Chapter 1

Vectors

1.1 Introduction

Physical quantities such as volume, area, length, temperature and time only have magnitude and each is completely characterized by a single real number (with an appropriate unit of measurement). These quantities are often referred to as *scalars*.

Quantities such as position, velocity, acceleration or force, each has a magnitude and a direction, and they are represented mathematically in a component-by-component fashion, or by means of an arrow, where the arrow points in a certain direction. The length of the arrow may be interpreted as the size of the relevant quantity. The term *vector* is given to such quantities.

1.2 Vector notation

Let us first only view vectors in 2D, i.e. in a plane. A vector in 2D may be described by its magnitude and its direction. (The direction is always taken relative to some fixed axis, e.g., the x-axis). Alternatively, given an xy coordinate (or Cartesian) system the vector may also be described by its two components in the x and y directions respectively. See Figure 1.1.

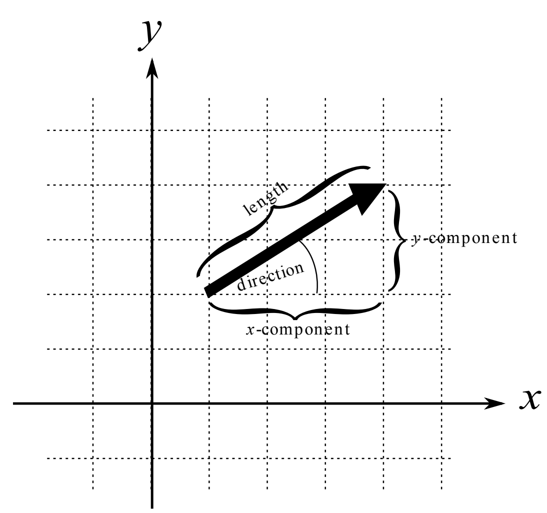

Figure 1.1. A vector as an arrow, showing its two components, as well as its magnitude and its direction. The direction is an angle.

It is the purpose of this chapter to establish the notation and rules by which vectors are manipulated mathematically.

In these notes vectors will be printed in boldface symbols such as **a**, **b**, **v**, **F**, etc.

We shall discuss and use mostly vectors in 2D (a plane) and vectors in 3D (space). We shall often alternate between working with vectors in 2D or in 3D, depending on the problem at hand, and it will usually be clear from the problem statement whether 2D or 3D vectors are required.

A vector in 2D, has two components. Consider the vector **a** with first component 4, and second component 3. We may write it in column form as follows:

$$\mathbf{a} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

There are also other notations used to denote vectors, for example:

$$\mathbf{a} \quad \text{or} \quad \underline{a} \quad \text{or} \quad \vec{a} \quad \text{or} \quad \overset{\rightharpoonup}{a}\ .$$

If the tail of the arrow is at A and the arrow head is at B, it may also be denoted by

$$\overrightarrow{AB} \quad \text{or} \quad \overset{\rightharpoonup}{AB}\ .$$

2D Vectors as displacement arrows:

The most straightforward use of a vector is to denote position or displacement. For example, the point in the 2D Cartesian plane with x-coordinate 4, and y-coordinate 3, may be viewed as a displacement from the origin given by the vector **a** above. In other words, the point lies at the arrow head of the vector.

If the vector **a** denotes position it is therefore an arrow starting at the origin and ending at the point $(4, 3)$. With this interpretation of a vector, *coordinates* of the point in xy-space correspond to *components* of a vector.[See Figure 1.2]

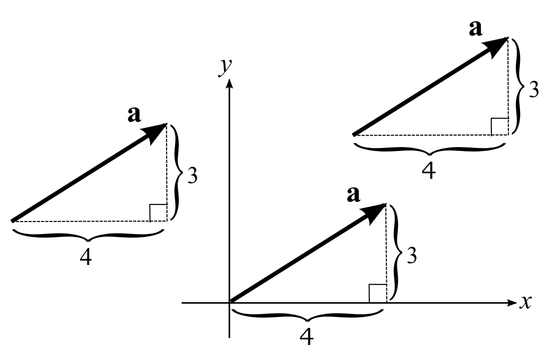

Figure 1.2. Graphical representation of the vector $\mathbf{a} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ in three different positions.

The vector starting at the origin is often used to describe position in 2D space and is then called a *position vector*. Vectors may however start at any coordinate point — its components denote shifts or displacements in the particular directions. With this notion it is clear that the arrow representation of a vector may be an arrow positioned anywhere in 2D space as long as it keeps the same two components. Consequently this means that the arrow also keeps its direction and length, irrespective of where it is placed on the xy-plane.

