Chapter 3: Euclidean Vector Spaces

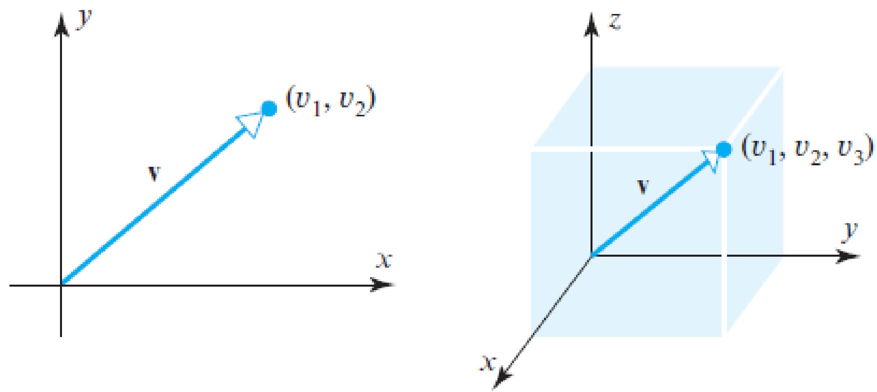
Elementary Linear Algebra (Anton & Rorres, 11th Edition)

3.1 Vectors in 2-Space, 3-Space, and n-Space

- Definition of vectors in R^2 and R^3
- Geometric representation of vectors
- Vector addition and scalar multiplication
- Generalization to R^n
- Examples of vectors in R^2, R^3, R^n

3.1 Vectors in 2-Space, 3-Space, and n-Space

- Definition of vectors in R^2 and R^3



▶ Figure 3.1.10

Vectors Whose Initial Point Is Not at the Origin

It is sometimes necessary to consider vectors whose initial points are not at the origin. If $\overrightarrow{P_1P_2}$ denotes the vector with initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$, then the components of this vector are given by the formula

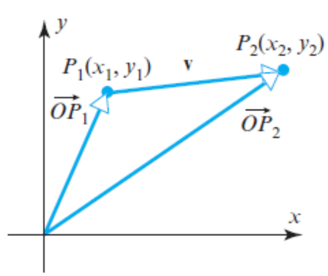
$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1) \tag{4}$$

That is, the components of $\overrightarrow{P_1P_2}$ are obtained by subtracting the coordinates of the initial point from the coordinates of the terminal point. For example, in Figure 3.1.12 the vector $\overrightarrow{P_1P_2}$ is the difference of vectors $\overrightarrow{OP_2}$ and $\overrightarrow{OP_1}$, so

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = (x_2, y_2) - (x_1, y_1) = (x_2 - x_1, y_2 - y_1)$$

As you might expect, the components of a vector in 3-space that has initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$ are given by

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1) \tag{5}$$

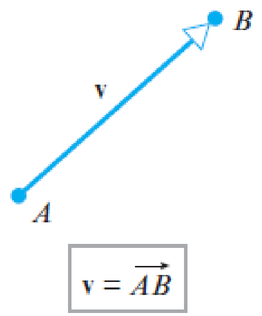


$\boxed{\mathbf{v} = \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}}$

▲ Figure 3.1.12

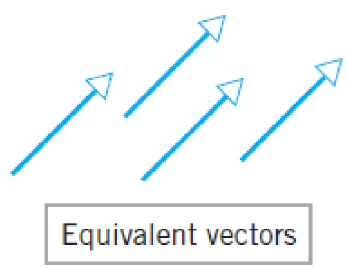
3.1 Vectors in 2-Space, 3-Space, and n-Space

- Geometric representation of vectors



▲ Figure 3.1.2

Vector Addition



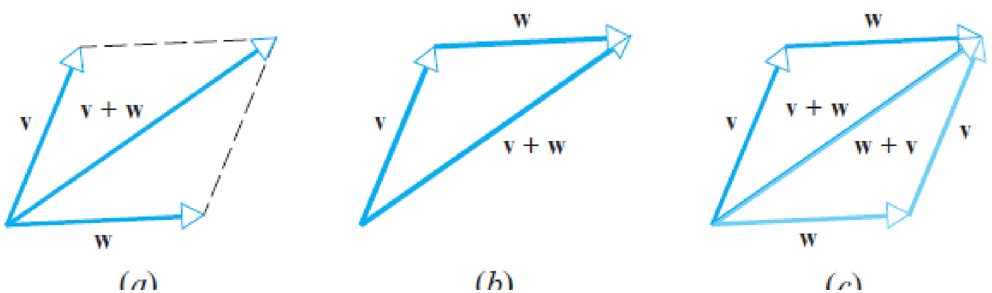
▲ Figure 3.1.3

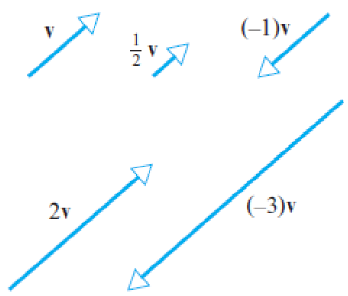
3.1 Vectors in 2-Space, 3-Space, and n-Space

- Vector addition and scalar multiplication

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} \tag{1}$$

and that the sum obtained by the triangle rule is the same as the sum obtained by the parallelogram rule.





