, \frac{1}{n\lambda+\tau}\right)$$

3

a)

a) Derive $p(\underline{\theta},\alpha,\beta|\underline{X})$

Lkhd:

$$p(\underline{X}|\underline{\theta}) = \prod^{10} p(x_j|\theta_j) = \prod^{10} \binom{n_j}{x_j} \theta_j^{x_j}(1-\theta_j)^{n_j-x_j}$$

Prior $\underline{\theta}$

$$p(\underline{\theta}|\alpha,\beta) = \prod^{10} p(\theta_j|\alpha,\beta) = \prod^{10} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta_j^{\alpha-1}(1-\theta_j)^{\beta-1}$$

Joint dist $p(\underline{X},\underline{\theta},\alpha,\beta) = p(\underline{X}|\underline{\theta},\alpha,\beta)\,p(\underline{\theta},\alpha,\beta) = p(\underline{X}|\underline{\theta},\alpha,\beta)\,p(\underline{\theta}|\alpha,\beta)$

$$= p(\underline{X}|\underline{\theta})\,p(\underline{\theta}|\alpha,\beta)\,p(\alpha,\beta)$$

$$p(\underline{\theta},\alpha,\beta,\underline{X}) \propto \prod^{10}\binom{n_j}{x_j}\theta_j^{x_j}(1-\theta_j)^{n_j-x_j}\prod^{10}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta_j^{\alpha-1}(1-\theta_j)^{\beta-1}\,(\alpha+\beta)^{-\frac{5}{2}}$$

$$p(\underline{\theta},\alpha,\beta|\underline{X}) \propto (\alpha+\beta)^{-\frac{5}{2}}\prod^{10}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta_j^{x_j}(1-\theta_j)^{n_j-x_j}\theta_j^{\alpha-1}(1-\theta_j)^{\beta-1}$$

$$= (\alpha+\beta)^{-\frac{5}{2}}\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)^{10}\prod^{10}\theta_j^{x_j+\alpha-1}(1-\theta_j)^{n_j-x_j+\beta-1} \quad \square$$

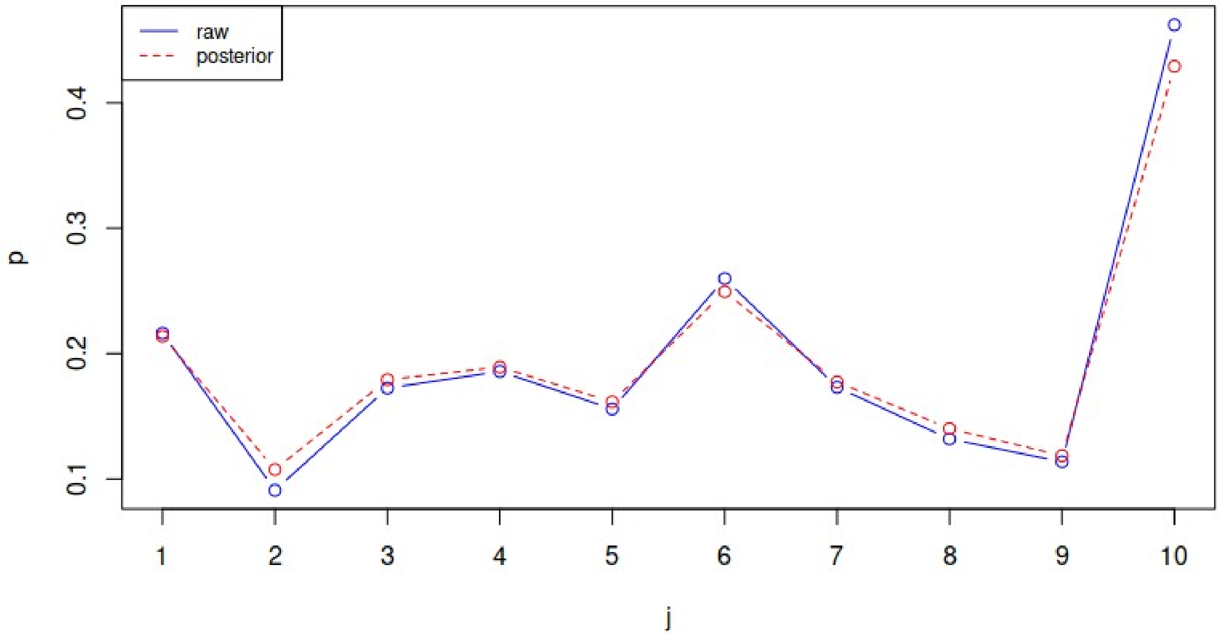
b)

b.) $p(\alpha,\beta|\underline{X}) = \dfrac{p(\underline{\theta},\alpha,\beta|X)}{p(\underline{\theta}|\alpha,\beta,\underline{X})} = \ldots$

$$\left[\begin{array}{l} p(\underline{\theta}|\alpha,\beta,\underline{X}) \propto \prod^{10}\theta_j^{x_j+\alpha-1}(1-\theta_j)^{n_j-x_j+\beta-1} \quad \therefore \theta_j|\alpha,\beta,\underline{X} \sim \mathrm{Beta}(x_j+\alpha, n_j-x_j+\beta) \\ \therefore p(\underline{\theta}|\alpha,\beta,\underline{X}) = \prod^{10}\dfrac{\Gamma(\alpha+\beta+n_j)}{\Gamma(x_j+\alpha)\Gamma(n_j-x_j+\beta)}\theta_j^{x_j+\alpha-1}(1-\theta_j)^{n_j-x_j+\beta-1} \end{array}\right]$$

$$\therefore \propto (\alpha+\beta)^{-\frac{5}{2}}\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)^{10}\prod^{10}\frac{\Gamma(x_j+\alpha)\Gamma(n_j-x_j+\beta)}{\Gamma(\alpha+\beta+n_j)}$$

d)



There seems to be shrinkage of posterior means towards the averages calculated from raw data.

e)

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     2.5%      97.5% 
0.1450659 0.2927260
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