3D Vectors as displacement arrows:

In three dimensions, there are three position coordinates, x, y, and z. A position vector in 3D must therefore have three components, where each component represents a shift (or displacement) in each of the three directions.

Consider the vector **a** with first component 3, and second component 1, and third component 2. We may write it in column form as follows:

$$\mathbf{a} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

Although more difficult to visualize, we shall still often try to depict such vectors as arrows in a system of three axes (x, y, and z), by adding a rectangular box with sides equal to the respective components of the vector. **[See Figure 1.3]** In this graphical representation it is recommended to keep the scales in the various directions relatively correct.

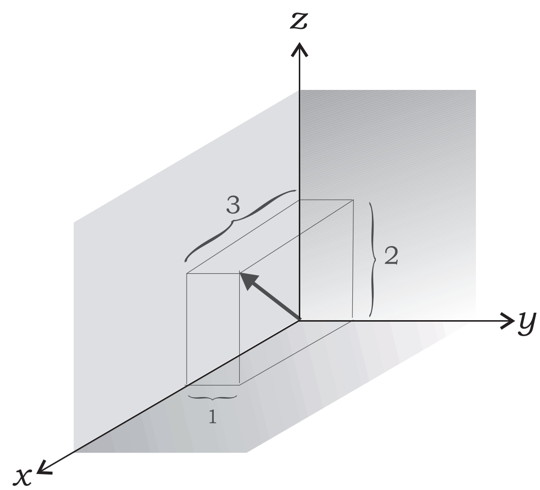

Figure 1.3. Graphical representation of the vector $\mathbf{a} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

Subscript notation for components:

In 3D we shall label the components of a vector with subscript indices, x, y, and z. The vector $\mathbf{a}$ will then have as *first* component a_x, and its *second* component is written as a_y, etc. This means that every vector can be written out fully showing all its components, or may be referred to by only using its boldface symbol, for example

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}.$$

We may also label the component rather with numbers, for example we may write

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

[Do Checkpoint 1.1]

Checkpoint 1.1:
If $\mathbf{a} = 2\mathbf{b}$, $b_1 = 3$, $a_2 = 8$, and $a_3 = -12$, find $\mathbf{b}$.

1.3 Basic vector algebra

We shall state the rules for vector algebra here for vectors in 3D. However, in Figure 1.4 the corresponding geometrical interpretation will be depicted only for 2D vectors.

Vector equality:

Vectors are equal if (and only if) each of the corresponding components are equal. Let

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

then:

$$\boxed{\mathbf{a} = \mathbf{b}, \quad \text{implies} \quad a_j = b_j, \quad j = 1, 2, 3.} \tag{1.1}$$

(From equation (1.1) it means that $\mathbf{a}$=$\mathbf{b}$, if a_1=b_1, a_2=b_2, a_3=b_3.)

As mentioned already, this implies that in the geometrical interpretation of a vector, the two arrows representing the two equal vectors, have the same corresponding x, y, and z-components. This means that the vectors are parallel (also pointing in the same direction) and have the same magnitudes. See Figure 1.4(a). **Note that**: Equality does not necessarily mean that the two equal vectors start at the same point.

<u>*Negative vectors:*</u>

The negative (additive inverse) of a vector is simply a vector where each component has changed sign:

$$\boxed{\mathbf{c} = -\mathbf{a}, \quad \text{implies} \quad c_j = -a_j, \quad j = 1, 2, 3.} \tag{1.2}$$

Using the arrow visualization, $\mathbf{c}$ is just $\mathbf{a}$ with its direction reversed. See Figure 1.4(b).

<u>*Vector addition:*</u>

Vectors are added by adding each component:

$$\boxed{\mathbf{c} = \mathbf{a} + \mathbf{b}, \quad \text{implies} \quad c_j = a_j + b_j, \quad j = 1, 2, 3.} \tag{1.3}$$

Using the geometrical interpretation of a vector as an arrow, vector addition may be viewed as joining the two vectors $\mathbf{a}$ and $\mathbf{b}$ head to tail. The result $\mathbf{c}$, is the arrow running from the free tail to the free head. See Figure 1.4(c). Note that the tail of $\mathbf{c}$ is where the free tail is, and its head is where the free head is. Check for yourself that adding $\mathbf{a}$ to $\mathbf{b}$, gives the same result as adding $\mathbf{b}$ to $\mathbf{a}$ — i.e. vector addition is commutative.