▲ Figure 3.1.7

3.1 Vectors in 2-Space, 3-Space, and n-Space

- Generalization to R^n

DEFINITION 2 Vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ in R^n are said to be ***equivalent*** (also called ***equal***) if

$$v_1 = w_1, \quad v_2 = w_2, \ldots, \quad v_n = w_n$$

We indicate this by writing $\mathbf{v} = \mathbf{w}$.

▶ **EXAMPLE 2 Equality of Vectors**

$$(a, b, c, d) = (1, -4, 2, 7)$$

if and only if $a = 1$, $b = -4$, $c = 2$, and $d = 7$. ◀

Linear Combination

DEFINITION 4 If $\mathbf{w}$ is a vector in R^n, then $\mathbf{w}$ is said to be a ***linear combination*** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in R^n if it can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r \tag{14}$$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the ***coefficients*** of the linear combination. In the case where $r = 1$, Formula (14) becomes $\mathbf{w} = k_1\mathbf{v}_1$, so that a linear combination of a single vector is just a scalar multiple of that vector.

3.1 Vectors in 2-Space, 3-Space, and n-Space

- Theory

THEOREM 3.1.1 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *and* m *are scalars, then:*

(*a*) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(*b*) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(*c*) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(*d*) $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

(*e*) $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$

(*f*) $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$

(*g*) $k(m\mathbf{u})=(km)\mathbf{u}$

(*h*) $1\mathbf{u}=\mathbf{u}$

3.2 Norm, Dot Product, and Distance in Rn

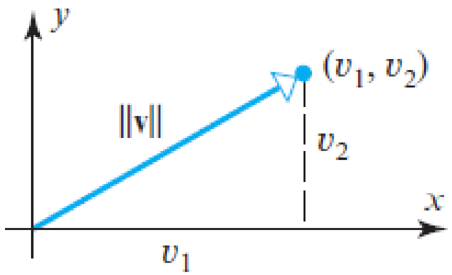
- Definition of norm (length of a vector)
- Dot product and its properties
- Geometric meaning of dot product
- Distance formula in R^n
- Examples and computations

3.2 Norm, Dot Product, and Distance in Rn

Definition of norm (length of a vector)

DEFINITION 1 If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the ***norm*** of $\mathbf{v}$ (also called the ***length*** of $\mathbf{v}$ or the ***magnitude*** of $\mathbf{v}$) is denoted by $\|\mathbf{v}\|$, and is defined by the formula

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \tag{3}$$



(a)

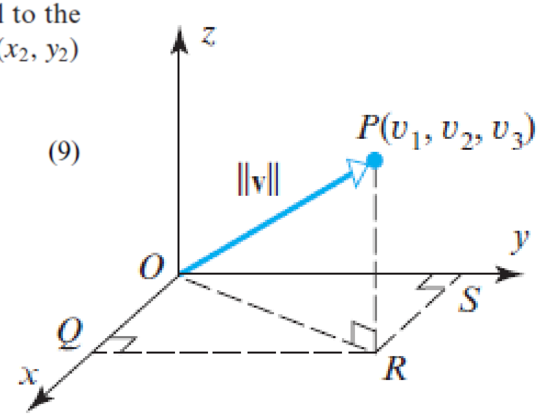
Distance in R^n

If P_1 and P_2 are points in R^2 or R^3, then the length of the vector $\overrightarrow{P_1P_2}$ is equal to the distance d between the two points (Figure 3.2.3). Specifically, if $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points in R^2, then Formula (4) of Section 3.1 implies that

$$d = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{9}$$

DEFINITION 2 If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are points in R^n, then we denote the ***distance*** between $\mathbf{u}$ and $\mathbf{v}$ by $d(\mathbf{u}, \mathbf{v})$ and define it to be

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2} \tag{11}$$



(b)

▲ Figure 3.2.1

3.2 Norm, Dot Product, and Distance in Rn

- Dot product and its properties

THEOREM 3.2.1 *If* $\mathbf{v}$ *is a vector in* R^n, *and if* k *is any scalar, then:*

(*a*) $\|\mathbf{v}\| \geq 0$

(*b*) $\|\mathbf{v}\| = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$

(*c*) $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$

Unit Vectors

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

Example 4

EXAMPLE 4 Calculating Distance in R^n

If

$$\mathbf{u} = (1, 3, -2, 7) \quad \text{and} \quad \mathbf{v} = (0, 7, 2, 2)$$

then the distance between **u** and **v** is

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(1-0)^2 + (3-7)^2 + (-2-2)^2 + (7-2)^2} = \sqrt{58}$$

3.2 Norm, Dot Product, and Distance in Rn

- Geometric meaning of dot product

DEFINITION 3 If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 or R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta \tag{12}$$

If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

THEOREM 3.2.5 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *then:*

(*a*) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ **[Triangle inequality for vectors]**

(*b*) $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ **[Triangle inequality for distances]**

Dot product

For computational purposes it is desirable to have a formula that expresses the dot product of two vectors in terms of components. We will derive such a formula for vectors in 3-space; the derivation for vectors in 2-space is similar.

