Chapter 2: Determinants (Expanded)

Based on Elementary Linear Algebra (Anton & Rorres, 11th Ed.)

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2.1 Introduction to Determinants

- Motivation: why determinants are important
- Connection to systems of equations and matrix invertibility
- Geometric interpretation: area and volume

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2×2 Determinant

Formula: det([[a, b], [c, d]]) = ad - bc

Properties of 2×2 determinant

$$\det(A) = ad - bc \quad \text{or} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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3×3 Determinant

Expansion formula (rule of Sarrus)

Examples with step-by-step expansion

Placeholder diagram for calculation

DEFINITION 1 If A is a square matrix, then the ***minor of entry*** a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A. The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the ***cofactor of entry*** a_{ij}.

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▶ EXAMPLE 1 Finding Minors and Cofactors

Let

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

The minor of entry a_{11} is

$$M_{11} = \begin{vmatrix} \cancel{3} & \cancel{1} & \cancel{-4} \\ \cancel{2} & 5 & 6 \\ \cancel{1} & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$$

The cofactor of a_{11} is

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 16$$

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General Definition

- n×n determinant definition
- Recursive definition via minors and cofactors
- Notation: det(A), |A|

DEFINITION 2 If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the ***determinant of*** A, and the sums themselves are called ***cofactor expansions of*** A. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \tag{7}$$

[cofactor expansion along the *j*th column]

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \tag{8}$$

[cofactor expansion along the *i*th row]

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▶ **EXAMPLE 4 Cofactor Expansion Along the First Column**

Let A be the matrix in Example 3, and evaluate $\det(A)$ by cofactor expansion along the first column of A.

Solution

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$= 3(-4) - (-2)(-2) + 5(3) = -1$$

This agrees with the result obtained in Example 3.

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▶ **EXAMPLE 5 Smart Choice of Row or Column**

If A is the 4×4 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

then to find $\det(A)$ it will be easiest to use cofactor expansion along the second column, since it has the most zeros:

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

For the 3×3 determinant, it will be easiest to use cofactor expansion along its second column, since it has the most zeros:

$$\det(A) = 1 \cdot -2 \cdot \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= -2(1 + 2)$$

$$= -6$$

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Determinant of Triangular Matrices

- Determinant is product of diagonal entries

▶ **EXAMPLE 6 Determinant of a Lower Triangular Matrix**

The following computation shows that the determinant of a 4×4 lower triangular matrix is the product of its diagonal entries. Each part of the computation uses a cofactor expansion along the first row.

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$= a_{11}a_{22} \begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33}|a_{44}| = a_{11}a_{22}a_{33}a_{44}$$ ◀

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Checking Attendace:

35. By inspection, what is the relationship between the following determinants?

$$d_1 = \begin{vmatrix} a & b & c \\ d & 1 & f \\ g & 0 & 1 \end{vmatrix} \quad \text{and} \quad d_2 = \begin{vmatrix} a+\lambda & b & c \\ d & 1 & f \\ g & 0 & 1 \end{vmatrix}$$

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Effect of Row Operations

Row swap multiplies determinant by -1

Row scaling multiplies determinant by scalar

Row replacement leaves determinant unchanged

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Effect of Row Operations

Table 1

Relationship	Operation
$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = k\det(A)$	In the matrix B the first row of A was multiplied by k.
$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = -\det(A)$	In the matrix B the first and second rows of A were interchanged.
$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = \det(A)$	In the matrix B a multiple of the second row of A was added to the first row.

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Effect of Row Operations (proof)

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ka_{11}C_{11} + ka_{12}C_{12} + ka_{13}C_{13}$$

$$= k(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13})$$

$$= k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

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2.2 Row Reduction Method

- Strategy: reduce matrix to triangular form
- Use row operation effects on determinant
- Advantages over cofactor expansion

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EXAMPLE 3 Using Row Reduction to Evaluate a Determinant

Evaluate det(A) where

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

Solution We will reduce A to row echelon form (which is upper triangular) and then apply Theorem 2.1.2.