Combining (1.2) and (1.3) enables us to subtract vectors:

$$\boxed{\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).} \tag{1.4}$$

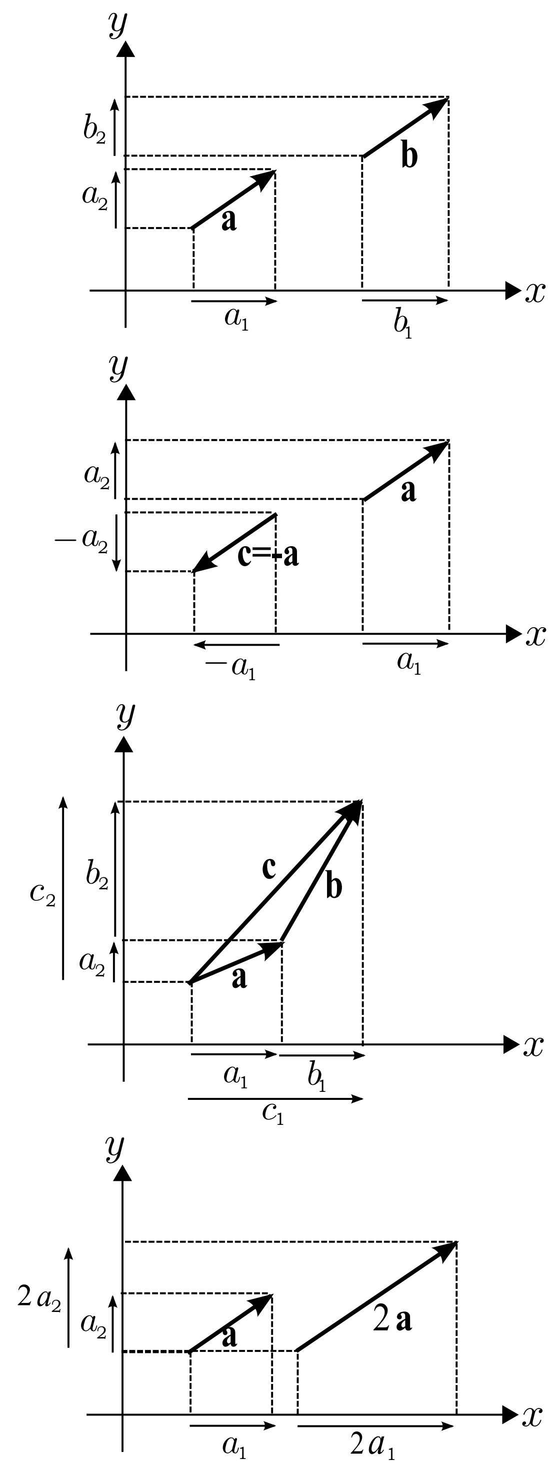

Figure 1.4. Graphical representation of (a) vector equality, (b) the negative of a vector, (c) vector addition, and (d) scalar multiplication of a vector.

Scaling a vector:

A vector multiplied by a scalar is a new vector where every component has been multiplied by that scalar. See Figure 1.4(d).

$$\boxed{\mathbf{c} = \lambda\mathbf{a}, \quad \text{implies} \quad c_j = \lambda a_j, \quad j = 1, 2, 3.} \tag{1.5}$$

We shall also use this opportunity to introduce another convention used in these notes: scalars will be denoted by Greek lower case letters such as α, β, λ, μ, etc.

A vector divided by a scalar is just another way of doing scalar multiplication:

$$\frac{\mathbf{a}}{\lambda} = \lambda^{-1}\mathbf{a}.$$

Here is a summary of the definitions and rules of vector operations, where we have generalized the definitions of the vector operations for vectors in n-dimensions.

DEFINITIONS: VECTOR OPERATIONS

Equality:	$\mathbf{a} = \mathbf{b}$	when	$a_j = b_j$	for all j
$\mathbf{a}$ and $\mathbf{b}$ are both in n dimensions				
Addition:	$\mathbf{c} = \mathbf{a} + \mathbf{b}$	when	$c_j = a_j + b_j$	for all j
$\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are all in n dimensions				
Scalar multiplication:	$\mathbf{b} = \lambda\mathbf{a}$	when	$b_j = \lambda a_j$	for all j

RULES: VECTOR OPERATIONS:

VECTOR ADDITION:	
Commutativity:	$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
Associativity:	$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
MULTIPLICATION BY A SCALAR:	
Commutativity:	$\lambda\mathbf{a} = \mathbf{a}\lambda$
Associativity:	$\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}$
Distributivity over vector addition:	$\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$
Distributivity over scalar addition:	$(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$

1.4 Magnitude and direction

Magnitude

The magnitude or length of vector **a** (more precisely the *Euclidean length*), also called the *norm* of the vector, is expressed as $\|\mathbf{a}\|$ or just a. Note that the letter 'a' is now employed in three different ways relating to the same vector: The vector is **a**, its j-th component is a_j and the magnitude of the vector is a.