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be two nonzero vectors. If, as shown in Figure 3.2.6, θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the law of cosines yields

$$\|\overrightarrow{PQ}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \tag{14}$$

Since $\overrightarrow{PQ} = \mathbf{v} - \mathbf{u}$, we can rewrite (14) as

$$\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = \tfrac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v} - \mathbf{u}\|^2)$$

or

$$\mathbf{u} \cdot \mathbf{v} = \tfrac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v} - \mathbf{u}\|^2)$$

Substituting

$$\|\mathbf{u}\|^2 = u_1^2 + u_2^2 + u_3^2, \qquad \|\mathbf{v}\|^2 = v_1^2 + v_2^2 + v_3^2$$

and

$$\|\mathbf{v} - \mathbf{u}\|^2 = (v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2$$

we obtain, after simplifying,

$$\boxed{\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3} \tag{15}$$

Proof

Proof (a)

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) &= (\mathbf{u}\cdot\mathbf{u}) + 2(\mathbf{u}\cdot\mathbf{v}) + (\mathbf{v}\cdot\mathbf{v}) \\
&= \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2 \\
&\le \|\mathbf{u}\|^2 + 2|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2 && \longleftarrow \text{Property of absolute value} \\
&\le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 && \longleftarrow \text{Cauchy–Schwarz inequality} \\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2
\end{aligned}$$

This completes the proof since both sides of the inequality in part (*a*) are nonnegative.

Proof (b) It follows from part (*a*) and Formula (11) that

$$\begin{aligned}
d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| &= \|(\mathbf{u}-\mathbf{w}) + (\mathbf{w}-\mathbf{v})\| \\
&\le \|\mathbf{u}-\mathbf{w}\| + \|\mathbf{w}-\mathbf{v}\| = d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v}) \quad \blacktriangleleft
\end{aligned}$$

It is proved in plane geometry that for any parallelogram the sum of the squares of the diagonals is equal to the sum of the squares of the four sides (Figure 3.2.10). The following theorem generalizes that result to R^n.

Proof

THEOREM 3.2.6 Parallelogram Equation for Vectors

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* R^n, *then*

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right) \tag{24}$$

Proof

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\ &= 2(\mathbf{u}\cdot\mathbf{u}) + 2(\mathbf{v}\cdot\mathbf{v}) \\ &= 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right) \quad \blacktriangleleft\end{aligned}$$

We could state and prove many more theorems from plane geometry that generalize to R^n, but the ones already given should suffice to convince you that R^n is not so different from R^2 and R^3 even though we cannot visualize it directly. The next theorem establishes a fundamental relationship between the dot product and norm in R^n.

proof

THEOREM 3.2.7 *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* R^n *with the Euclidean inner product, then*

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2 \tag{25}$$

Proof

$$\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \|\mathbf{u}\|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^2$$
$$\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^2$$

from which (25) follows by simple algebra. ◀

Other forms of u dot v

Table 1

Form	Dot Product	Example	
$\mathbf{u}$ a column matrix and $\mathbf{v}$ a column matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^T\mathbf{v}=\mathbf{v}^T\mathbf{u}$	$\mathbf{u}=\begin{bmatrix}1\\-3\\5\end{bmatrix}$ $\mathbf{v}=\begin{bmatrix}5\\4\\0\end{bmatrix}$	$\mathbf{u}^T\mathbf{v}=[1\quad -3\quad 5]\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}^T\mathbf{u}=[5\quad 4\quad 0]\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$
$\mathbf{u}$ a row matrix and $\mathbf{v}$ a column matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\mathbf{v}=\mathbf{v}^T\mathbf{u}^T$	$\mathbf{u}=[1\quad -3\quad 5]$ $\mathbf{v}=\begin{bmatrix}5\\4\\0\end{bmatrix}$	$\mathbf{u}\mathbf{v}=[1\quad -3\quad 5]\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}^T\mathbf{u}^T=[5\quad 4\quad 0]\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$
$\mathbf{u}$ a column matrix and $\mathbf{v}$ a row matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\mathbf{u}=\mathbf{u}^T\mathbf{v}^T$	$\mathbf{u}=\begin{bmatrix}1\\-3\\5\end{bmatrix}$ $\mathbf{v}=[5\quad 4\quad 0]$	$\mathbf{v}\mathbf{u}=[5\quad 4\quad 0]\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$ $\mathbf{u}^T\mathbf{v}^T=[1\quad -3\quad 5]\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$
$\mathbf{u}$ a row matrix and $\mathbf{v}$ a row matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\mathbf{v}^T=\mathbf{v}\mathbf{u}^T$	$\mathbf{u}=[1\quad -3\quad 5]$ $\mathbf{v}=[5\quad 4\quad 0]$	$\mathbf{u}\mathbf{v}^T=[1\quad -3\quad 5]\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}\mathbf{u}^T=[5\quad 4\quad 0]\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$

