EE2211 Tutorial 4

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Q1

Given $Xw = y$ where $X = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)

(b) Is X invertible? Why?

(c) Solve for w if it is solvable.

Q1(a) What kind of system is this?

The number of equations and unknowns determines the type of system:

- Even-determined: Number of equations = number of unknowns.
- Over-determined: More equations than unknowns.
- Under-determined: Fewer equations than unknowns.

Here, we need to compare the number of equations (rows in X) and the number of unknowns (columns in X).

- X has 2 rows and 2 columns.
- Since the number of equations equals the number of unknowns, this is an even-determined system (or a square system).

Q1(b) Is X invertible? Why?

To determine if X is invertible, we need to check if its determinant is non-zero. The determinant of a 2×2 matrix $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by:

$$\det(X) = ad - bc$$

For $X = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, we calculate:

$$\det(X) = (1 \times 4) - (1 \times 3) = 4 - 3 = 1$$

Since the determinant is non-zero, X is invertible.

Q1(c) Solve for w if it is solvable.

Since X is invertible, we can solve the system $Xw = y$ by multiplying both sides by X^{-1}:

$$w = X^{-1}y$$

To find X^{-1}, the inverse of a 2×2 matrix $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by:

$$X^{-1} = \frac{1}{\det(X)} \begin{bmatrix} d & -b \\ -c & c \end{bmatrix}$$

For $X = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, we already found that $\det(X) = 1$. Thus:

$$X^{-1} = \frac{1}{1} \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$$

Now, multiply X^{-1} by y:

$$w = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Q1

```
[ ]: import numpy as np
     m_list = [[1, 1], [3, 4]]
```

This line converts m_list, which is expected to be a list (or a list of lists for a matrix), into a NumPy array. NumPy arrays are more efficient than Python lists for numerical operations.

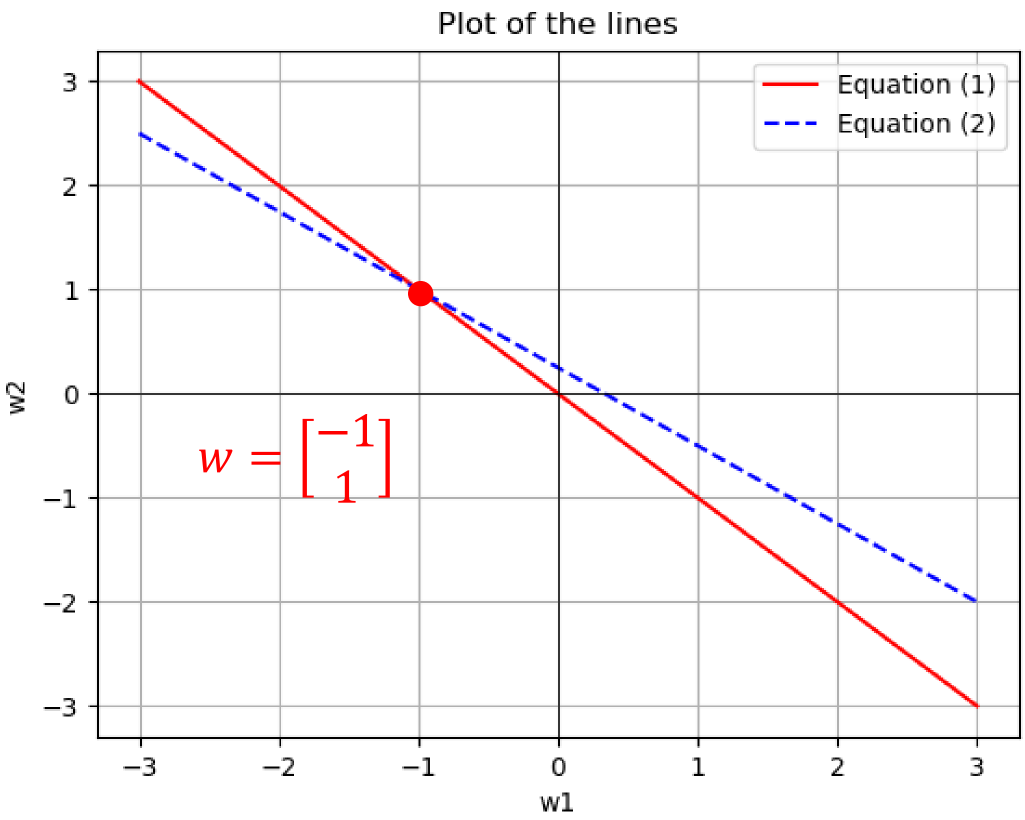
```
[4]: X = np.array(m_list)
     inv_X = np.linalg.inv(X)
```

np.linalg.inv(X) computes the inverse of the matrix X using NumPy's linear algebra module (np.linalg).

```
[6]: y = np.array([0, 1])
     w = inv_X.dot(y)
     print(w)

[-1.  1.]
```

Q1



Plot of the lines
Equation (1)
Equation (2)
$w = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
w1
w2
−3
−2
−1
0
1
2
3

Q2

Given $Xw = y$ where $X = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)

(b) Is X invertible? Why?