For vectors in 2D it is easy to use Pythagoras' theorem to show that the magnitude of the vector is given by

$$\|\mathbf{a}\| = \sqrt{(a_x)^2 + (a_y)^2}. \tag{1.6}$$

For vectors in 3D, Pythagoras' theorem must be applied twice to show that the magnitude of the vector is given by

$$\|\mathbf{a}\| = \sqrt{(a_x)^2 + (a_y)^2 + (a_z)^2}. \tag{1.7}$$

[Do Checkpoint 1.2]

Checkpoint 1.2:

1. If $\mathbf{a} = \begin{bmatrix} 5 \\ -12 \end{bmatrix}$, calculate a.
2. If $\mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$, calculate b.
3. What must c_x be, so that the magnitude of $\mathbf{c} = \begin{bmatrix} 2 \\ 6 \\ c_x \end{bmatrix}$ is 7?
4. Is $a - b$ the same as $\|\mathbf{a} - \mathbf{b}\|$? Supply an example or give a counter example.

Unit vectors of Cartesian coordinates

A *unit vector* is vector of magnitude 1. Unit vectors may be used to represent the axes of a Cartesian coordinate system. The unit vectors in the direction of the x, y, and z axes of a three dimensional Cartesian coordinate system are defined, respectively, as:

The unit vectors:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

When working in 2D (in Cartesian coordinates), we shall use only **i** and **j**, and then the definitions are of course

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Using the notion of vector addition, any vector in 3D can be expressed in terms of these unit vectors, for example

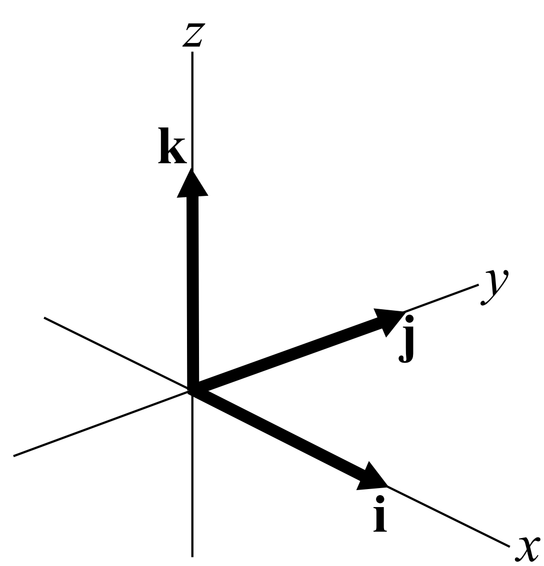

Figure 1.5. Unit vectors of Cartesian coordinates **i**, **j** and **k**.

$$4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} = 4\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}.$$

The two notations (i.e. using a column of components, or expressing it as a sum of scaled unit vectors) are equivalent.

The direction of a vector in 2D:

For the purpose of these notes, we shall use the convention that the direction of a vector in 2D is given by the angle (labeled θ here) between the vector and the positive x-axis. **[See Figure 1.6]** The angle is positive if it is taken counter clockwise from the positive x-axis, and it is negative if it is in a clockwise direction.

By the use of trigonometry, the relationship between the components of a vector and its magnitude and direction is

$$\begin{aligned} a_x &= a\cos\theta \\ a_y &= a\sin\theta. \end{aligned}$$

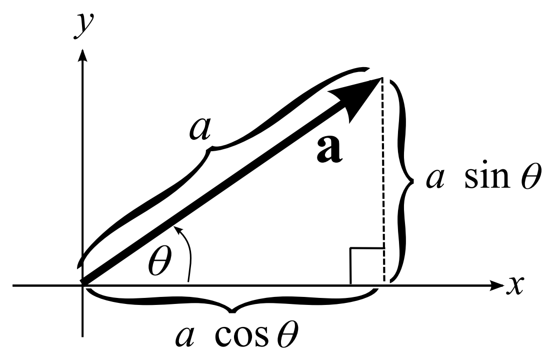

Figure 1.6. The relationship between the direction (θ) and the two components of a vector in 2D.

