MS-E2121 Homework 5

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Problem 5.1: IP formulations

Let

$$X = \{(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),(0,1,0,1),(0,0,1,1)\}$$
$$= \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_7\}$$

(a)

(1): For P_1

Let $P_1 = \left\{\mathbf{x} \in \mathbb{R}^4 : 83x_1 + 61x_2 + 49x_3 + 20x_4 \leq 100, 0 \leq \mathbf{x} \leq 1\right\}$

Clearly $X \subseteq \{0,1\}^4$. Because each coefficient in the inequality of P_1 is < 100, then $\mathbf{x}_1, ..., \mathbf{x}_5 \in P_1$. In addition, because $49 + 20 \leq 61 + 20 < 100$, then $\mathbf{x}_7, \mathbf{x}_6 \in P_1$. Therefore $X \subseteq P_1 \cap \mathbb{Z}^4$.

Let $\mathbf{x} \in P_1 \cap \mathbb{Z}^4$. Then $\mathbf{x} \in \{0,1\}^4$ because $0 \leq \mathbf{x} \leq 1$ and $\mathbf{x} \in \mathbb{Z}^4$. Let $C = \{83, 61, 49, 20\}$. Because $\forall c \in C : c < 100$ and $\forall c_i, c_j, c_k \in C : c_i + c_j + c_k > 100$, then $\sum_i \mathbf{x}_i \leq 2$. Because $83 + c_i > 100$ and $61 + 49 > 100$, then $\mathbf{x} \in X$, i.e. $P_1 \cap \mathbb{Z}^4 \subseteq X$.

Therefore $X = P_1 \cap \mathbb{Z}^4$, i.e. P_1 is formulation of X.

(2): For P_2

Let $P_2 = \left\{\mathbf{x} \in \mathbb{R}^4 : 4x_1 + 3x_2 + 2x_3 + x_4 \leq 4, 0 \leq \mathbf{x} \leq 1\right\}$

Clearly $X \subseteq \{0,1\}^4$. Because each coefficient in the inequality of P_2 is ≤ 4, then $\mathbf{x}_1, ..., \mathbf{x}_5 \in P_2$. In addition, because $2 + 1 < 3 + 1 \leq 4$ then $\mathbf{x}_7, \mathbf{x}_6 \in P_2$. Therefore $X \subseteq P_2 \cap \mathbb{Z}^4$.

Lets compare P_1 and P_2. Moreover, P_2 can be expressed as

$$P_2 = \left\{\mathbf{x} \in \mathbb{R}^4 : 100x_1 + 75x_2 + 50x_3 + 25x_4 \leq 100, 0 \leq \mathbf{x} \leq 1\right\}$$

by multiplying both sides of inequality by 25. Because every factor in the inequality of above formulation of P_2 is smaller than corresponding factors in P_1, we can write

$$83x_1 + 61x_2 + 49x_3 + 20x_4 \leq 100x_1 + 75x_2 + 50x_3 + 25x_4, \forall 0 \leq \mathbf{x} \leq 1$$

This implies that $P_2 \subseteq P_1$ i.e., $P_2 \cap \mathbb{Z}^4 \subseteq P_1 \cap \mathbb{Z}^4$. Moreover, from part (1) we also have $P_2 \cap \mathbb{Z}^4 \subseteq P_1 \cap \mathbb{Z}^4 \subseteq X$.

We have now proved that $X \subseteq P_2 \cap \mathbb{Z}^4$ and $P_2 \cap \mathbb{Z}^4 \subseteq X$, i.e. $X = P_2 \cap \mathbb{Z}^4$, which means that P_2 is formulation of X.

(3): For P_3

Let $P_3 = \left\{\mathbf{x} \in \mathbb{R}^4 : 4x_1 + 3x_2 + 2x_3 + x_4 \leq 4, x_1 + x_2 + x_3 \leq 1, x_1 + x_4 \leq 1, 0 \leq \mathbf{x} \leq 1\right\}$

Clearly $X \subseteq \{0,1\}^4 \subseteq \mathbb{Z}^4$. In addition

(1) Each coefficient in the first inequality is ≤ 4 hence $\mathbf{x}_1, ..., \mathbf{x}_5$ satisfy the first inequality. Because $2 + 1 < 3 + 1 \leq 4$, then also $\mathbf{x}_6, \mathbf{x}_7$ satisfy the first inequality.