Example

If A is an $n \times n$ matrix and $\mathbf{u}$ and $\mathbf{v}$ are $n \times 1$ matrices, then it follows from the first row in Table 1 and properties of the transpose that

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{v}^T(A\mathbf{u}) = (\mathbf{v}^TA)\mathbf{u} = (A^T\mathbf{v})^T\mathbf{u} = \mathbf{u} \cdot A^T\mathbf{v}$$
$$\mathbf{u} \cdot A\mathbf{v} = (A\mathbf{v})^T\mathbf{u} = (\mathbf{v}^TA^T)\mathbf{u} = \mathbf{v}^T(A^T\mathbf{u}) = A^T\mathbf{u} \cdot \mathbf{v}$$

The resulting formulas

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v} \tag{26}$$

$$\mathbf{u} \cdot A\mathbf{v} = A^T\mathbf{u} \cdot \mathbf{v} \tag{27}$$

provide an important link between multiplication by an $n \times n$ matrix A and multiplication by A^T.

Example

EXAMPLE 9 Verifying that $A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v}$

Suppose that

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$$

Then

$$A\mathbf{u} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix}$$

$$A^T\mathbf{v} = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 4 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ -1 \end{bmatrix}$$

from which we obtain

$$A\mathbf{u} \cdot \mathbf{v} = 7(-2) + 10(0) + 5(5) = 11$$
$$\mathbf{u} \cdot A^T\mathbf{v} = (-1)(-7) + 2(4) + 4(-1) = 11$$

Thus, $A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v}$ as guaranteed by Formula (26). We leave it for you to verify that Formula (27) also holds.

3.3 Orthogonality

- Definition of orthogonality
- Orthogonal vectors in R^2 and R^3
- Orthogonal sets of vectors
- Applications of orthogonality
- Examples

DEFINITION 1 Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are said to be ***orthogonal*** (or ***perpendicular***) if $\mathbf{u} \cdot \mathbf{v} = 0$. We will also agree that the zero vector in R^n is orthogonal to *every* vector in R^n.

- Definition of orthogonality

3.3 Orthogonality

EXAMPLE 1 Orthogonal Vectors

(a) Show that $\mathbf{u} = (-2, 3, 1, 4)$ and $\mathbf{v} = (1, 2, 0, -1)$ are orthogonal vectors in R^4.

(b) Let $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be the set of standard unit vectors in R^3. Show that each ordered pair of vectors in S is orthogonal.

Solution (a) The vectors are orthogonal since

$$\mathbf{u} \cdot \mathbf{v} = (-2)(1) + (3)(2) + (1)(0) + (4)(-1) = 0$$

Solution (b) It suffices to show that

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0$$

Example

3.3 Orthogonality

Orthogonal vectors in R^2 and R^3

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0$$

$$a(x - x_0) + b(y - y_0) = 0 \quad \text{[line]} \tag{2}$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{[plane]} \tag{3}$$

These are called the ***point-normal*** equations of the line and plane.

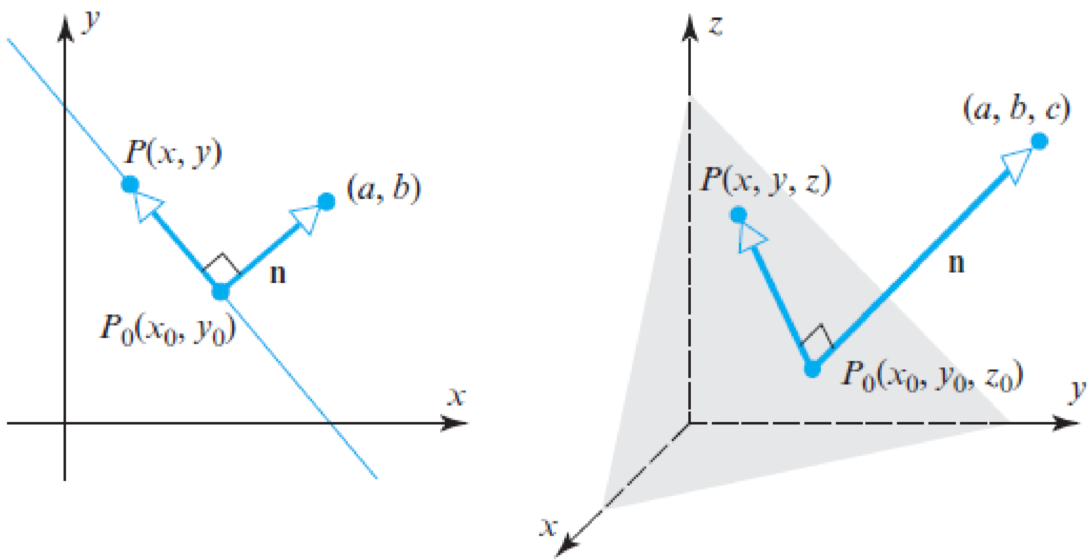
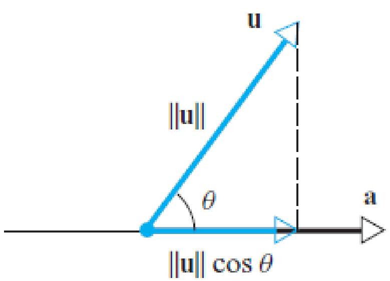


