You are requested to provide a report in pdf format answering problems 6.1 - 6.3 and a notebook for problems 6.2 and 6.3

Please submit your answers to this page before the given deadline. Late submissions will be penalized by 5 points per day after the deadline.

Problem 6.1: Knapsack covers

Consider the following Knapsack set S:

$$S = \{x \in \{0,1\}^5 : 79x_1 + 5x_2 + 53x_3 + 45x_4 + 45x_5 \leq 143\}.$$

(a) List all the minimal covers for S.

(b) Write the separation problem for finding a violated cover inequality for the point $\overline{x} = (\frac{1}{5}, \frac{2}{5}, 1, 1, \frac{3}{5})$. Can you find a violated cover inequality?

(c) Derive an extended cover inequality that cuts off $\overline{x}$ starting from the covers of point (a).

Problem 6.2: Improving TSP formulation with valid inequalities

Recall the following MTZ formulation for the Travelling Salesman Problem (TSP) presented in Exercise 8.2. Letting $N = \{1, \dots, n\}$ be the set of cities, $A = \{(i,j) : i,j \in N, i \neq j\}$ the arcs between the cities, and c_{ij} the distance between cities $i \in N$ and $j \in N$, the problem MTZ can be formulated as

$$\begin{aligned}
\text{(MTZ)} \quad \min_{x,u} \quad & \sum_{i \in N} \sum_{j \in N} c_{ij} x_{ij} & & \\
\text{s.t.:} \quad & \sum_{j \in N \setminus \{i\}} x_{ij} = 1, & \forall i \in N, & \\
& \sum_{j \in N \setminus \{i\}} x_{ji} = 1, & \forall i \in N, & \\
& u_i - u_j + (n-1)x_{ij} \leq n-2, & \forall i,j \in N \setminus \{1\} : i \neq j, & \quad (1)\\
& x_{ij} \in \{0,1\}, & \forall i,j \in N : i \neq j. &
\end{aligned}$$

where $x_{ij} = 1$ if city $j \in N$ is visited immediately after city $i \in N$, and $x_{ij} = 0$ otherwise. Constraints (1) with the variables $u_i \in \mathbb{R}$ for all $i \in N \setminus \{1\}$ are the *Miller-Tucker-Zemlin* subtour elimination constraints.

In this exercise, we will add valid inequalities to the formulation MTZ *a priori* (i.e., before calling the solver), and see if we can solve a "harder" 16-city instance faster with the help of these inequalities.

(a) Show that the inequalities (2) - (5) are valid for the problem MTZ, assuming that $n > 2$:

$$\begin{aligned}
& x_{ij} + x_{ji} \leq 1, & \forall i,j \in N : i \neq j & \quad (2)\\
& u_i - u_j + (n-1)x_{ij} + (n-3)x_{ji} \leq n-2, & \forall i,j \in N \setminus \{1\} : i \neq j & \quad (3)\\
& u_j - 1 + (n-2)x_{1j} \leq n-1 & \forall j \in N \setminus \{1\} & \quad (4)\\
& 1 - u_i + (n-1)x_{i1} \leq 0 & \forall i \in N \setminus \{1\} & \quad (5)
\end{aligned}$$

Hint: You can assume that the variables $u_2, \dots, u_n$ take unique integer values from the set $\{2, \dots, n\}$. That is, we have $u_i \in \{2, \dots, n\}$ for all $i = 2, \dots, n$ with $u_i \neq u_j$ for all $i, j \in 2, \dots, n$. This holds for any TSP solution of problem MTZ as we showed in Exercise 8.2. If we fix city 1 as the starting city, then the value of each u_i represents the position of city i in the TSP tour, i.e., $u_i = t$ for $t = 2, \dots, n$ if city $i \neq 1$ is the t:th city in the tour. You have to check that each of the inequalities

(2) - (5) hold (individually) for any arc $(i, j) \in A$ and city $i \in N$ that are part of the inequality, by checking that the following two cases are satisfied: either $x_{ij} = 0$ or $x_{ij} = 1$.

(b) Add all four sets of inequalities (2) - (5) to the MTZ formulation and check if solving the "harder" 16-city instance provided here takes less time than with no extra inequalities. The skeleton notebook contains only the TSP MTZ formulation without the valid inequalities (2) - (5). Fill in the skeleton notebook with the valid inequalities and return your final Julia code in notebook format (.ipynb).

Problem 6.3: Vehicle routing problem with time windows

Consider the Exercise 11.2 with the vehicle routing problem (VRP) statement and formulation. Now, consider an extension for this problem where the clients have specific time windows in which they can accept the deliveries. Considering all the former constraints stated in Exercise 11.2 hold, reformulate the problem to consider the specific delivery time windows from each customer using the additional parameters:

- T_j^{LB}: Lower bound, minimal time, for the delivery in client j
- T_j^{UB}: Upper bound, maximal time, for the delivery in client j
- T_{ij}: Transportation time from node i to node j
- S_i: Service time for node i

Using the notebook skeleton found here, and the previous formulation presented for Exercise 11.2,

(a) Formulate the linear programming considering the changes needed to conform the deliveries to specific time windows $[T_j^{LB}, T_j^{UB}]$. *HINT*: you can use the formulation presented in Exercise 11.2 and complement it with the time windows extensions, considering all new variables, constraints, and/or costs needed.

(b) Use the notebook skeleton to find a solution for the capacitated vehicle routing problem with time windows using the given instance

(c) Consider the four different solver configurations provided in the notebook skeleton and see how they perform, also comparing with the default configuration. In your report, present your insights about these results and the significance of the different parts of a MIP solver in this particular problem.