(2) Each coefficient in the second inequality is ≤ 1, hence $\mathbf{x}_1, ..., \mathbf{x}_5$ satisfy the second inequality. Because x_4 does not appear in inequality, then also $\mathbf{x}_6, \mathbf{x}_7$ satisfy the second inequality.

(3) There doesn't exists $\mathbf{x} \in X$ such that $x_1 > 0$ and $x_4 > 0$. Therefore X satisfies the third inequality.

(1),(2) and (3) further imply that $X \subseteq P_3 \cap \mathbb{Z}^4$.

Lets compare P_2 and P_3. Because P_3 is more restricted than P_2 (have extra constraints) we can immediately see that $P_3 \subseteq P_2$, i.e. $P_3 \cap \mathbb{Z}^4 \subseteq P_2 \cap \mathbb{Z}^4$. Moreover, from part (2) we also have $P_3 \cap \mathbb{Z}^4 \subseteq P_2 \cap \mathbb{Z}^4 \subseteq X$.

We have now proved that $X \subseteq P_3 \cap \mathbb{Z}^4$ and $P_3 \cap \mathbb{Z}^4 \subseteq X$, i.e. $X = P_3 \cap \mathbb{Z}^4$, which means that P_3 is formulation of X.

(b)

During part (a), we showed that $P_3 \subseteq P_2 \subseteq P_1$. This means that P_3 is the best formulation.

Problem 5.2: B&B formulation for knapsack

(a)

Consider following LP-relaxation of 0-1 Knapsack problem. Both problems are equivalent.

$$\begin{aligned} \text{max.} \quad & \sum_{i=1}^{n} c_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} a_i x_i \leq b_i \\ & 0 \leq x \leq 1 \end{aligned} \qquad\qquad \begin{aligned} \text{max.} \quad & c^T x \\ \text{s.t.} \quad & \begin{bmatrix} a^T \\ I \end{bmatrix} x \leq \begin{bmatrix} b \\ \mathbf{1} \end{bmatrix} \\ & x \geq 0 \end{aligned} \tag{1}$$

Lets assume that the optimal solution for (1) is $x \in \mathbb{R}^n$ that is formulated as follows

$$x_i = \begin{cases} 1 & \forall i < r \\ \frac{1}{a_r}(b - \sum_{i=1}^{r-1} a_i) & i = r \\ 0 & \forall i > r \end{cases} \tag{2}$$

The objective value z_P of (1) for x equals to

$$\begin{aligned} z_P &= \sum_{i=1}^{n} c_1 x_i \\ &= \sum_{i=1}^{r-1} c_i x_i + c_r x_r + \sum_{i=r+1}^{n} c_i x_i \\ &= \sum_{i=1}^{r-1} c_i + \frac{c_r}{a_r}(b - \sum_{i=1}^{r-1} a_i) \\ &= \sum_{i=1}^{r-1} c_i - \frac{c_r}{a_r} \sum_{i=1}^{r-1} a_i + \frac{c_r}{a_r} b \\ &= \sum_{i=1}^{r-1} (c_i - \frac{c_r}{a_r} a_i) + \frac{c_r}{a_r} b \end{aligned} \tag{3}$$

Using the second primal problem formulation in (1), we can form a following equivalent dual problems:

$$\begin{aligned} \text{min.} \quad & p^T \begin{bmatrix} b \\ \mathbf{1} \end{bmatrix} \\ \text{s.t.} \quad & \left[\, a \;\middle|\; I \,\right] p \geq c \\ & p \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \text{min.} \quad & p_1 b + \sum_{i=2}^{n+1} p_i \\ \text{s.t.} \quad & a_i p_1 + p_{i+1} \geq c_i, \forall i \in \{1, ..., n\} \\ & p \geq 0 \end{aligned} \tag{4}$$

Lets try to make a suggestion for the optimal solution p of the dual problem (4) by equating the corresponding optimal values z_D and z_P.

$$z_D = p_1 b + \sum_{i=2}^{n+1} p_i = \sum_{i=1}^{r-1}(c_i - \frac{c_r}{a_r}a_i) + \frac{c_r}{a_r}b = z_P \tag{5}$$