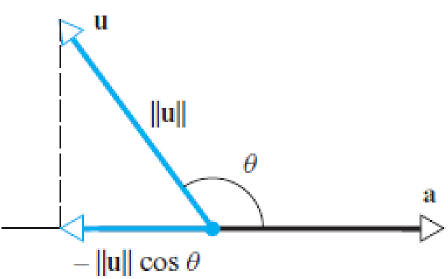
Figure 3.3.1

3.3 Orthogonality

- Orthogonal sets of vectors



(a) $0 \leq \theta < \frac{\pi}{2}$



(b) $\frac{\pi}{2} < \theta \leq \pi$

▲ Figure 3.3.4

Sometimes we will be more interested in the *norm* of the vector component of $\mathbf{u}$ along $\mathbf{a}$ than in the vector component itself. A formula for this norm can be derived as follows:

$$\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \left\|\frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}\right\| = \left|\frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\right|\|\mathbf{a}\| = \frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|^2}\|\mathbf{a}\|$$

where the second equality follows from part (*c*) of Theorem 3.2.1 and the third from the fact that $\|\mathbf{a}\|^2 > 0$. Thus,

$$\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|} \tag{12}$$

If θ denotes the angle between $\mathbf{u}$ and $\mathbf{a}$, then $\mathbf{u}\cdot\mathbf{a} = \|\mathbf{u}\|\|\mathbf{a}\|\cos\theta$, so (12) can also be written as

$$\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \|\mathbf{u}\||\cos\theta| \tag{13}$$

(Verify.) A geometric interpretation of this result is given in Figure 3.3.4.

3.3 Orthogonality

- Applications of orthogonality

THEOREM 3.3.4

(*a*) *In* R^2 *the distance* D *between the point* $P_0(x_0, y_0)$ *and the line* $ax + by + c = 0$ *is*

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \tag{15}$$

(*b*) *In* R^3 *the distance* D *between the point* $P_0(x_0, y_0, z_0)$ *and the plane* $ax + by + cz + d = 0$ *is*

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{16}$$

proof

Proof (b) The underlying idea of the proof is illustrated in Figure 3.3.6. As shown in that figure, let $Q(x_1, y_1, z_1)$ be any point in the plane, and let $\mathbf{n} = (a, b, c)$ be a normal vector to the plane that is positioned with its initial point at Q. It is now evident that the distance D between P_0 and the plane is simply the length (or norm) of the orthogonal projection of the vector $\overrightarrow{QP_0}$ on $\mathbf{n}$, which by Formula (12) is

$$D = \|\text{proj}_{\mathbf{n}}\overrightarrow{QP_0}\| = \frac{|\overrightarrow{QP_0} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

But

$$\overrightarrow{QP_0} = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$$

$$\overrightarrow{QP_0} \cdot \mathbf{n} = a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)$$

$$\|\mathbf{n}\| = \sqrt{a^2 + b^2 + c^2}$$

Thus

$$D = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}} \tag{17}$$

Since the point $Q(x_1, y_1, z_1)$ lies in the given plane, its coordinates satisfy the equation of that plane; thus

$$ax_1 + by_1 + cz_1 + d = 0$$

or

$$d = -ax_1 - by_1 - cz_1$$

Substituting this expression in (17) yields (16). ◀

▶ **EXAMPLE 7 Distance Between a Point and a Plane**

Find the distance D between the point $(1, -4, -3)$ and the plane $2x - 3y + 6z = -1$.

Solution Since the distance formulas in Theorem 3.3.4 require that the equations of the line and plane be written with zero on the right side, we first need to rewrite the equation of the plane as

$$2x - 3y + 6z + 1 = 0$$

from which we obtain

$$D = \frac{|2(1) + (-3)(-4) + 6(-3) + 1|}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{|-3|}{7} = \frac{3}{7} \blacktriangleleft$$

3.4 The Geometry of Linear Systems

- Linear equations as lines and planes
- Geometric interpretation of solutions
- Consistent vs inconsistent systems
- Intersection of lines and planes
- Examples