(c) Solve for w if it is solvable.

Q2(a) What kind of system is this? (even-, over-, or under-determined?)

The number of equations and unknowns determines the type of system:

- Even-determined: Number of equations = number of unknowns.
- Over-determined: More equations than unknowns.
- Under-determined: Fewer equations than unknowns.

Here, X is a 2×2 matrix, meaning there are 2 equations and 2 unknowns in the system. This is an even-determined system.

Q2(b) Is X invertible? Why?

To check if X is invertible, we need to find its determinant. For a 2x2 matrix, the determinant is given by:

$$\det(X) = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\right) = 1 \times 6 - 2 \times 3 = 6 - 6 = 0$$

Since the determinant of X is 0, the matrix is not invertible. A matrix is invertible only if its determinant is non-zero.

Q2(c) Solve for w if it is solvable.

The system is $Xw = y$, or:

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We can write the system as two linear equations:

$$w_1 + 2w_2 = 0$$

$$3w_1 + 6w_2 = 1$$

Now let's analyze the system:

- The first equation is $w_1 + 2w_2 = 0$, so $w_1 = -2w_2$.
- Substituting $w_1 = -2w_2$ into the second equation:

$$3(-2w_2) + 6w_2 = 1 \implies -6w_2 + 6w_2 = 1 \implies 0 = 1$$

This is a contradiction, meaning the system is inconsistent and has no solution. Therefore, it is not solvable.

Q2(c) Solve for *w* if it is solvable.

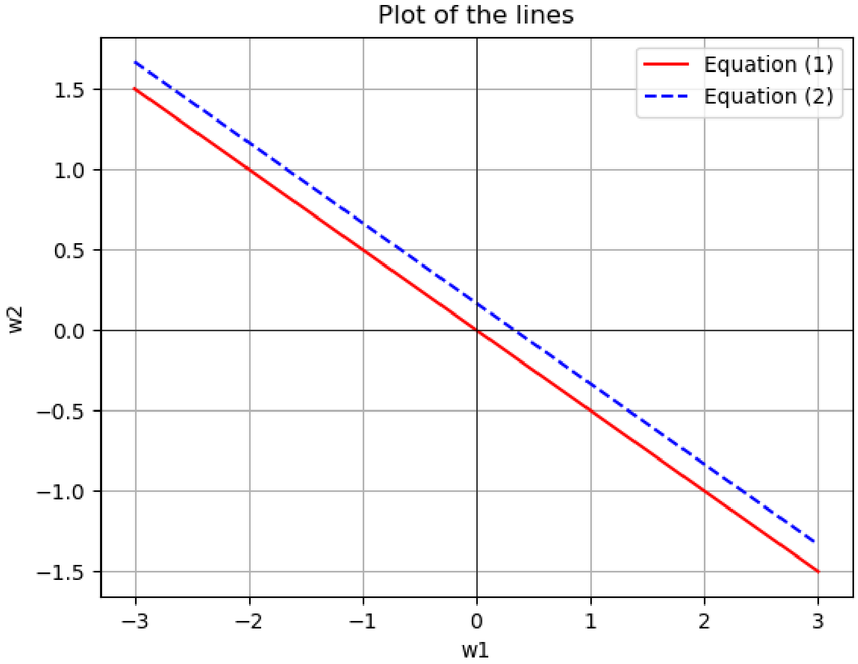
```
[1]: import numpy as np
     import matplotlib.pyplot as plt

     # Define the range for w1
     w1 = np.linspace(-3, 3, 400)

     # Define the equations of the lines
     w2_1 = -w1 / 2  # From w1 + 2 * w2 = 0, w2 = -w1 /2
     w2_2 = (1 - 3*w1) / 6  # From 3*w1 + 6 * w2 = 1, w2 = (1 - 3*w1) / 6

     # Plot the lines
     plt.plot(w1, w2_1, 'r-', label='Equation (1)')
     plt.plot(w1, w2_2, 'b--', label='Equation (2)')
```

Q2(c) Solve for w if it is solvable.