This means that a position vector that lies in the first quadrant, i.e. its direction satisfies $\theta \in (0, 90^\circ)$, has both components positive. A position vector that lies in the second quadrant, i.e. its direction satisfies $\theta \in (90^\circ, 180^\circ)$, has a_x negative and a_y positive. A position vector that lies in the third quadrant, i.e. its direction satisfies $\theta \in (180^\circ, 270^\circ)$, has both components negative. Lastly, a position vector **a** that lies in the fourth quadrant, i.e. its direction satisfies $\theta \in (270^\circ, 360^\circ)$, has a_x positive and a_y negative.

The direction of a vector in 3D:

For vectors in 3D the direction cannot be associated with a single angle, and often direction vectors, i.e. a unit vector that points in the desired direction, are used. Describing the direction of a 3D vector is discussed elsewhere.

The null vector:

What is $\mathbf{a} - \mathbf{a}$? If we use the arrow interpretation, it must be a vector with no magnitude at all — it is just a point. This vector will be called the *null vector*, and will be denoted by the symbol $\mathbf{0}$. Written out this vector (in 3D) is

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The null vector is the only vector that has no particular direction. All other vectors have specific magnitudes and directions. Geometrically the null vector describes the point at the origin of the axes.

Example 1.1

Two vectors **a** and **b** are shown. [See Figure 1.7]. First add the vectors graphically. Call the result **c**, and find the magnitude of **c** by measuring it, and also find the direction of **c** using a protractor.
Then write down the components of **a** and **b**, and calculate $\mathbf{c} = \mathbf{a} + \mathbf{b}$. Find the direction of **c** as well as its magnitude using the formulae.

Graphical Solution:
The figure [See Figure 1.8] shows the graphical addition of $\mathbf{c} = \mathbf{a} + \mathbf{b}$. Vector **b** is positioned so that its tail touches the arrow head of **a**. Remember that when moving a vector about, you must keep its magnitude and orientation constant. Measuring its length yields $c \approx 5$ and the direction can be measured as $\theta \approx 217^\circ$.

Calculated solution (column notation):
Reading the components off the figure yields

$$\mathbf{a} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}.$$

Then

$$\mathbf{c} = \mathbf{a} + \mathbf{b} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$$

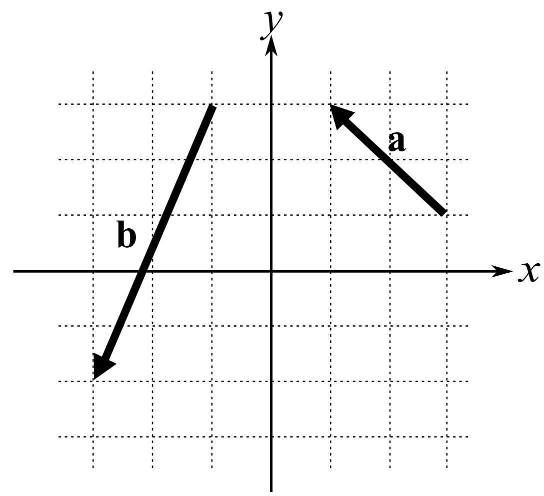

Figure 1.7. Vectors **a** and **b**.

The length of **c** is

$$c = \sqrt{(-3)^2 + (-4)^2} = 5,$$

and its direction is

$$\theta = 180^\circ + \arctan\left(\frac{-3}{-4}\right) = 216.87^\circ \ .$$

Calculated solution (unit vector notation):
Using the unit vectors **i** and **j**, the solution is as follows:

$$\mathbf{a} = -2\mathbf{i} + 2\mathbf{j}, \qquad \mathbf{b} = -2\mathbf{i} - 5\mathbf{j}$$

Then **c** is as follows

$$\begin{aligned}
\mathbf{c} &= (-2\mathbf{i} + 2\mathbf{j}) + (-2\mathbf{i} - 5\mathbf{j}), \\
&\quad \text{... after removing the brackets and rearaging the terms} \\
&= -2\mathbf{i} - 2\mathbf{i} + 2\mathbf{j} - 5\mathbf{j} \\
&= (-2 - 2)\mathbf{i} + (2 - 5)\mathbf{j} \\
&= -4\mathbf{i} - 3\mathbf{j}.
\end{aligned}$$