3.4 The Geometry of Linear Systems

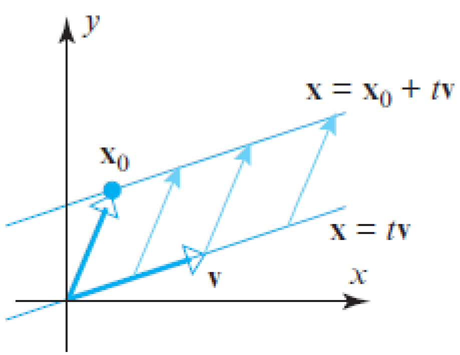
- Linear equations as lines and planes

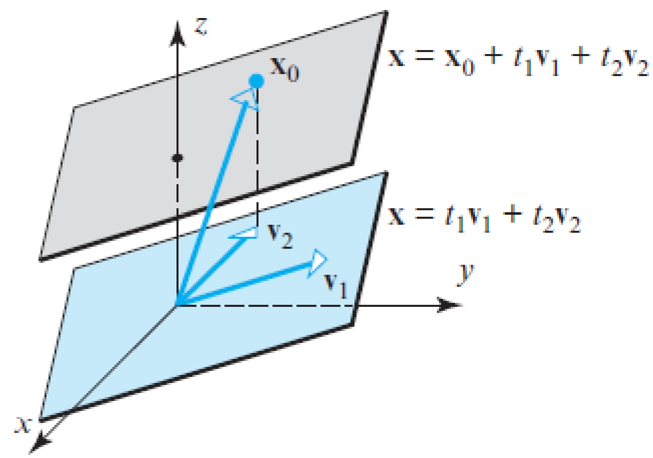
THEOREM 3.4.1 *Let L be the line in R^2 or R^3 that contains the point $\mathbf{x}_0$ and is parallel to the nonzero vector $\mathbf{v}$. Then the equation of the line through $\mathbf{x}_0$ that is parallel to $\mathbf{v}$ is*

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \tag{1}$$

If $\mathbf{x}_0 = \mathbf{0}$, then the line passes through the origin and the equation has the form

$$\mathbf{x} = t\mathbf{v} \tag{2}$$





Example

▶ EXAMPLE 2 Vector and Parametric Equations of a Plane in R^3

Find vector and parametric equations of the plane $x - y + 2z = 5$.

Solution We will find the parametric equations first. We can do this by solving the equation for any one of the variables in terms of the other two and then using those two variables as parameters. For example, solving for x in terms of y and z yields

$$x = 5 + y - 2z \tag{8}$$

and then using y and z as parameters t_1 and t_2, respectively, yields the parametric equations

$$x = 5 + t_1 - 2t_2, \quad y = t_1, \quad z = t_2$$

To obtain a vector equation of the plane we rewrite these parametric equations as

$$(x, y, z) = (5 + t_1 - 2t_2, t_1, t_2)$$

or, equivalently, as

$$(x, y, z) = (5, 0, 0) + t_1(1, 1, 0) + t_2(-2, 0, 1)$$

3.4 The Geometry of Linear Systems

- Geometric interpretation of solutions

DEFINITION 1 If $\mathbf{x}_0$ and $\mathbf{v}$ are vectors in R^n, and if $\mathbf{v}$ is nonzero, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \tag{5}$$

defines the ***line through*** $\mathbf{x}_0$ ***that is parallel to*** $\mathbf{v}$. In the special case where $\mathbf{x}_0 = \mathbf{0}$, the line is said to ***pass through the origin***.

DEFINITION 2 If $\mathbf{x}_0$, $\mathbf{v}_1$, and $\mathbf{v}_2$ are vectors in R^n, and if $\mathbf{v}_1$ and $\mathbf{v}_2$ are not collinear, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \tag{6}$$

defines the ***plane through*** $\mathbf{x}_0$ ***that is parallel to*** $\mathbf{v}_1$ ***and*** $\mathbf{v}_2$. In the special case where $\mathbf{x}_0 = \mathbf{0}$, the plane is said to ***pass through the origin***.

3.4 The Geometry of Linear Systems

- Consistent vs inconsistent systems

Except for a notational change from $\mathbf{n}$ to $\mathbf{a}$, Formula (18) is the extension to R^n of Formula (6) in Section 3.3. This equation reveals that *each solution vector* $\mathbf{x}$ *of a homogeneous equation is orthogonal to the coefficient vector* $\mathbf{a}$. To take this geometric observation a step further, consider the homogeneous system

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & 0 \end{array}$$

If we denote the successive row vectors of the coefficient matrix by $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$, then we can rewrite this system in dot product form as

$$\begin{array}{c} \mathbf{r}_1 \cdot \mathbf{x} = \mathbf{0} \\ \mathbf{r}_2 \cdot \mathbf{x} = \mathbf{0} \\ \vdots \quad\quad \vdots \\ \mathbf{r}_m \cdot \mathbf{x} = \mathbf{0} \end{array} \tag{19}$$

$$\mathbf{a} \cdot \mathbf{x} = b \tag{17}$$

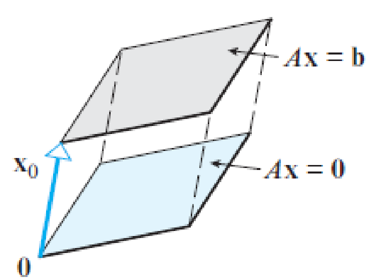
and Formula (16) as

$$\mathbf{a} \cdot \mathbf{x} = 0 \tag{18}$$

3.4 The Geometry of Linear Systems

- Intersection of lines and planes

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 6 \end{bmatrix}$$



▲ Figure 3.4.7 The solution set of $A\mathbf{x} = \mathbf{b}$ is a translation of the solution space of $A\mathbf{x} = \mathbf{0}$.

