Tightening bounds / redundant constraints

Consider the LP below,

$$\begin{array}{rrcrcrcr}
\max & 2x_1 & + & x_2 & - & x_3 & & \\
\text{s.t.} & 5x_1 & - & 2x_2 & + & 8x_3 & \leq & 15 \\
& 8x_1 & + & 3x_2 & - & x_3 & \leq & 9 \\
& x_1 & + & x_2 & + & x_3 & \leq & 6 \\
& \multicolumn{3}{c}{0 \leq x_1 \leq 3} & & & & \\
& \multicolumn{3}{c}{0 \leq x_2 \leq 1} & & & & \\
& \multicolumn{3}{c}{x_3 \geq 1} & & & &
\end{array}$$

Derive tightened bounds for variables x_1 and x_3 from the first constraint and eliminate redundant constraints after that.

Solution: We can obtain upper bounds for x_1 and x_3 from the first constraint. For that, we need the minimum activity $\alpha_{min} = \sum_{j:a_j>0} a_j l_j + \sum_{j:a_j<0} a_j u_j$. That is, the variables with a positive coefficient are at their lower bounds and the ones with negative coefficients at their upper bounds. This leads to the smallest possible value $a^\top x$. From the lecture material, we know that $a^\top x \leq b$ can (for a positive a_j) be written as $x_j \leq \frac{b-(a^\top x - a_j x_j)}{a_j}$. This is just the original constraint rearranged. The part $a^\top x - a_j x_j$ is the original LHS without the variable x_j. Using the idea of minimum activity, we get $x_j \leq \frac{b-(a^\top x - a_j x_j)}{a_j} \leq \frac{b-(\alpha_{min} - a_j l_j)}{a_j}$, where $\alpha_{min} - a_j l_j$ is the minimum activity without variable x_j.

The minimum activity for the first constraint is $\alpha_{min} = 5*0 - 2*1 + 8*1 = 6$. Thus, $x_1 \leq \frac{15-(6-5*0)}{5} = \frac{9}{5}$ and $x_3 \leq \frac{15-(6-8*1)}{8} = \frac{17}{8}$.

Now that we have a lower and upper bound for all constraints, minimum and maximum activities are all bounded. In the original formulation, minimum activity for the second constraint would be $\alpha_{min} = 8*0 + 3*0 - 1*\infty = -\infty$. Using the bounds

$$0 \leq x_1 \leq \frac{9}{5}$$
$$0 \leq x_2 \leq 1$$
$$1 \leq x_3 \leq \frac{17}{8},$$

we can obtain the minimum and maximum activity for all constraints:

Constraint	α_{min}	α_{max}	b
1	6	26	15
2	-2.125	16.4	9
3	1	4.925	6

For the first and second constraint, b is between α_{min} and α_{max}, but for the last constraint, $b > \alpha_{max}$. This means that the last constraint is redundant, since even the largest possible value for the LHS considering the variable bounds is less than the RHS.