DATA.ML.200, Exercise set 1.

Task 1.

$x_n = Acos(n) + w_n,$ where $n \in \{0, \cdots, N-1\}$ and $w_n \sim N(0, \sigma^2)$

Because $Acos(n)$ can be thought as a constant it holds that $x_n \sim N(Acos(n), \sigma^2)$.

Therefore the likelihood function of *n* independent samples takes the form:

$$
\begin{aligned}
p(\mathbf{x}; A) &= \prod_{n=0}^{N-1} p(x_n; A) \\
&= \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2\sigma^2}(x_n - Acos(n))^2) \\
&= \frac{1}{(2\pi)^{N/2}} * exp(-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x_n - Acos(n))^2)
\end{aligned}
$$

$p(\mathbf{x}; A)$ is maximized, when $\sum_{n=0}^{N-1} (x_n - Acos(n))^2$ is minimized in terms of *A*. Therefore

$$
\begin{aligned}
\frac{\partial}{\partial A} \sum_{n=0}^{N-1} (x_n - Acos(n))^2 &= \sum_{n=0}^{N-1} \frac{\partial}{\partial A} (x_n - Acos(n))^2 \\
&= \sum_{n=0}^{N-1} -2 * cos(n) * (x_n - Acos(n)) \\
&= -2 \left[\sum_{n=0}^{N-1} cos(n) * x_n - A * \sum_{n=0}^{N-1} cos(n)^2 \right]
\end{aligned}
$$

When the bottommost line is equated to zero we get

$$
-2 \left[\sum_{n=0}^{N-1} cos(n) * x_n - A * \sum_{n=0}^{N-1} cos(n)^2 \right] = 0
$$

$$
\sum_{n=0}^{N-1} cos(n) * x_n - A * \sum_{n=0}^{N-1} cos(n)^2 = 0
$$

$$
A * \sum_{n=0}^{N-1} cos(n)^2 = \sum_{n=0}^{N-1} cos(n) * x_n
$$

$$
A = \frac{\sum_{n=0}^{N-1} cos(n) * x_n}{\sum_{n=0}^{N-1} cos(n)^2}
$$

$$
\therefore A_{MLE} = \frac{\sum_{n=0}^{N-1} cos(n) * x_n}{\sum_{n=0}^{N-1} cos(n)^2}
$$

$p(\mathbf{x}; A)$

$= \prod_{n=0}^{N-1} p(x_n; A)$

$= \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2\sigma^2}(x_n - Acos(n))^2)$

$= 1_{\overline{(2\pi)^{N/2}}}$
$\prod_{n=0}^{N-1}$

$$\frac{1}{(2\pi)^{N/2}} \, 1_{\overline{(2\pi)^{N/2}}}$$