Homework 2

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Problem 2.1

(a)

Let $A = \begin{pmatrix} | & & | \\ A_1 & ... & A_n \\ | & & | \end{pmatrix} \in \mathbb{R}^{m \times n}$, and C be the cone generated by A i.e.

$$
\begin{aligned}
C &= \left\{ \sum_{i=1}^{n} \lambda_i A_i : \text{¸}\lambda_1, ..., \lambda_n \geq 0 \right\} \\
&= \{ b \in \mathbb{R}^m : A\lambda = b\text{¸}\lambda \geq 0 \}
\end{aligned}
$$

Let $b \in C$. From λ's perspective, we can consider the polyhedron

$$
\Lambda = \{ \lambda \in \mathbb{R}^n : A\lambda = b\text{¸}\lambda \geq 0 \}
$$

Because Λ is a standard form polyhedral set it does not contain a line, which implies that $\exists \lambda^* \in \Lambda$ that is an extreme point. In optimization setting, extreme point equals to basic feasible solution (BFS), and hence we can utilize properties of BFS to show the following.

Because $A \in \mathbb{R}^{m \times n}$, it has $k \leq m$ linearly independent rows. From the properties of BFS, we know that there exist an index set I such that $|I| = k$ and

$$
\forall j \in \{1, ..., n\} \setminus I : \lambda_j^* = 0
$$

Because $|I| \leq m$, then $|\{1, ..., n\} \setminus I| \geq n - m$, i.e. λ^* has at least $n - m$ zero elements. This implies that at most m of the elements of λ^* are nonzero.

To summarize: $\forall b \in C$ we found $\lambda^* \in \Lambda$, i.e. $A\lambda^* = b, \lambda^* \geq 0$, such that at most m of the elements of λ^* are zero.

(b)

Let $A = \begin{pmatrix} | & & | \\ A_1 & ... & A_n \\ | & & | \end{pmatrix} \in \mathbb{R}^{m \times n}$, and P be the convex hull of the vectors A_i

$$\begin{aligned} P &= \left\{ \sum_{i=1}^{n} \lambda_i A_i : \sum_{i=1}^{n} \lambda_i = 1, \lambda_1, ..., \lambda_n \geq 0 \right\} \\ &= \left\{ b \in \mathbb{R}^m : A\lambda = b, 1^T \lambda = 1, \lambda \geq 0 \right\} \\ &= \left\{ b \in \mathbb{R}^m : \begin{pmatrix} A \\ 1^T \end{pmatrix} \lambda = \begin{pmatrix} b \\ 1 \end{pmatrix}, \lambda \geq 0 \right\} \end{aligned}$$

More preciously, if $b \in P$ then $\exists \lambda \geq 0$ such that

$$\begin{pmatrix} | & & | \\ A_1 & ... & A_n \\ | & & | \\ 1 & ... & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \\ 1 \end{pmatrix} \tag{1}$$

The proof is analogous with the one given in (a). The λ's satisfying (1) form a standard form polyhedral set, so extreme point λ^* exists. Because there are now an extra row in the constraint matrix, replacing m by $m+1$ in (a) completes the proof.

Problem 2.2

Assumption:
$f : \mathbb{R}^n \leftarrow \mathbb{R}$ is a convex function.
$S \subset \mathbb{R}^n$ is a convex set.
$\exists x^* \in S, \exists \epsilon > 0, \forall x \in B_\epsilon(x^*) : f(x^*) \leq f(x)$, i.e., x^* is local optimum.

Claim:
$\forall x \in S : f(x^*) \leq f(x)$, i.e., x^* is global optimum.

Proof:
Proceed by proof by contradiction. Suggest that $\exists \bar{x} \in S$ such that $f(\bar{x}) < f(x^*)$.
Define $\epsilon > 0$ such that x^* is local optimum in its closed neighbourhood $\bar{B}_\epsilon(x^*)$.
Define $\lambda = \min \left\{ 1, \frac{\epsilon}{|\bar{x} - x^*|} \right\}$, i.e., $\lambda \in [0, 1]$.

Due to convexity of S, $\lambda\bar{x} - (1-\lambda)x^* \in S$. In addition

$$\begin{aligned}
|\lambda\bar{x} + (1-\lambda)x^* - x^*| &= |\lambda\bar{x} + 1x^* - \lambda x^* - x^*| \\
&= |\lambda\bar{x} - \lambda x^*| \\
&= \lambda|\bar{x} - x^*| \\
&\leq \frac{\epsilon}{|\bar{x} - x^*|}|\bar{x} - x^*| \\
&= \epsilon
\end{aligned}$$

and hence $\lambda\bar{x} - (1-\lambda)x^* \in \bar{B}_\epsilon(x^*)$.

