DATA.ML.420

Lab Exercise 5

1.

a)

Show $E(Y) = \dfrac{1}{2\sqrt{\lambda}}$.

$$
\begin{aligned}
E(Y) &= \int_0^\infty yf(y)dy \\
&= \text{in terms of cdf} \\
&= \int_0^\infty (1 - F(y))dy - \int_{-\infty}^0 F(y)dy \\
&= \int_0^\infty (1 - F(y))dy \\
&= \int_0^\infty exp(-\lambda\pi y^2)dy \\
&= (*1) \\
&= \frac{1}{2\sqrt{\lambda}}
\end{aligned}
$$

$(*1)$: $exp(-\lambda\pi y^2)$ is the kernel of $y \sim N(0, \sigma^2 = \dfrac{1}{2\lambda\pi})$. Hence $\displaystyle\int_{-\infty}^\infty f(y)dy = \sqrt{2\pi\sigma^2}$ and $\displaystyle\int_0^\infty f(y)dy = \frac{1}{2}\sqrt{2\pi\sigma^2} = \frac{1}{2}\sqrt{2\pi\frac{1}{2\pi\lambda}} = \frac{1}{2}\sqrt{\frac{1}{\lambda}} = \frac{1}{2\sqrt{\lambda}}$

Show $Var(Y) = \dfrac{4-\pi}{4\lambda\pi}$.

$$
\begin{aligned}
E(Y^2) &= \int_0^\infty y^2 f(y)dy \\
&= \text{in terms of cdf} \\
&= 2\int_0^\infty y(1 - F(y))dy \\
&= 2\left[\int_0^\infty ydy - \int_0^\infty yF(y)dy\right] \\
&= 2\left[\int_0^\infty ydy - \int_0^\infty ydy + \int_0^\infty yexp(-\lambda\pi y^2)dy\right] \\
&= 2\int_0^\infty yexp(-\lambda\pi y^2)dy \\
&= 2\frac{1}{2\lambda\pi} \\
&= \frac{1}{\lambda\pi}
\end{aligned}
$$

$$E(Y)^2 = \frac{1}{4\lambda}$$

$$Var(Y) = E(Y^2) - E(Y)^2 = \frac{1}{\lambda\pi} - \frac{1}{4\lambda} = \frac{4-\pi}{4\lambda\pi}$$

b)

c)

$$\begin{aligned}
E(Y^2) &= \int_{r_0}^{\infty} y^2 2\lambda\pi y exp(-\lambda\pi(y^2 - r_0^2))dy \\
&= exp(\lambda\pi r_0^2) \int_{r_0}^{\infty} 2\lambda\pi y^3 exp(-\lambda\pi y^2)dy \\
&= exp(\lambda\pi r_0^2) \int_{r_0^2}^{\infty} \lambda\pi u exp(-\lambda\pi u)du \\
&= exp(\lambda\pi r_0^2)\frac{1}{\lambda\pi}(\lambda\pi r_0^2 + 1)exp(-\lambda\pi r_0^2) \\
&= \frac{\lambda\pi r_0^2 + 1}{\lambda\pi} \\
&= r_0^2 + \frac{1}{\lambda\pi}
\end{aligned}$$

$$Var(Y) = E(Y^2) - E(Y)^2 = r_0^2 + \frac{1}{\lambda\pi} - E(Y)^2$$