We can immediately spot that to above equation hold, we must set

$$p_1 = \frac{c_r}{a_r} \tag{6}$$

and

$$\sum_{i=2}^{n+1} p_i = \sum_{i=1}^{r-1}(c_i - \frac{c_r}{a_r}a_i) \tag{7}$$

Therefore, for example, we can suggest following formulation for p such that (6) and (7) both hold:

$$p_i = \begin{cases} \frac{c_r}{a_r} & i = 1 \\ c_{i-1} - \frac{c_r}{a_r}a_{i-1} & i \in \{2, ..., r\} \\ 0 & i \in \{r+1, ..., n+1\} \end{cases} \tag{8}$$

Next, lets verify that suggested p is indeed a feasible solution for the dual problem. Because $\frac{c_{i-1}}{a_{i-1}} \geq \frac{c_r}{a_r}, \forall i \in \{2, ..., r\}$ we have

$$\begin{aligned} 1 &\geq \frac{c_r}{a_r}\frac{a_{i-1}}{c_{i-1}} \\ c_{i-1} &\geq \frac{c_r}{a_r}a_{i-1} \\ c_{i-1} - \frac{c_r}{a_r}a_{i-1} &\geq 0, \qquad \forall i \in \{2, ..., r\} \end{aligned} \tag{9}$$

In addition, $p_1 \geq 0$ and $p_{r+1} = ... = p_{n+1} = 0$. Therefore p satisfies feasibility condition $p \geq 0$ in (4).
For the other feasibility condition in in (4), we consider following cases:

Case 1: $i \in \{1, ..., r-1\}$
Follows directly from the definition (8) of p.

$$\begin{aligned} c_i &= c_i \\ a_i\frac{c_r}{a_r} + c_i - a_i\frac{c_r}{a_r} &= c_i \\ a_i p_1 + p_{i+1} &= c_i, \qquad \forall i \in \{1, ..., r-1\} \end{aligned} \tag{10}$$

Case 2: $i \in \{r, ..., n\}$

Follows directly from the definition (8) of p and assumption $\frac{c_1}{a_1} \geq ... \geq \frac{c_n}{a_n}$.

$$\begin{aligned}
\frac{c_r}{a_r} &\geq \frac{c_i}{a_i} \\
a_i \frac{c_r}{a_r} &\geq c_i \\
a_i \frac{c_r}{a_r} + p_{i+1} &\geq c_i \\
a_i p_1 + p_{i+1} &\geq c_i, \qquad \forall i \in \{r, ..., n\}
\end{aligned} \tag{11}$$

Therefore, p satisfies feasibility condition $a_i p_1 + p_{i+1} \geq c_i, \forall i \in \{1, ..., n\}$, meaning that proposed definition (8) for p is dual feasible.

In addition, from equations (5) - (8) we have that primal and dual optimal values match

$$c^T x = p^T \begin{bmatrix} b \\ \mathbf{1} \end{bmatrix} \tag{12}$$

Finally, the strong duality implies that solutions x and p are optimum for LP-relaxed 0-1 Knapsack problem (1) and dual problem (4), respectively. $\square$

(b)

The starting point is to set the root node of the BB-tree that equals to the LP-relaxation of the original problem, i.e.

$$\text{max.} \ \{17x_1 + 10x_2 + 25x_3 + 17x_4 : 5x_1 + 3x_2 + 8x_3 + 7x_4 \leq 12, 0 \leq x \leq 1\} \tag{13}$$

We note that $5 + 3 + 8 = 16 > 12 \Rightarrow r = 3$, therefore the optimal solution for (13) is $\bar{x} = (1, 1, \frac{1}{8}(12 - 5 - 3), 0, 0)^T = (1, 1, 0.5, 0, 0)^T$. The objective value is then $z = 39.5$. Because x_3 is fractional, we split the tree to two branches, where $x_3 = 0$ and $x_3 = 1$, respectively. This would yield a following tree:

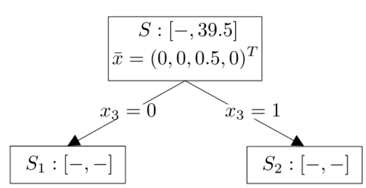

Figure 1: BB-tree after solving (13)

We proceed to solve S_2, i.e.

$$\text{max.} \ \{17x_1 + 10x_2 + 25 + 17x_4 : 5x_1 + 3x_2 + 7x_4 \leq 4, 0 \leq x \leq 1\} \tag{14}$$

