# MS-E2121 Homework 4

Tuomas Porkamaa

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## Problem 4.1: Implementing the full-scale problem

### (a)

I obtained following optimal values for server locations: $\bar{x} = (0, 0, 0, 1, 1)^T$, i.e. servers are located at locations $j = 4$ and $j = 5$.

### (b)

The total cost for locating the servers as $\bar{x}$ indicates plus expected costs of server demands and shortage capacities equals to $-1327.37$.

## Problem 4.2: Single-cut Benders decomposition

### (a)

Main problem formulation:

$$\begin{aligned} \text{min.} \quad & \sum_{j \in J} C_j x_j + \theta \\ \text{s.t.} \quad & \sum_{j \in J} x_j \leq V \\ & x_j \in \{0, 1\}, \forall j \in J \end{aligned} \tag{1}$$

## (b)

Primal subproblem formulation:

$$\begin{aligned}
\text{min.} \ & \sum_{s\in S} P_s \sum_{j\in J} F z_{js} - \sum_{s\in S} P_s \sum_{i\in I, j\in J} Q_{ij} D_{is} y_{ijs} \\
\text{s.t.} \ & \sum_{i\in I} D_{is} y_{ijs} - z_{js} \le U x_j, \quad \forall j \in J, s \in S \\
& \sum_{j\in J} y_{ijs} = H_{is}, \quad \forall j \in J, s \in S \\
& y_{ijs} \ge 0 \quad \forall i \in I, j \in J, s \in S \\
& z_{js} \ge 0 \quad \forall j \in J, s \in S
\end{aligned} \tag{2}$$

## (c)

Dual formulation of the subproblem:

$$\begin{aligned}
\text{max.} \ & \sum_{s\in S} \sum_{i\in I} H_{is} v_{is} + \sum_{s\in S} \sum_{j\in J} U x_j u_{js} \\
\text{s.t.} \ & D_{is} u_{js} + v_{is} \le -P_s DisQij, \quad \forall i \in I, j \in J, s \in S \\
& - u_{js} \le P_s F, \quad \forall j \in J, s \in S \\
& v_{is} \in \mathbb{R}, \quad \forall i \in I, s \in S \\
& u_{js} \le 0, \quad \forall j \in, s \in S
\end{aligned} \tag{3}$$

## (d)

Optimality cut formulation to be added to the constraints of main problem after each iteration:

$$\sum_{s\in S} \sum_{i\in I} H_{is} \bar{v}_{is}^l + \sum_{s\in S} \sum_{j\in J} U x_j \bar{u}_{js}^l \le \theta \tag{4}$$

where $\bar{v}_{is}^l$ and $\bar{u}_{js}^l$ are solutions from (3) at iteration $l$.

For the single-cut Benders decomposition, we doesn't have to consider the feasibility cuts because the subproblem (2) always contains at least one solution that is feasible on each iteration $l$. To show this, let $\bar{x} \in \{0,1\}^J$ and define $y_{ijs}$ as follows:

$$y_{ijs} = \begin{cases} \frac{1}{J} & \text{if } i \text{ is active} \\ 0 & \text{otherwise} \end{cases}$$

Clearly $y_{ijs} \ge 0$.
Now, when client $i$ is active, we have

$$\sum_j y_{ijs} = \sum_j \frac{1}{J} = 1 = H_{is}$$

Alternatively, when client $i$ is inactive, we have

$$\sum_j y_{ijs} = \sum_j 0 = 0 = H_{is}$$

Therefore, this $y$ satisfies the second constraint of (2).
Lets next define $z_{js}$ as follows:

$$z_{js} = \frac{1}{J} \sum_i D_{is}$$

Clearly $z_{js} \geq 0$, and for the first constraint of (2) we obtain:

$$\begin{aligned}
\sum_i D_{is} y_{ijs} - z_{js} &\leq \frac{1}{J} \sum_i D_{is} - z_{js} \\
&= \frac{1}{J} \sum_i D_{is} - \frac{1}{J} \sum_i D_{is} \\
&= 0 \\
&\leq U \bar{x}_j
\end{aligned}$$

Hence, there always exists at least one pair of $(y, z)$ such that the primal subproblem (2) has a feasible solution, i.e. subproblem can't be infeasible in this problem formulation.

# Problem 4.3: Multi-cut Benders decomposition

## (a)

Formulation of the modified main problem:

$$\begin{aligned}
\text{min.} \quad & \sum_{j \in J} C_j x_j + \sum_{s \in S} \theta_s \\
\text{s.t.} \quad & \sum_{j \in J} x_j \leq V \\
& x_j \in \{0, 1\}, \quad \forall j \in J
\end{aligned} \tag{5}$$

## (b)

Optimality cut formulation to be added to the constraints of modified main problem (5) after each iteration:

$$\sum_{i \in I} H_{is} \bar{v}_{is}^l + \sum_{j \in J} U x_j \bar{u}_{js}^l \leq \theta_s, \quad \forall s \in S \tag{6}$$

where $\bar{v}_{is}^l$ and $\bar{u}_{js}^l$ are solutions from (3) at iteration $l$.

### (c)

Table 1:

| Problem formulation | obj. value | #iterations | time | avg. iteration time |
|---|---|---|---|---|
| Full model | -1327.3 | 43715 | 22.75s | 0.0005s |
| Single-cut | -1327.3 | 20 | 2.82s | 0.141s |
| Multi-cut | -1327.37) | 11 | 5.42s | 0.5s |

First of all, all implemented methods converged to the same optimal value. In terms of iterations until convergence, the multi-cut Benders decomposition performed best before single-cut and full model, respectively. This behaviour is expectable since adding more cuts will make each iteration more complex in the sense that solving the problem goes further on each iteration. For the total time taken until convergence, the single-cut method is fastest before multi-cut and full model, respectively even though actual multiprocessing weren't used. Per iteration times are in favor of full-scale model since there is no additional complexity present unlike in single-cut and multi-cut methods. The extra complexity of multi-cut method shows in these per-iteration times, and hence mirrors it to the total time taken as well.