3.4 The Geometry of Linear Systems

- Examples

3.5 Cross Product

- Definition of cross product
- Properties of cross product
- Geometric meaning (area of parallelogram)
- Applications in physics and engineering
- Examples

3.5 Cross Product

Definition of cross product

DEFINITION 1 If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the ***cross product*** $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$$

or, in determinant notation,

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \tag{1}$$

3.5 Cross Product

Properties of cross product

THEOREM 3.5.1 Relationships Involving Cross Product and Dot Product

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in 3-space, then*

(*a*) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ [$\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{u}$]

(*b*) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ [$\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{v}$]

(*c*) $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ [*Lagrange's identity*]

(*d*) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ [*vector triple product*]

(*e*) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$ [*vector triple product*]

Example

Proof (a) Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Then

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= (u_1, u_2, u_3) \cdot (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \\ &= u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0\end{aligned}$$

Proof (b) Similar to (a).

Proof (c) Since

$$\|\mathbf{u} \times \mathbf{v}\|^2 = (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \tag{2}$$

and

$$\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \tag{3}$$

the proof can be completed by "multiplying out" the right sides of (2) and (3) and verifying their equality.

properties

THEOREM 3.5.2 Properties of Cross Product

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are any vectors in 3-space and* k *is any scalar, then:*

(*a*) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(*b*) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(*c*) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(*d*) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(*e*) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(*f*) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

3.5 Cross Product

Geometric meaning (area of parallelogram)

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in 3-space, then the norm of $\mathbf{u} \times \mathbf{v}$ has a useful geometric interpretation. Lagrange's identity, given in Theorem 3.5.1, states that

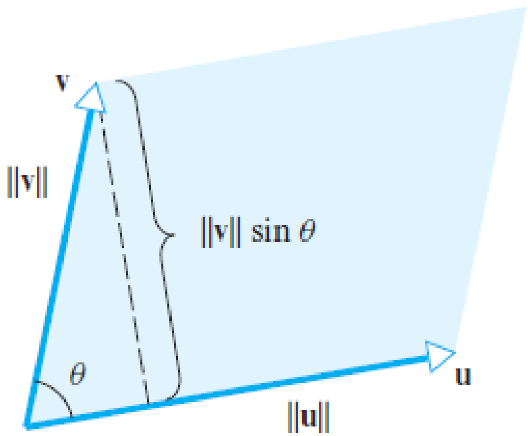
$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 \tag{5}$$

If θ denotes the angle between $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\| \cos\theta$, so (5) can be rewritten as

$$\begin{aligned} \|\mathbf{u} \times \mathbf{v}\|^2 &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2\|\mathbf{v}\|^2\cos^2\theta \\ &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2(1 - \cos^2\theta) \\ &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2\sin^2\theta \end{aligned}$$

Since $0 \le \theta \le \pi$, it follows that $\sin\theta \ge 0$, so this can be rewritten as

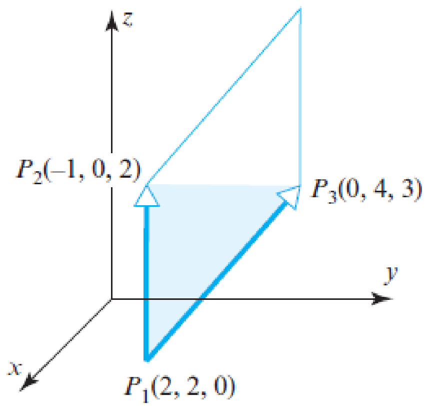
$$\boxed{\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta} \tag{6}$$



▲ Figure 3.5.4

3.5 Cross Product

Applications in physics and engineering



▲ Figure 3.5.5

▶ **EXAMPLE 4 Area of a Triangle**

Find the area of the triangle determined by the points $P_1(2, 2, 0)$, $P_2(-1, 0, 2)$, and $P_3(0, 4, 3)$.

Solution The area A of the triangle is $\frac{1}{2}$ the area of the parallelogram determined by the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ (Figure 3.5.5). Using the method discussed in Example 1 of Section 3.1, $\overrightarrow{P_1P_2} = (-3, -2, 2)$ and $\overrightarrow{P_1P_3} = (-2, 2, 3)$. It follows that

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = (-10, 5, -10)$$

(verify) and consequently that

$$A = \frac{1}{2}\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\| = \frac{1}{2}(15) = \frac{15}{2}$$ ◀

3.5 Cross Product

Examples

Geometric Interpretation of Determinants

The next theorem provides a useful geometric interpretation of 2×2 and 3×3 determinants.