Then the length is

$$c = \sqrt{(-3)^2 + (-4)^2} = 5,$$

and the direction of **c** is

$$\theta = 180^\circ + \arctan\left(\frac{-3}{-4}\right) = 216.87^\circ \ .$$

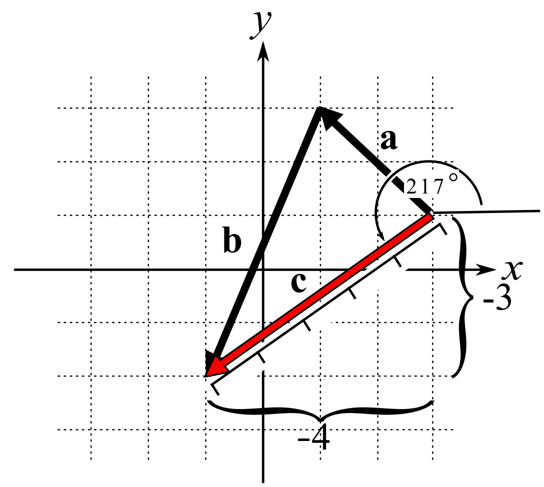

Figure 1.8. Adding **a** and **b** graphically.

1.5 The dot product

The *dot product* also called the *scalar product* or the *inner product* is a special operation between two vectors that returns a scalar as answer. The dot product $\mathbf{a} \cdot \mathbf{b}$ may either be defined geometrically or algebraically. In these notes we shall use the algebraic approach.

The dot product:

The dot product $\mathbf{a} \cdot \mathbf{b}$ of $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

In 2D the dot product is just $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$.

The dot product with unit vectors

It is fundamentally important to first determine the dot products of the unit vectors, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, with each other. This information is used to calculate dot products of general vectors (especially when we shall later work in coordinate systems other than Cartesian). Let us for example check the result of $\mathbf{i} \cdot \mathbf{i}$ as well as $\mathbf{i} \cdot \mathbf{j}$:

$$\mathbf{i} \cdot \mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 + 0 + 0 = 1,$$

$$\mathbf{i} \cdot \mathbf{j} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 + 0 + 0 = 0.$$

It should be clear that in general the dot product of any unit vector with itself is one, and the dot product between two orthogonal unit vectors is zero. (This is only the case in special coordinate systems that are called *orthogonal*, however, we shall not further explain this technicality at this stage.) The full set of rules for dot products of the unit vectors is as follows:

Dot products of i, j and k:

$$\begin{aligned} \mathbf{i} \cdot \mathbf{i} &= 1 \qquad & \mathbf{i} \cdot \mathbf{j} &= 0 \\ \mathbf{j} \cdot \mathbf{j} &= 1 & \mathbf{j} \cdot \mathbf{k} &= 0 \\ \mathbf{k} \cdot \mathbf{k} &= 1 & \mathbf{k} \cdot \mathbf{i} &= 0 \end{aligned}$$

Distributivity of the dot product over addition:

The dot product is distributive over vector addition,

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}. \tag{1.8}$$

Confirm this yourself with a simple example or do the checkpoint. **[Do Checkpoint 1.3]**

When expressing a vector in the unit vector notation, these identities together with the distributivity rule are used to calculate the dot product.

Angles between vectors

There is a close relationship between the dot product and the angle between two vectors. Consider the two vectors **a** and **b** in Figure 1.9. If we join the arrow heads of **a** and **b** then it completes a triangle. This is true in both 2D and 3D. The completing line segment is itself a vector — it is $\mathbf{a}-\mathbf{b}$. We can calculate the lengths of all three vectors. The cosine rule for triangles may then be used to find the angle between two sides if we know the lengths of all three sides of the triangle.

Let the angle between **a** and **b** be θ. Then

$$\begin{aligned}
\|\mathbf{a}-\mathbf{b}\|^2 &= a^2+b^2-2ab\cos\theta \\
(a_x-b_x)^2+(a_y-b_y)^2+(a_z-b_z)^2 &= a_x^2+a_y^2+a_z^2+b_x^2+b_y^2+b_z^2-2ab\cos\theta \\
&\text{....... Cancelling similar terms on both sides.} \\
-2(a_xb_x+a_yb_y+a_zb_z) &= -2ab\cos\theta \\
a_xb_x+a_yb_y+a_zb_z &= ab\cos\theta
\end{aligned}$$

Notice that the expression on the left hand side is a scalar quantity that we have seen before — it is the dot product between **a** and **b**:

$$\mathbf{a}\cdot\mathbf{b} = a_xb_x+a_yb_y+a_zb_z. \tag{1.9}$$

We therefore have the following convenient relationship between the dot product of two vectors and the angle between them:

Dot product of two vectors:

Let θ be the angle between **a** and **b** (both are in 3D). Then

$$\mathbf{a}\cdot\mathbf{b} = ab\cos\theta.$$

In some texts the above statement is used as the definition of the dot product as it has an interesting geometric interpretation. The disadvantage of this definition is that it difficult to extend to more than three dimensions and also it is cumbersome to use it to prove some of the properties of a dot product.

