3. a)

i) Show that $\phi V_{\underline{i}} = \underline{w}$

$$V_{\underline{i}} = \frac{1}{1-\phi^2}\begin{pmatrix} 1 & \phi & \phi^2 & \cdots & \phi^{t-2} & \phi^{t-1} \\ \phi & 1 & \phi & \cdots & \phi^{t-3} & \phi^{t-2} \\ \vdots & & & \ddots & & \\ \phi^{t-2} & \phi^{t-3} & \phi^{t-4} & \cdots & 1 & \phi \\ \phi^{t-1} & \phi^{t-2} & \phi^{t-3} & \cdots & \phi & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \phi^{t-1} \\ \phi^{t-2} \\ \phi^{t-3} \\ \vdots \\ \phi \\ 1 \end{pmatrix} \frac{1}{1-\phi^2}$$

$$\phi V_{\underline{i}} = \frac{\phi}{1-\phi^2}\begin{pmatrix} \phi^{t-1} \\ \phi^{t-2} \\ \vdots \\ \phi \\ 1 \end{pmatrix} = \frac{1}{1-\phi^2}\begin{pmatrix} \phi^{t} \\ \phi^{t-1} \\ \vdots \\ \phi^2 \\ \phi \end{pmatrix} = w$$

ii) Show that BLUP of y_{t+1} is $\hat{y}_{t+1} = (1-\phi)\hat{\beta}_0 + \phi y_t$

$$\hat{y}_{t+1} = x'_{t+1}\hat{\beta} + \underline{w}'V^{-1}(y - X\hat{\beta})$$
$$= 1 \cdot \hat{\beta}_0 + w'V^{-1}(y - 1\hat{\beta}_0)$$
$$= \hat{\beta}_0 + \phi \underline{i}'(y - \underline{1}\hat{\beta}_0)$$
$$= \hat{\beta}_0 - \phi i \underline{1}\hat{\beta}_0 + \phi i' y$$
$$= (1-\phi)\hat{\beta}_0 + \phi y_t \quad \square$$

$\phi V_{\underline{i}} = \underline{w} \iff \phi \underline{i}' = \underline{w}'V^{-1}$