MS-E2121 - Linear optimization **DEADLINE:**
Homework 4 **14.03.2021 @23:55**

In this homework, you will be requested to complete the implementation of a Benders decomposition for a mixed-integer programming problem, using the skeleton code provided here.

In addition you are requested to provide a report in pdf format answering the questions below. The report must be written using, e.g. LaTeX, or Word.

Please submit your answers to this page before the given deadline. Late submissions will be penalised by 5 points per day after the deadline.

Problem 4.1: Implementing the full-scale problem

We will consider a variant of the stochastic server location problem (SSLP), which is employed to plan the emergency response operations in man-made or natural disasters that potentially affect several locations simultaneously.

In the SSLP, the server locations are chosen such that they try to satisfy the demand of all locations in need. The uncertain aspect is related to the demand of each location, which is either zero or a scenario-dependent random number representing, e.g., a fraction of the population exposed to the disaster.

To model the variant of the SSLP considered, we define the following elements. Let I be the set of locations to be served, and J the candidate locations for the servers (which are assumed to not be affected in the disaster situation). To model the uncertainty related with the nodes affected by the disaster, we consider a set of scenarios $s \in S$, for which a probability P_s is given. Each of these scenarios s is one possible unfolding of the disaster situation. The parameter H_{is} indicates which locations are affected in scenario s, and thus $H_{is} = 1$ if location i is affected in scenario s, and 0 otherwise. To formulate the model, we consider the following parameters:

- C_j - cost of installing server $j \in J$;
- D_{is} - demand of client i in scenario s;
- F - cost per unit of unplanned extra capacity (same for all locations i and scenarios s);
- H_{is} - a binary parameter with value 1 if client i is active in scenario s, i.e., the demand D_{is} has to be fulfilled;
- P_s - probability associated with scenario $s \in S$;
- Q_{ij} - benefit, measured in the same unit of C_j and F, per one unit of demand of client i served by server j;
- U - maximum capacity of a server (same for all servers and scenarios);
- V - maximum allowed number of servers.

Let us define the following decision variables:

- x_j - binary variable with value 1 if a server is located, i.e., built or installed, at location j;
- y_{ijs} - the proportion of demand D_{is} fulfilled by server j. The total demand of i served by j is thus $y_{ijs}D_{is}$;
- z_{js} - shortage of capacity for server j. Represents an emergency extra capacity procured for a specific scenario $s \in S$.

The mixed-integer programming model representing the described variant of the SSLP is given by:

$$\min \sum_{j \in J} C_j x_j + \sum_{s \in S} P_s \left(\sum_{j \in J} F z_{js} - \sum_{i \in I, j \in J} Q_{ij} D_{is} y_{ijs} \right) \tag{1}$$

$$\text{s.t.:} \sum_{j \in J} x_j \leq V \tag{2}$$

$$\sum_{i \in I} D_{is} y_{ijs} \leq U x_j + z_{js}, \ \forall j \in J, s \in S \tag{3}$$

$$\sum_{j \in J} y_{ijs} = H_{is}, \ \forall i \in I, s \in S \tag{4}$$

$$x_j \in \{0, 1\}, \ \forall j \in J \tag{5}$$

$$y_{ijs} \geq 0, \ \forall i \in I, \ \forall j \in J, \forall s \in S \tag{6}$$

$$z_{js} \geq 0, \ \forall j \in J, \ \forall s \in S \tag{7}$$

The objective function (1) represents the costs for locating the servers, plus the expected benefits from supplying the demand and the costs with extra emergency capacity. Constraint (2) establishes the limit on the total number of servers. Constraint (3) models the capacity of a server located in $j \in J$, provided that it has been located, also considering the extra emergency capacity procured. Notice that the extra capacity is assumed to be available even if the location j is not chosen for having a server. Constraint (4) states that if the location $i \in I$ is affected in scenario $s \in S$, then all of its demand D_{is} must be satisfied. It also states that non-affected locations $i \in I$ are not served. Finally, constraints (5)–(7) define the domain of the decision variables.

Complete the implementation of the large-scale version of the SSLP problem. As you will see in the skeleton provided, the data has been generated and provided in a JLD2-file that should be downloaded and placed into the same directory with the notebook. It is then read into a structure containing the relevant information for the problem instance. Your task is to implement the model (1)–(7), using the parameters provided in the skeleton code. Pay attention to the skeleton code comments, as they will serve as guidelines for correctly implementing the model.

Once you have implemented the model, provide answers to the following questions in your report. You will need these values in the next tasks to validate your implementation of the Benders decomposition.

a) What are the locations selected to have a server?

b) What is the total cost of the system?

Problem 4.2: Single-cut Benders decomposition

Implement a Benders decomposition approach for the SSLP model implemented in Task 1. In the context of this assignment, our implementation is likely to be less efficient than simply solving the large-scale problem since we will not be relying on parallelisation or any more sophisticated methods to exploit the decomposition.

The complete implementation of the Benders decomposition requires the tasks below to be performed. Make sure you present in your report the formulation for each of the elements required. Notice that the two-stage nature of the problem provides us with a natural setting for a Benders decomposition, i.e., a main problem with the first-stage decisions (i.e., the location decisions) and a single subproblem with the second-stage decisions (those depending on the scenario realisation).

Specifically, you are required to provide in your report:

a) The formulation of the main problem at the first iteration of the Benders decomposition, i.e., without any cuts.

b) The formulation of the primal subproblem, and

c) The dual formulation of the subproblem.

Once you have correctly implemented the dual subproblem, which can be verified against its primal formulation from (b), you can provide

d) The formulation of the optimality cut to be added at each iteration of the method; explain why in this case one does not need to consider the generation of cuts for the case when the subproblem is infeasible (HINT: can the subproblem ever be infeasible in this case?).

Problem 4.3: Multi-cut Benders decomposition

Modify the Benders decomposition such that the formulation of the main problem is strengthened. To accomplish that, notice that the original problem defined in Task 1 has a similar form to that presented in Lecture 7, that is

$$\begin{aligned}(P_R) : \min_{x}.\ & c^\top x + \sum_{k=1}^{K} z_k(x) \\ \text{s.t.: } & Ax = b \\ & x \geq 0.\end{aligned}$$

where $z_k(x) = \min._y \left\{ f_k^\top y_k : D_k y_k = e_k - C_k x \right\}$.

Therefore, instead of creating a single subproblem encompassing the second-stage decision variables and constraints, we can, following the same reasoning presented in Lecture 7, generate one cut for each scenario $s \in S$. This would be equivalent to generating one cut for each subproblem $k \in K$, as described in Lecture 7.

With the above in mind, you are requested to provide in your report:

a) The formulation of the modified main problem at the first iteration of the Benders decomposition, i.e., without any cuts.

b) The formulation of the optimality cuts to be added at each iteration of the method.

c) Analytical evidence comparing the performance of the method implemented in Task 2 and the modified Benders decomposition. You can base your comparison on indicators such as the number of total iterations, the total time, the time per iteration and so forth.