[Do Checkpoint 1.4]

The dot product provides a convenient notation to express the length of a vector. Since the angle between **a** and itself

Checkpoint 1.3:

Let

$$\mathbf{a}=\begin{bmatrix}a_1\\a_2\end{bmatrix},\quad \mathbf{b}=\begin{bmatrix}b_1\\b_2\end{bmatrix},\quad \mathbf{c}=\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

Add **b** and **c** first and then multiply out

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}).$$

Alternatively, multiply out $\mathbf{a}\cdot\mathbf{b}$ as well as $\mathbf{a}\cdot\mathbf{c}$ and then add them. Is the result the same?

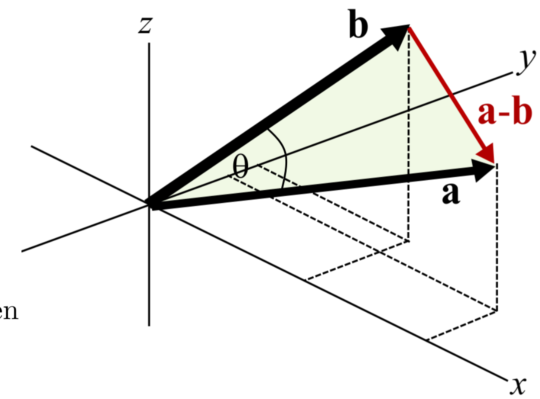

Figure 1.9. Calculating the angle θ between **a** and **b**.

Checkpoint 1.4:

1. Find the angle between $\mathbf{a}=\mathbf{i}+\mathbf{j}$ and $\mathbf{b}=\mathbf{i}$. Also check your answer geometrically.
2. Find the angle between $\mathbf{a}=\mathbf{i}+\mathbf{j}$ and $\mathbf{b}=-\mathbf{i}$. Also check your answer geometrically.

is 0° , and $\cos 0^\circ = 1$,

$$\mathbf{a} \cdot \mathbf{a} = a^2.$$

When calculating expressions that involve the length of a vector, it is often convenient to rather work with the length squared and to express it as the dot product of the vector with itself.

Therefore the length of a vector may be expressed as

$$a = \sqrt{\mathbf{a} \cdot \mathbf{a}}. \tag{1.10}$$

It now makes sense that, for example $\mathbf{i} \cdot \mathbf{i} = 1$, since this product simple gives the length-squared of the unit vector. Also is makes sense that the dot product between two different (orthogonal) unit vectors must be zero because they have a 90° angle between them.

Example 1.2

Calculate the dot product between **a** and **b** and also calculate the angle between them.

$$\mathbf{a} = \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Solution (Column notation):

$$\mathbf{a} \cdot \mathbf{b} = \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = (3)(2) + (-2)(1) + (6)(2) = 16$$

$$\|\mathbf{a}\| = \sqrt{3^2 + (-2)^2 + 6^2} = 7$$

$$\|\mathbf{b}\| = \sqrt{2^2 + 1^2 + 2^2} = 3.$$

Therefore

$$3 \times 7 \times \cos\theta = 16$$

and therefore $\theta = 40.37°$.

Solution (Unit vector notation):

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \\ &= (3)(2)\mathbf{i} \cdot \mathbf{i} + (3)(1)\mathbf{i} \cdot \mathbf{j} + (3)(2)\mathbf{i} \cdot \mathbf{j} + (-2)(2)\mathbf{j} \cdot \mathbf{i} + (-2)(1)\mathbf{j} \cdot \mathbf{j} \\ &\quad +(-2)(2)\mathbf{j} \cdot \mathbf{k} + (6)(2)\mathbf{k} \cdot \mathbf{i} + (6)(1)\mathbf{k} \cdot \mathbf{j} + (6)(2)\mathbf{k} \cdot \mathbf{k} \\ &\text{.... remember that } \mathbf{i} \cdot \mathbf{j} = 0\text{, etc, therefore most of the terms are multiplied by zero.} \\ &= 6 + 0 + 0 + 0 + (-2) + 0 + 0 + 0 + 12 \\ &= 16 \end{aligned}$$

The rest follows as above.

Orthogonality:

An important special case is when the dot product between two vectors is zero – the vectors are said to be *orthogonal*. A zero dot product means the angle between the vectors is 90° , i.e. the two vectors are perpendicular in space.

