DATA.ML.200 Pattern Recognition and Machine Learning

Exercise Set 1: October 20 – October 27, 2020

- Exercises consist of both `pen&paper` and `python` assignments.
- Prepare a single PDF and return to Moodle on Tuesday, October 27th at 23:55 at the latest.
- Mark on 1st page which exercises you did.

Before you start, load all the data for all questions from the following links. We will need them in later weeks as well.

```
http://www.cs.tut.fi/courses/SGN-41007/Ex1_data.zip
http://www.cs.tut.fi/courses/SGN-41007/least_squares_data.zip
http://www.cs.tut.fi/courses/SGN-41007/locationData.zip
```

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1. `pen&paper` *Maximum likelihood estimation.* Consider the model

$$x[n] = A\cos[n] + w[n], \qquad n = 0, 1, \ldots, N-1,$$

where $w[n] \sim \mathcal{N}(0, \sigma^2)$. In other words, we assume that our measurement is a sinusoid at fixed frequency and phase and want to estimate the amplitude A. Derive the maximum likelihood estimator of A.

Hint: The same procedure as on the Thursday 10.1. lecture applies: Just write the probability of observing $\mathbf{x} = (x[0], \ldots, x[N-1])$, and maximize with respect to A. The only difference to lecture case is that you have the known signal $\cos[n]$ as an additional variable. However, it is known, so just treat is at as a constant (it will be part of the end result).

2. `pen&paper` *Maximum likelihood estimation 2.* The *Poisson distribution* is a discrete probability distribution that expresses the probability of a number of events $x \geq 0$ occurring in a fixed period of time:

$$p(x; \lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$$

We measure N samples: $x_0, x_1, \ldots, x_{N-1}$ and assume they are Poisson distributed and independent of each other. Find the maximum likelihood estimator of λ, *i.e.*, do the following steps.

   a) Compute the probability $p(\mathbf{x}; \lambda)$ of observing the samples $\mathbf{x} = (x_0, x_1, \ldots, x_{N-1})$.

   b) Compute the natural logarithm of p, *i.e.*, $\log p(\mathbf{x}; \lambda)$.

   c) Differentiate the result with respect to λ.

d) Find the maximum of the function, *i.e.*, the value where $\frac{\partial}{\partial \lambda} \log p(\mathbf{x}; \lambda) = 0$.

3. $\boxed{\texttt{python}}$ *Load Matlab data into Python and plot it.*

   a) Load the file `twoClassData.mat` into Python (found in `Ex1_data.zip`). This can be done as follows.

```
>>> from scipy.io import loadmat
>>> mat = loadmat("twoClassData.mat")
```

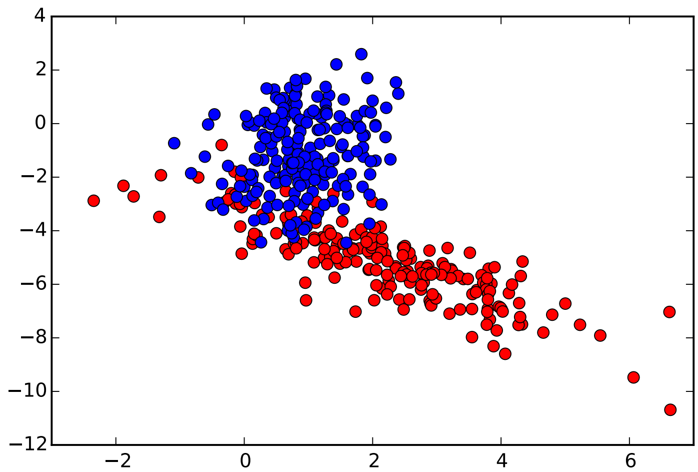
   This generates a **`dict`** structure, whose elements can be accessed through their names.

```
>>> print(mat.keys()) # Which variables mat contains?
['y', 'X', '__version__', '__header__', '__globals__']
>>> X = mat["X"] # Collect the two variables.
>>> y = mat["y"].ravel()
```

   The function `ravel()` transforms y from 400×1 matrix into a 400-length array. In Python these are different things unlike Matlab.

   b) The matrix X contains two-dimensional samples from two classes, as defined by y. Plot the data as a scatter plot like the picture below. Hints:

   - You can access all class 0 samples from X as: `X[y == 0, :]`.
   - The samples can be plotted like: `plt.plot(X[:, 0], X[:, 1], 'ro')`



4. $\boxed{\texttt{python}}$ *Remove the uneven illumination from a microscope image using least squares fitting like we did at the lecture.*

   Load the following Python template and test image:

```
http://www.cs.tut.fi/courses/SGN-41007/exercises/task5_template.py
http://www.cs.tut.fi/courses/SGN-41007/exercises/uneven_illumination.jpg
```

Load the image into numpy array `Z` using the `imread` function of `matplotlib.image`[1]. Finally, show the image on screen (using `matplotlib.pyplot.imshow`) and check that the image shape is 1300×1030. Let's next fit a 2nd order surface to the grayscales.

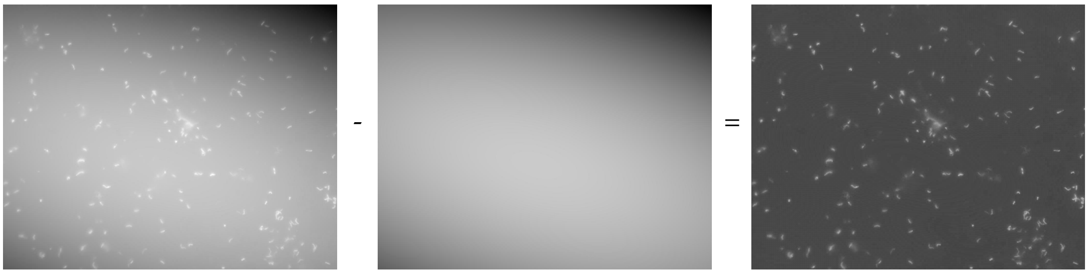
a) Create the explanatory variables (all x,y-coordinates in a matrix): `X, Y = np.meshgrid(range(1300), range(1030))`

b) Vectorize the matrices $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ using `ravel`, *e.g.*, `z = Z.ravel()`.

c) Prepare the design matrix $\mathbf{H}$ like in the lectures and solve the LS coefficients $\mathbf{c}$.

d) Compute the model prediction as `z_pred = np.dot(H, c)` and resize the vector to the original size.

e) Subtract the model prediction from the original image and show the result on screen.



5. **`python`** *Estimate sinusoidal parameters.*

a) Generate a 100-sample long synthetic test signal from the model:

$$x[n] = \sin(2\pi f_0 n) + w[n], \qquad n = 0, 1, \ldots, 99$$

with $f_0 = 0.015$ and $w[n] \sim \mathcal{N}(0, 0.3)$. Note that $w[n]$ is generated by `w = numpy.sqrt(0.3) * numpy.random.randn(100)`. Plot the result.

b) Implement code from estimating the frequency of `x` using the maximum likelihood estimator:

$$\hat{f}_0 = \text{value of } f \text{ that maximizes } \left| \sum_{n=0}^{N-1} x(n) e^{-2\pi i f n} \right|.$$

Implementation is straightforward by noting that the sum expression is in fact a dot product:

$$\hat{f}_0 = \text{value of } f \text{ that maximizes } |\mathbf{x} \cdot \mathbf{e}|,$$

with $\mathbf{x} = (x_0, x_1, \ldots, x_{N-1})$ and $\mathbf{e} = (e^{-2\pi i f \cdot 0}, e^{-2\pi i f \cdot 1}, \ldots, e^{-2\pi i f \cdot (N-1)})$.

Use the following template and fill in the blanks.

[1] Note: There are several other ways of doing this, such as: `matplotlib.image.imread`, `scipy.ndimage.imread`, `PIL.Image.open` or `cv2.imread`

```
scores = []
frequencies = []

for f in numpy.linspace(0, 0.5, 1000):

    # Create vector e. Assume data is in x.
    n = numpy.arange(100)
    z = # <compute -2*pi*i*f*n. Imaginary unit is 1j>
    e = numpy.exp(z)

    score = # <compute abs of dot product of x and e>
    scores.append(score)
    frequencies.append(f)

fHat = frequencies[np.argmax(scores)]
```

c) Run parts (a) and (b) a few times. Are the results close to true $f_0 = 0.015$?