We note that $5 > 4 \Rightarrow r = 1$, therefore the optimal solution for (14) is $\bar{x} = (\frac{1}{5}(4-0), 0, 1, 0)^T = (0.8, 0, 1, 0)^T$. The objective value is then $z = 38.6$. Because

x_1 is fractional, we split the node S_2 to two branches, where $x_1 = 0$ and $x_1 = 1$, respectively. This would yield a following tree:

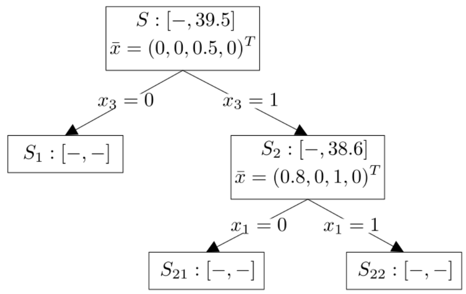

Figure 2: BB-tree after solving (14)

We proceed to solve S_{22}, i.e.

$$\text{max. } \{17 + 10x_2 + 25 + 17x_4 : 3x_2 + 7x_4 \leq -1, 0 \leq x \leq 1\} \tag{15}$$

It can be immediately seen that constraint makes (15) infeasible. Therefore this branch can be pruned. That said, we proceed to solve S_{21}, i.e.

$$\text{max. } \{10x_2 + 25 + 17x_4 : 3x_2 + 7x_4 \leq 4, 0 \leq x \leq 1\} \tag{16}$$

We note that $3 + 7 > 4 \Rightarrow r = 2$, therefore the optimal solution for (16) is $\bar{x} = (0, 1, 1, \frac{1}{7}(4 - 3))^T = (0, 1, 1, \frac{1}{7})^T$. The objective value is then $z = \frac{262}{7} = 37.42$. Because x_4 is factorial, we split the node S_{21} to two branches, where $x_4 = 0$ and $x_4 = 1$, respectively. This would yield a following tree:

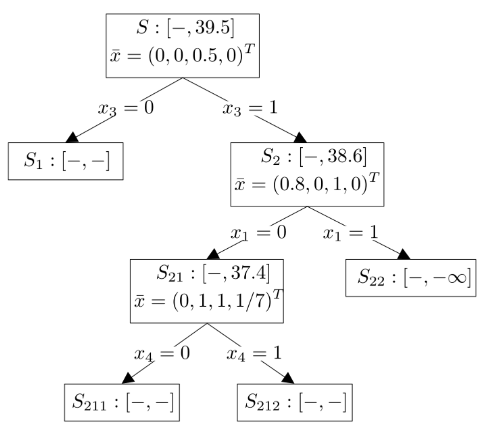

Figure 3: BB-tree after solving (15) and (16)

We proceed to solve S_{211}, i.e.

$$\max. \ \{10x_2 + 25 : 3x_2 \leq 4, 0 \leq x \leq 1\} \tag{17}$$

We note that $3 < 4 \Rightarrow r > 1$, therefore the optimal solution for (17) is $\bar{x} = (0, 1, 1, 0)^T$. The objective value is then $z = 35$. Because $\bar{x}$ is feasible solution for original 0-1 KP, the objective value z becomes the global primal bound. After solving S_{211}, we proceed to solve S_{212}, i.e.

$$\max. \ \{10x_2 + 25 + 17 : 3x_2 \leq -3, 0 \leq x \leq 1\} \tag{18}$$

From the constraint of (18)) we can immediately see that the problem is infeasible. Therefore this branch can be pruned. After these two steps, the BB-tree looks like following:

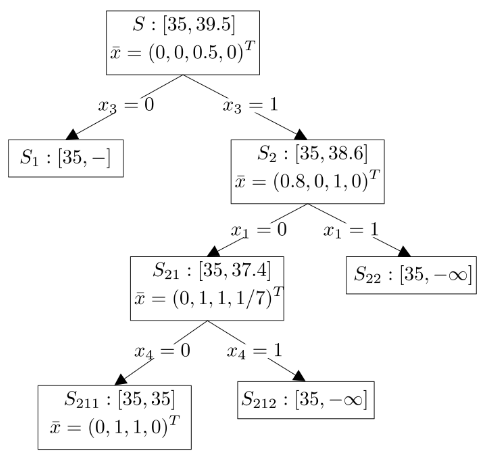

Figure 4: BB-tree after solving (17) and (18)

Now the whole left branch of the tree is processed so we proceed to solve S_1, i.e.

