Spatial Data Analysis

Lab Exercise 2

Pen & paper problems

1.

a) Show that $Z(s)$ is intrinsic stationary

i) The mean is constant

By definition, the mean is constant, say $E(Z(s)) = 0$.

ii) $Var(Z(s) - Z(t))$ depends only on the lag difference.

$$\begin{aligned} Cov(Z(s), Z(t)) &= E(Z(s)Z(t)) - E(Z(s))E(Z(t)) \\ &= E(Z(s)Z(t)) \\ &= \sigma^2 min(s,t) \end{aligned}$$

$$\begin{aligned} Var(Z(s) - Z(t)) &= E((Z(s) - Z(t))^2) - E(Z(s) - Z(t))^2 \\ &= E((Z(s) - Z(t))^2) - (E(Z(s)) - E(Z(t)))^2 \\ &= E((Z(s) - Z(t))^2) \\ &= E(Z(s)^2) - 2E(Z(s)Z(t)) + E(Z(t)^2) \\ &= Var(Z(s)) - 2E(Z(s)Z(t)) + Var(Z(t)) \\ &= \sigma^2 s - \sigma^2 min(s,t) + \sigma^2 t \\ &= \begin{cases} -\sigma^2(s-t) \text{ if } s < t \\ \ \ \sigma^2(s-t) \text{ if } s \geq t \end{cases} \end{aligned}$$

Variance of difference depend only on the lag difference $s - t$ and hence $Z(s)$ is intrinsic stationary.

Here $Z(s)$ is not second-order stationary since the covariance depends also on the actual positions, say, in general it holds that

$$\begin{aligned} Cov(Z(s), Z(t)) &= \sigma^2 min(s,t) \\ &\neq \sigma^2 min(s+h, t+h) \\ &= Cov(Z(s+h), Z(t+h)) \end{aligned}$$

b) Show that $\{Y(s) = Z(s+1) - Z(s) : s \in \mathbb{R}\}$ is second order stationary

i) The mean is constant.

$E(Y(s)) = E(Z(s+1)) - E(Z(s)) = 0 - 0 = 0$

ii) Covariance between variate at two sites depends only on the site's relative positions.

$$
\begin{aligned}
Cov(Y(s),Y(t)) &= E(Y(s)Y(t)) - E(Y(s))E(Y(t)) \\
&= E(Y(s)Y(t)) \\
&= E\left[(Z(s+1)-Z(s))(Z(t+1)-Z(t))\right] \\
&= E\left[Z(s+1)Z(y+1) - Z(s+1)Z(t) - Z(t+1)Z(s) + Z(s)Z(t)\right] \\
&= E\left[Z(s+1)Z(y+1)\right] - E\left[Z(s+1)Z(t)\right] - E\left[Z(t+1)Z(s)\right] + E\left[Z(s)Z(t)\right] \\
&= \sigma^2(min(s+1,t+1) - min(s+1,t) - min(s,t+1) + min(s,t)) \\
&= \sigma^2((min(s+1,t+1)+h) - (min(s+1,t)+h) - (min(s,t+1)+h) + (min(s,t)+h)) \\
&= \sigma^2(min(s+h+1,t+h+1) - min(s+h+1,t+h) - min(s+h,t+h+1) + min(s+h,t+h)) \\
&= Cov(Y(s+h),Y(t+h))
\end{aligned}
$$

c)

Derive covariance function $C_Y(h)$ of $\{Y(s) = Z(s+1) - Z(s) : s \in \mathbb{R}\}$

$$
\begin{aligned}
C_Y(h) &= Cov(Y(s+h),Y(s)) \\
&= \sigma^2\left[min(s+h+1,s+1) - min(s+h+1,s) - min(s+h,s+1) + min(s+h,s)\right] \\
&= \cdots
\end{aligned}
$$

If $h < 0$:

$$
\begin{aligned}
\cdots &= \sigma^2\left[s+h+1 - min(s+h+1,s) - s - h + s + h\right] \\
&= \sigma^2\left[s+h+1 - min(s+h+1,s)\right]
\end{aligned}
$$

If $-1 < h < 0$:

$$
\begin{aligned}
\sigma^2\left[s+h+1 - min(s+h+1,s)\right] &= \sigma^2\left[s+h+1-s\right] \\
&= \sigma^2(h+1)
\end{aligned}
$$

If $h < -1$:

$$
\begin{aligned}
\sigma^2\left[s+h+1 - min(s+h+1,s)\right] &= \sigma^2\left[s+h+1-s-h-1\right] \\
&= 0
\end{aligned}
$$

If $h \geq 0$:

$$
\begin{aligned}
\cdots &= \sigma^2\left[s+1-s-min(s+h,s+1)+s\right] \\
&= \sigma^2\left[s+1-min(s+h,s+1)\right]
\end{aligned}
$$

If $0 \leq h < 1$:

$$
\begin{aligned}
\sigma^2\left[s+1-min(s+h,s+1)\right] &= \sigma^2(s+1-s-h) \\
&= \sigma^2(1-h)
\end{aligned}
$$

If $1 \leq h$:

$$
\begin{aligned}
\sigma^2\left[s+1-min(s+h,s+1)\right] &= \sigma^2(s+1-s-1) \\
&= 0
\end{aligned}
$$

Therefore the covariance function takes the form of

$$C_Y(h) = \begin{cases} \sigma^2(h+1), \text{ if } -1 < h < 0 \\ \sigma^2(1-h), \text{ if } 0 \le h < 1 \\ \qquad 0, \text{ otherwise} \end{cases}$$

Because of the second-order stationary of the random field, the semivariogram is

$$\begin{aligned} \gamma(h) &= C_Y(0) - C_Y(h) \\ &= \sigma^2 - C_Y(h) \\ &= \begin{cases} \sigma^2 - \sigma^2(h+1), \text{ if } -1 < h < 0 \\ \sigma^2 - \sigma^2(1-h), \text{ if } 0 \le h < 1 \\ \qquad \sigma^2, \text{ otherwise} \end{cases} \quad = \begin{cases} -\sigma^2 h, \text{ if } -1 < h < 0 \\ \sigma^2 h, \text{ if } 0 \le h < 1 \\ \sigma^2, \text{ otherwise} \end{cases} \end{aligned}$$

Properties of semivariogram:

nugget effect: 0

sill: σ^2

range: 1