Finally, due to convexity of f, it holds that

$$f(\lambda\bar{x} - (1-\lambda)x^*) \leq \lambda f(\bar{x}) - (1-\lambda)f(x^*) < \lambda f(x^*) - (1-\lambda)f(x^*) = f(x^*)$$

which contradicts the original assumption that x^* is local optimum. Therefore the original claim is valid.

Problem 2.3

Consider problem min. $\mathbf{c}^T\mathbf{x}$ s.t. $\mathbf{x} \in P$

(a)

Direction $\Rightarrow$:
Let $\mathbf{y} \in P$ and $\mathbf{x} \in P$ be an optimal feasible solution, i.e. $\mathbf{c}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} \geq 0$. Let $\mathbf{d}$ be a feasible direction at $\mathbf{x}$, i.e. $\mathbf{d} = \mathbf{y} - \mathbf{x}$. Then
$\mathbf{c}^T\mathbf{d} = \mathbf{c}^T(\mathbf{y} - \mathbf{x}) = \mathbf{c}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} \geq 0$.

Direction $\Leftarrow$:
Let $\mathbf{y}, \mathbf{x} \in P$ and $\mathbf{d} = \mathbf{y} - \mathbf{x} \geq 0$ be a feasible direction. Then

$$\begin{aligned}
\mathbf{c}^T\mathbf{d} &\geq 0 \\
\mathbf{c}^T(\mathbf{y} - \mathbf{x}) &\geq 0 \\
\mathbf{c}^T\mathbf{y} - \mathbf{c}^T\mathbf{y} &\geq 0 \\
\mathbf{c}^T\mathbf{y} &\geq \mathbf{c}^T\mathbf{y}
\end{aligned}$$

(b)

The proof below is analogous to the one presented above, but due to uniqueness of the optimal solution, the inequalities $\geq$ are replaces by strict inequalities $>$.
Direction $\Rightarrow$:
Let $\mathbf{y} \in P$ and $\mathbf{x} \in P$ be an unique optimal feasible solution, i.e. $\mathbf{c}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} > 0$. Let $\mathbf{d}$ be a feasible direction at $\mathbf{x}$, i.e. $\mathbf{d} = \mathbf{y} - \mathbf{x}$. Then
$\mathbf{c}^T\mathbf{d} = \mathbf{c}^T(\mathbf{y} - \mathbf{x}) = \mathbf{c}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} > 0$.

Direction $\Leftarrow$:
Let $\mathbf{y}, \mathbf{x} \in P$ and $\mathbf{d} = \mathbf{y} - \mathbf{x} \geq 0$ be a feasible direction. Then

$$\begin{aligned} \mathbf{c}^T \mathbf{d} &> 0 \\ \mathbf{c}^T (\mathbf{y} - \mathbf{x}) &> 0 \\ \mathbf{c}^T \mathbf{y} - \mathbf{c}^T \mathbf{y} &> 0 \\ \mathbf{c}^T \mathbf{y} &> \mathbf{c}^T \mathbf{y} \end{aligned}$$

Problem 2.4

Let x be a BFS with basis B.

(a)

Assumption: Reduced cost of every nonbasic variable is positive.
Claim: x is unique optimal solution.
Proof:
Let x_j, $j \in I_N$ be nonbasic variable with positive reduced cost

$$\bar{c}_j = c_j - c^T B^{-1} A_j > 0$$

Let $d = y - x$ be a feasible direction at x, where $y \in P$.
The objective function change in direction d equals to

$$c^T d = \sum_{j \in I_N} \bar{c}_j d_j = \bar{c}_j > 0$$

because of the definition of feasible direction components associated with non-basic variables. Hence

$$c^T y - c^T x = c^T (y - x) = c^T d > 0$$

i.e. $c^T y > c^T x$, so x is unique optimal solution.

(b)

Assumption: x is nondegenerate unique optimal solution.
Claim: Reduced cost of every nonbasic variable is positive.
Proof:
Proceed by proof by contradiction. Assume that $\exists \bar{c}_j \leq 0, j \in I_N$. This implies that

$$\begin{aligned} \bar{c}_j = c^T d &\leq 0 \\ c^T (y - x) &\leq 0 \\ c^T y - c^T x &\leq 0 \\ c^T y &\leq c^T x \end{aligned}$$

i.e. x is not an unique optimal solution. This contradicts the original assumption and therefore the original claim is valid.