$$\max. \ \{17x_1 + 10x_2 + 17x_4 : 5x_1 + 3x_2 + 7x_4 \leq 12, 0 \leq x \leq 1\} \tag{19}$$

We note that $5 + 3 + 7 = 15 > 12 \Rightarrow r > 3$, therefore the optimal solution for (19) is $\bar{x} = (1, 1, 0, \frac{1}{7}(12 - 5 - 3))^T = (1, 1, 0, \frac{4}{7})^T$. The objective value is then $z = \frac{257}{7} = 36.7$. Because x_4 is factorial, we split the node S_1 to two branches, where $x_4 = 0$ and $x_4 = 1$, respectively. This would yield a following tree:

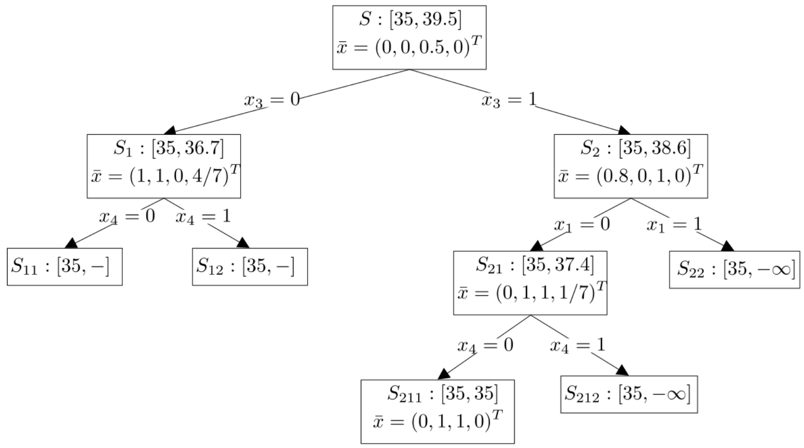

Figure 5: BB-tree after solving (19)

We proceed to solve S_{11}, i.e.

$$\max.\ \{17x_1 + 10x_2 : 5x_1 + 3x_2 \leq 12, 0 \leq x \leq 1\} \tag{20}$$

We note that $5 + 3 < 12 \Rightarrow r > 2$, therefore the optimal solution for (20) is $\bar{x} = (1, 1, 0, 0)^T$. The objective value is then $z = 27$ that is smaller than previously found 35, hence this branch can be pruned by bound. Finally, we proceed to solve S_{12}, i.e.

$$\max.\ \{17x_1 + 10x_2 + 17 : 5x_1 + 3x_2 \leq 5, 0 \leq x \leq 1\} \tag{21}$$

We note that $5 + 3 > 5 \Rightarrow r = 2$, therefore the optimal solution for (21) is $\bar{x} = (1, \frac{1}{3}(5 - 5), 0, 1)^T = (1, 0, 0, 1)^T$. The objective value is then $z = 34$ that is smaller than previously found 35, hence this branch can be pruned by bound. This yields a following tree:

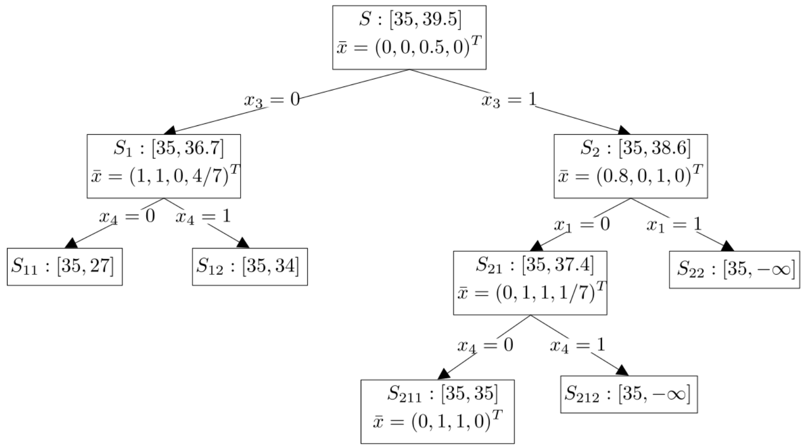

Figure 6: BB-tree after solving (20) and (21)

We can now see from Figure 6 that every branch is pruned, and therefore the optimal solution for the original 0-1 Knapsack problem is $\bar{x} = (0, 1, 1, 0)^T$.

Problem 5.3: GAP formulation

Implementation and results in *hw5_problem53.ipynb*.