There is no solution for w since the rows/columns of X are inter-dependent. The two lines shown in the plot are parallel and has no intersection.

Q3

Given $Xw = y$ where $X = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$, $y = \begin{bmatrix} 0 \\ 0.1 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)

(b) Is X invertible? Why?

(c) Solve for w if it is solvable.

Q3(a) What kind of system is this? (even-, over-, or under-determined?)

The number of equations and unknowns determines the type of system:

- Even-determined: Number of equations = number of unknowns.
- Over-determined: More equations than unknowns.
- Under-determined: Fewer equations than unknowns.

Here, we need to compare the number of equations (rows in X) and the number of unknowns (columns in X).

- X is a 3×2 matrix (3 rows, 2 columns).
- There are 3 equations and 2 unknowns.

Since there are more equations than unknowns, the system is over-determined.

Q3 (b): Is X invertible? Why?

- For a matrix to be invertible, it must be a square matrix (i.e., the number of rows must equal the number of columns).
- Since X is a 3×2 matrix (not square), it is not invertible.
- But $X^T X = \begin{bmatrix} 6 & 9 \\ 9 & 21 \end{bmatrix}$ is invertible.

Q3 (c) Solve for w if it is solvable.

We want to solve:

$$Xw = y$$

where

$$X = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}, \qquad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \qquad y = \begin{bmatrix} 0 \\ 0.1 \\ 1 \end{bmatrix}$$

Since this system is over-determined, it may not have an exact solution. The most common way to solve an over-determined system is to find a least squares solution. The least squares solution minimizes the error $\parallel Xw - y \parallel$, and it is given by:

$$w = (X^T X)^{-1} X^T y$$

Let's calculate it step by step

Step 1: Compute $X^T X$:

$$X^T X = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 9 & 21 \end{bmatrix}$$

Step 2: Compute $\left(X^T X\right)^{-1}$

$$(X^T X)^{-1} = \begin{bmatrix} 6 & 9 \\ 9 & 21 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4667 & -0.2 \\ -0.2 & 0.1333 \end{bmatrix}$$

Step 3: Compute w

$$w = (X^T X)^{-1} X^T y = \begin{bmatrix} 0.4667 & -0.2 \\ -0.2 & 0.1333 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.68 \\ -0.32 \end{bmatrix}$$

Q3 (c) Solve for *w* if it is solvable.

[2]:
```
import numpy as np
```

This imports the NumPy library, which is used for numerical computations, especially with arrays and matrices.

[4]:
```
from numpy.linalg import inv
```

The NumPy linear algebra functions rely on BLAS and LAPACK to provide efficient low level implementations of standard linear algebra algorithms

This imports the inv function from the numpy.linalg module, which is used to compute the inverse of a matrix.

[6]:
```
X = np.array([[1, 2], [2, 4], [1, -1]])
```

This creates a 3x2 matrix XX:

[8]:
```
y = np.array([0, 0.1, 1])
```

This creates a vector yy representing the target values:

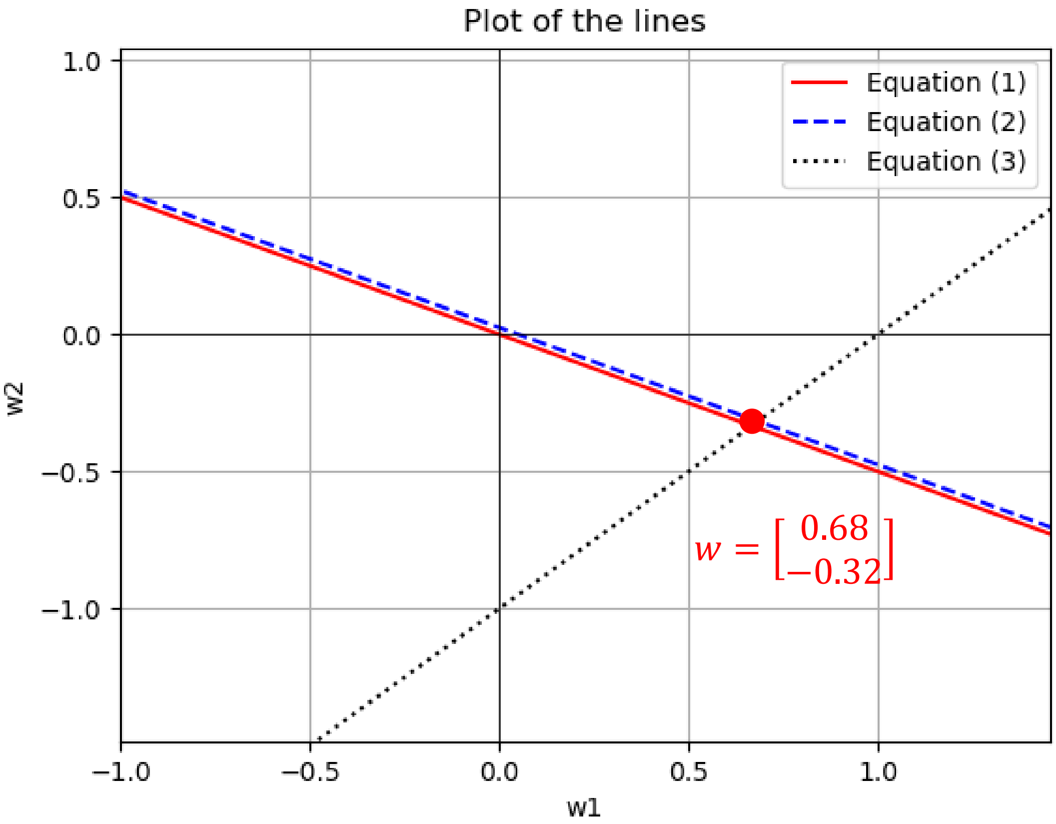
[22]:
```
w = inv(X.T @ X) @ X.T @ y
```