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

The first and second rows of A were interchanged.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

A common factor of 3 from the first row was taken through the determinant sign.

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$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

-2 times the first row was added to the third row.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

-10 times the second row was added to the third row.

$$= (-3)(-55)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$

A common factor of -55 from the last row was taken through the determinant sign.

$$= (-3)(-55)(1) = 165$$

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EXAMPLE 5 Row Operations and Cofactor Expansion

Evaluate $\det(A)$ where

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

Solution By adding suitable multiples of the second row to the remaining rows, we obtain

$$\det(A) = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} \quad \text{Cofactor expansion along the first column}$$

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix} \quad \text{We added the first row to the third row.}$$

$$= -(-1)\begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix} \quad \text{Cofactor expansion along the first column}$$

$$= -18$$

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In Exercises 25–28, confirm the identities without evaluating the determinants directly.

25. $$\begin{vmatrix} a_1 & b_1 & a_1 + b_1 + c_1 \\ a_2 & b_2 & a_2 + b_2 + c_2 \\ a_3 & b_3 & a_3 + b_3 + c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Exercise1:

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Zero Determinants

- Determinant zero when rows/columns are dependent
- Examples of singular matrices
- Connection to linear dependence

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Special Cases

- Upper/lower triangular matrices
- Singular matrices during reduction
- Zero determinant detection

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Summary of Row Reduction

- Efficient method for computation
- Ties to matrix invertibility
- Preparation for eigenvalue problems

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2.3 Properties of Determinants

- Linearity in rows/columns
- Multilinearity and alternating property
- Det(I) = 1

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Multiplicative Property

det(AB) = det(A)det(B)

Proof sketch

Applications

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Proof We divide the proof into two cases that depend on whether or not A is invertible. If the matrix A is not invertible, then by Theorem 1.6.5 neither is the product AB. Thus, from Theorem 2.3.3, we have $\det(AB) = 0$ and $\det(A) = 0$, so it follows that $\det(AB) = \det(A)\det(B)$.

Now assume that A is invertible. By Theorem 1.6.4, the matrix A is expressible as a product of elementary matrices, say

$$A = E_1 E_2 \cdots E_r \tag{5}$$

so

$$AB = E_1 E_2 \cdots E_r B$$

Applying (3) to this equation yields

$$\det(AB) = \det(E_1)\det(E_2)\cdots\det(E_r)\det(B)$$

and applying (3) again yields

$$\det(AB) = \det(E_1 E_2 \cdots E_r)\det(B)$$

which, from (5), can be written as $\det(AB) = \det(A)\det(B)$. ◀

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▶ **EXAMPLE 1 det($A + B$) $\neq$ det(A) + det(B)**

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad A + B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$$

We have $\det(A) = 1$, $\det(B) = 8$, and $\det(A + B) = 23$; thus

$$\det(A + B) \neq \det(A) + \det(B)$$ ◀

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For example,

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} = k^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Unfortunately, no simple relationship exists among $\det(A)$, $\det(B)$, and $\det(A + B)$. In particular, $\det(A + B)$ will usually *not* be equal to $\det(A) + \det(B)$. The following example illustrates this fact.

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Transpose Property

$\det(A^T) = \det(A)$

Geometric interpretation

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DEFINITION 1 If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the ***matrix of cofactors from A***. The transpose of this matrix is called the ***adjoint of A*** and is denoted by $\mathrm{adj}(A)$.

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EXAMPLE 5 **Entries and Cofactors from Different Rows**

Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

We leave it for you to verify that the cofactors of A are

$$\begin{array}{lll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\ C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{array}$$

so, for example, the cofactor expansion of $\det(A)$ along the first row is

$$\det(A) = 3C_{11} + 2C_{12} + (-1)C_{13} = 36 + 12 + 16 = 64$$

and along the first column is

$$\det(A) = 3C_{11} + C_{21} + 2C_{31} = 36 + 4 + 24 = 64$$

Suppose, however, we multiply the entries in the first row by the corresponding cofactors from the *second row* and add the resulting products. The result is

$$3C_{21} + 2C_{22} + (-1)C_{23} = 12 + 4 - 16 = 0$$

Or suppose we multiply the entries in the first column by the corresponding cofactors from the *second column* and add the resulting products. The result is again zero since

$$3C_{12} + 1C_{22} + 2C_{32} = 18 + 2 - 20 = 0$$

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Invertibility Criterion

- A invertible <-> det(A) ≠ 0
- Connection to matrix rank
- Examples

THEOREM 2.3.6 **Inverse of a Matrix Using Its Adjoint**

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) \tag{6}$$

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EXAMPLE 7 Using the Adjoint to Find an Inverse Matrix

Use Formula (6) to find the inverse of the matrix A in Example 6.

Solution We showed in Example 5 that $\det(A) = 64$. Thus,

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & -\frac{10}{64} \\ -\frac{16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix}$$

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Block Matrices and Determinants

- Special block structures
- Triangular block determinants
- Applications

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Volume Interpretation

- Determinant as volume scaling factor
- Examples with parallelograms/parallelepipeds
- Diagram placeholder

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Cramer's Rule Introduction

- Solve Ax = b using determinants
- Formula for $x_i = \det(A_i)/\det(A)$
- Requirements: $\det(A) \neq 0$

THEOREM 2.3.7 Cramer's Rule

If $A\mathbf{x} = \mathbf{b}$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

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Proof If $\det(A) \neq 0$, then A is invertible, and by Theorem 1.6.2, $\mathbf{x} = A^{-1}\mathbf{b}$ is the unique solution of $A\mathbf{x} = \mathbf{b}$. Therefore, by Theorem 2.3.6 we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)}\operatorname{adj}(A)\mathbf{b} = \frac{1}{\det(A)}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Multiplying the matrices out gives

$$\mathbf{x} = \frac{1}{\det(A)}\begin{bmatrix} b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1} \\ b_1C_{12} + b_2C_{22} + \cdots + b_nC_{n2} \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ b_1C_{1n} + b_2C_{2n} + \cdots + b_nC_{nn} \end{bmatrix}$$

The entry in the jth row of $\mathbf{x}$ is therefore

$$x_j = \frac{b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}}{\det(A)} \tag{9}$$

Now let

$$A_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn} \end{bmatrix}$$

Since A_j differs from A only in the jth column, it follows that the cofactors of entries $b_1, b_2, \ldots, b_n$ in A_j are the same as the cofactors of the corresponding entries in the jth column of A. The cofactor expansion of $\det(A_j)$ along the jth column is therefore

$$\det(A_j) = b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}$$

Substituting this result in (9) gives

$$x_j = \frac{\det(A_j)}{\det(A)}$$ ◀

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▶ EXAMPLE 8 Using Cramer's Rule to Solve a Linear System

Use Cramer's rule to solve

$$\begin{aligned} x_1 + \quad\quad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

Solution

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

Therefore,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11},$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$ ◀

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Limitations of Cramer's Rule

- Inefficiency for large systems
- Numerical stability concerns
- Still useful for theory and small systems

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Applications of Cramer's Rule and Exercise2:

- Theoretical proofs
- Small system solving
- Examples in physics and economics

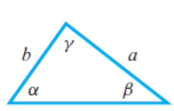
29. (a) For the triangle in the accompanying figure, use trigonometry to show that

$$b\cos\gamma + c\cos\beta = a$$
$$c\cos\alpha + a\cos\gamma = b$$
$$a\cos\beta + b\cos\alpha = c$$

and then apply Cramer's rule to show that

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

(b) Use Cramer's rule to obtain similar formulas for $\cos\beta$ and $\cos\gamma$.



◀ Figure Ex-29

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Summary of Chapter 2

- Cofactor expansion definition
- Row reduction method for efficiency
- Key determinant properties
- Cramer’s Rule

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Next Steps

- Transition to Chapter 3: Euclidean Vector Spaces
- Importance of determinants in linear algebra
- Preview: orthogonality and vector geometry

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