THEOREM 3.5.4

(*a*) *The absolute value of the determinant*

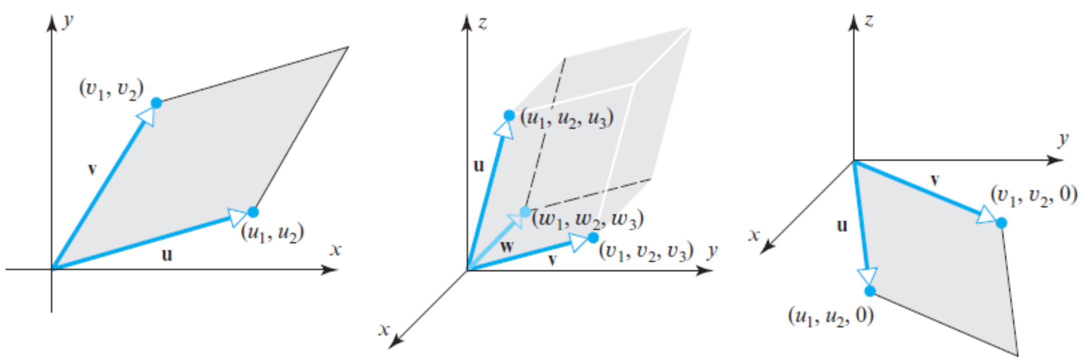
$$\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$$

is equal to the area of the parallelogram in 2-space determined by the vectors $\mathbf{u} = (u_1, u_2)$ *and* $\mathbf{v} = (v_1, v_2)$. *(See Figure 3.5.7a.)*

(*b*) *The absolute value of the determinant*

$$\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

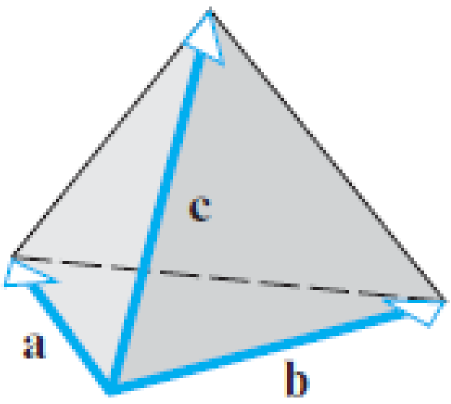
is equal to the volume of the parallelepiped in 3-space determined by the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, *and* $\mathbf{w} = (w_1, w_2, w_3)$. *(See Figure 3.5.7b.)*



(*a*) (*b*) (*c*)

Exercise 1

38. It is a theorem of solid geometry that the volume of a tetrahedron is $\frac{1}{3}$(area of base) $\cdot$ (height). Use this result to prove that the volume of a tetrahedron whose sides are the vectors **a**, **b**, and **c** is $\frac{1}{6}|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ (see accompanying figure).



◀ Figure Ex-38

Exercise 2

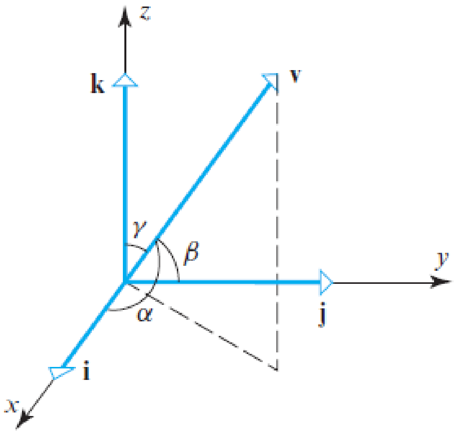
29. Simplify $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$.

Problems:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \tag{13}$$

21. Use Formula (13) to show that the direction cosines of a vector $\mathbf{v} = (v_1, v_2, v_3)$ in R^3 are

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|}, \qquad \cos\beta = \frac{v_2}{\|\mathbf{v}\|}, \qquad \cos\gamma = \frac{v_3}{\|\mathbf{v}\|}$$

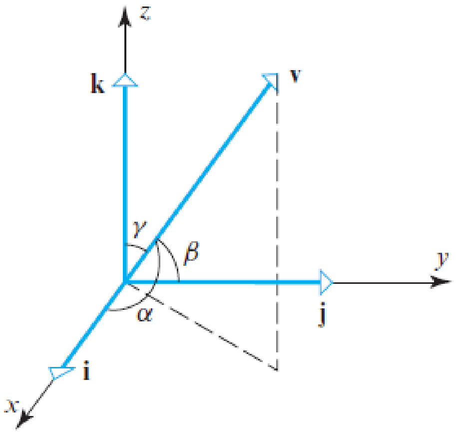


◀ Figure Ex-21

Problems:

22. Use the result in Exercise 21 to show that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$



◀ Figure Ex-21

Problems:

▶ **Exercises 21–25** The direction of a nonzero vector **v** in an xyz-coordinate system is completely determined by the angles α, β, and γ between **v** and the standard unit vectors **i**, **j**, and **k** (Figure Ex-21). These are called the ***direction angles*** of **v**, and their cosines are called the ***direction cosines*** of **v**. ◀

23. Show that two nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in R^3 are orthogonal if and only if their direction cosines satisfy

$$\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2=0$$