This line computes the least squares solution for ww, using the following formula:

[18]:
```
print(w)
```

```
[ 0.68 -0.32]
```

Q3 (c) Solve for w if it is solvable.



$$w = \begin{bmatrix} 0.68 \\ -0.32 \end{bmatrix}$$

Question 4

Given $Xw = y$ where $X = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)

(b) Is X invertible? Why?

(c) Solve for w if it is solvable.

Q4 (a) What kind of system is this? (even-, over- or under-determined?)

The number of equations and unknowns determines the type of system:

- Even-determined: Number of equations = number of unknowns.
- Over-determined: More equations than unknowns.
- Under-determined: Fewer equations than unknowns.

To determine the type of system, we compare the number of equations (rows in X) and the number of unknowns (columns in X).

- X is a 3×4 matrix (3 rows, 4 columns).
- There are 3 equations and 4 unknowns.

In this case, there are 3 equations and 4 unknowns. So, the system is under-determined.

Q4 (b) Is X invertible? Why?

- For a matrix to be invertible, it must be a square matrix (i.e., the number of rows must equal the number of columns).
- Since X is a 3×4 matrix (not square), it is not invertible.
- But XX^T is invertible.

Q4 (c) Solve for w if it is solvable.

We want to solve:

$$Xw = y$$

where

$$X = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, y = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Since this system is over-determined, it may have infinite solutions. We want to use the right inverse to find a constrained solution for w :

$$w = X^T (XX^T)^{-1} y$$

Let's calculate it step by step

Step 1: Compute XX^T:

$$XX^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Step 2: Compute $(XX^T)^{-1}$

$$(XX^T)^{-1} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 0.75 & 0.5 \\ -1 & 0.5 & 1 \end{bmatrix}$$

Step 3: Compute w

$$w = X^T (XX^T)^{-1} y = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 0.75 & 0.5 \\ -1 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

Q5

Given $\mathrm{w}^{\mathrm{T}}X = y^{\mathrm{T}}$ where $X = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)

(b) Is X invertible? Why?

(c) Solve for w if it is solvable.

Q5 (a) What kind of system is this? (even-, over- or under-determined?)

A canonical formulation is given by

$$w^T X = y^T \rightarrow X^T w = y \rightarrow X_1 w = y$$

where

$$X^T = X_1 = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

Here, we compare the number of equations (rows) with the number of unknowns (columns).

- Number of equations: The number of rows in matrix X_1, which is 2.
- Number of unknowns: The number of columns in matrix X_1, which is 2.

Since the number of equations equals the number of unknowns, the system is even-determined.

Q5 (b) Is X invertible? Why?

A matrix is invertible if it is square (number of rows equals the number of columns) and its determinant is non-zero.

To check if X is invertible, calculate its determinant

$$\det(X) = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\right) = 1 \times 6 - 2 \times 3 = 0$$

Since the determinant of X is zero, the matrix is **not invertible**.

Q5 (c) Solve for w if it is solvable.

Similar to Question 2, the matrix is not invertible, and therefore no solution exists.

Q6

Given $\mathrm{w}^{\mathrm{T}}X = y^{\mathrm{T}}$ where $X = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$, $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)

(b) Is X invertible? Why?

(c) Solve for w if it is solvable to obtain a constrained solution.

Q6 (a) What kind of system is this? (even-, over- or under-determined?)

A canonical formulation is given by taking transpose of both sides

$$w^T X = y^T \ \rightarrow \ X^T w = y$$

where

$$X^T = X_1 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix}$$

To determine the type of system, we compare the number of equations (rows in X^T) and the number of unknowns (columns in X^T).

- X^T is a 3×2 matrix (3 rows, 2 columns).
- There are 3 equations and 2 unknowns.

Since there are more unknows than equations, the system is under-determined.

Q6 (b) Is X invertible? Why?

- For a matrix to be invertible, it must be square (i.e., the number of rows must equal the number of columns). X is a 3×2 matrix, so it is not square and therefore not invertible.
- But $X^T X$ is invertible.

$$\det\left(\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 6 & 9 \\ 9 & 21 \end{bmatrix}\right) = 6 \times 21 - 9 \times 9 = 45$$

Q6 (c) Solve for w if it is solvable.

A canonical formulation is given by

$$w^T X = y^T \rightarrow X^T w = y$$

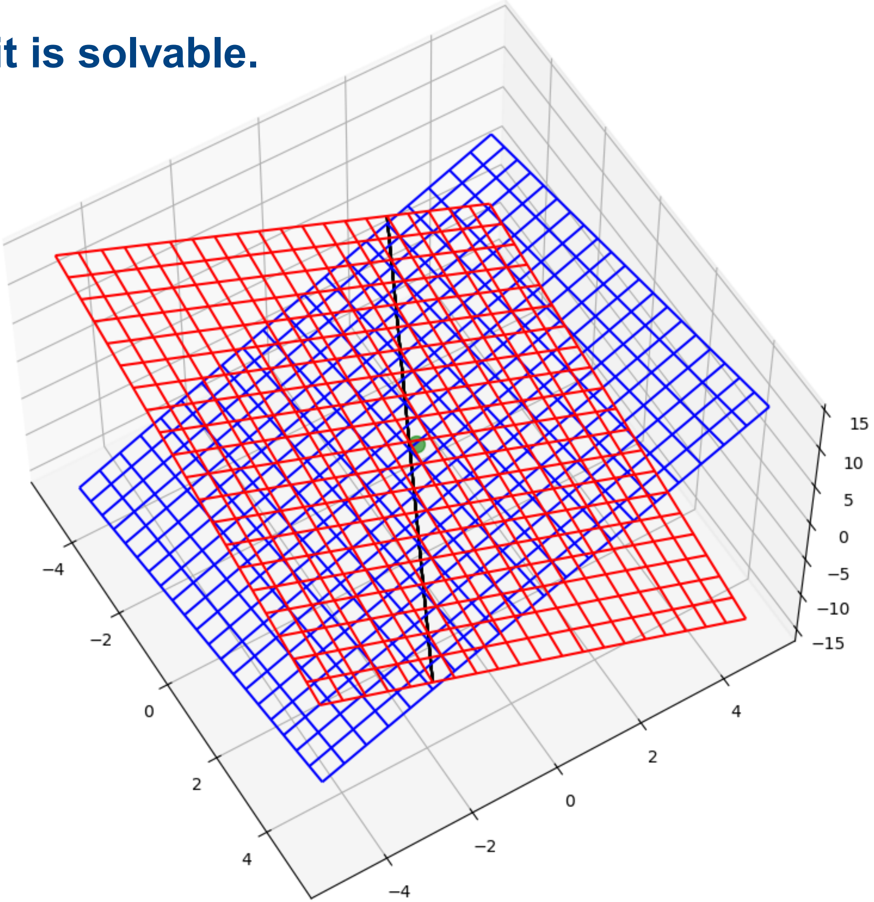
where

$$X^T = X_1 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix}$$

A constrained solution (exact) is given by

$$\hat{w} = X_1^T (X_1 X_1^T)^{-1} y = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix}^T \left(\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix}^T \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 0.066667 \\ 0.133333 \\ -0.33333 \end{bmatrix}$$

Q6 (c) Solve for w if it is solvable.



Question 7

This question is related to determination of types of system where an appropriate solution can be found subsequently. The following matrix has a left inverse.

$$X = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) True

b) False

Q8

MCQ: Which of the following is/are true about matrix A below? There could be more than one answer.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

a) A is invertible

b) A is left invertible

c) A is right invertible

d) A has no determinant

e) None of the above

THANK YOU