Can three vectors in 3D be mutually perpendicular? Geometrically the vectors are like three sides of a cube meeting at the vertex of the cube.[See Figure 1.10] When a number of vectors are mutually perpendicular, we say that they form an *orthogonal* set.

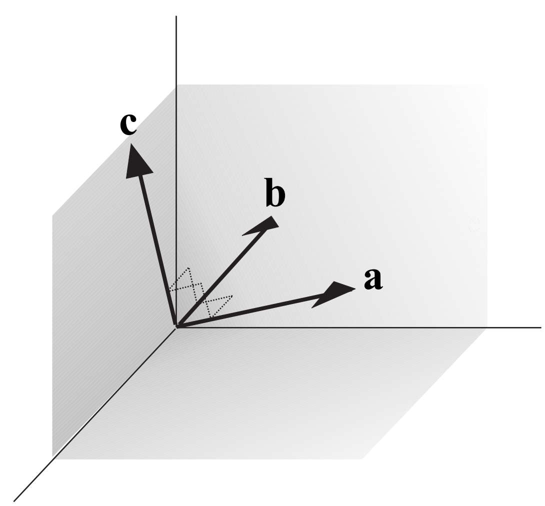

Figure 1.10. A set of three orthogonal vectors in 3D.

Expressed in symbols, this is

$$\boxed{\begin{array}{l} \underline{\textbf{Orthogonal vectors:}} \\ \{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \ \textit{is an orthogonal set, if} \\ \quad \mathbf{a} \cdot \mathbf{b} \ = 0, \\ \quad \mathbf{b} \cdot \mathbf{c} \ = 0, \\ \quad \mathbf{c} \cdot \mathbf{a} \ = 0. \end{array}} \tag{1.11}$$

Think of the unit vectors **i**, **j**, and **k**. They are certainly mutually perpendicular. Check for yourself by calculating the dot products between them. **[Do Checkpoint 1.5]**

Rules for dot products

Let us formally define the dot product in n dimensions:

DEFINITION: DOT PRODUCT	
Dot product:	$\mathbf{a} \cdot \mathbf{b} = \sum_{j=1}^{n} a_j b_j$
a and **b** are both in n-dimensions	

Check for yourself that the following rules are correct:

RULES: DOT PRODUCT	
Commutativity:	$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
Associativity with scalar multiplication:	$\mathbf{a} \cdot (\lambda \mathbf{b}) = (\mathbf{a}\lambda) \cdot \mathbf{b}$
Distributivity over addition:	$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

Checkpoint 1.5:

Which of the following vectors are orthogonal?

(a) $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -10 \\ 5 \end{bmatrix}$

(b) $\mathbf{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{d} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(c) $\mathbf{p} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$, and $\mathbf{q} = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$

(c) Find any vector that is perpendicular to $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$.

Example 1.3

Vector $\mathbf{a}$ has length a, and vector $\mathbf{b}$ has twice the length of $\mathbf{a}$, but they point in different directions. The length of $\mathbf{a}+\mathbf{b}$ is $\sqrt{8}a$. Find the angle between $\mathbf{a}$ and $\mathbf{b}$.
Solution:
We can write $\|\mathbf{a}\| = a$ and $\|\mathbf{b}\| = 2a$.
The length squared of $\mathbf{a}+\mathbf{b}$ is

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) &= \mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}\\ &= a^2+2ab\cos\theta+b^2\\ &= a^2+2a(2a)\cos\theta+(2a)^2\\ &= 5a^2+4a^2\cos\theta\end{aligned}$$

But it is given that the length squared of $\mathbf{a}+\mathbf{b}$, is $(\sqrt{8}a)^2$, therefore

$$5a^2+4a^2\cos\theta = 8a^2$$

and then

$$\cos\theta = \frac{3}{4}, \quad \text{i.e.} \quad \theta = 41.1^\circ\ .$$

1.6 Points and vectors

The position of any point with respect to a predefined coordinate system, may be denoted by a position vector pointing from the origin of the coordinate system to the point. The position vectors of two points A and B with coordinates $A(x_A, y_A)$ and $B(x_B, y_B)$ are given by $\mathbf{r}_A = x_A\mathbf{i}+y_A\mathbf{j}$ and $\mathbf{r}_B = x_B\mathbf{i}+y_B\mathbf{j}$, respectively. In the more general case, a vector may be directed from point A to point B designated by $\mathbf{r}_{AB}$. From the head-to-tail vector addition,

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{AB}.$$

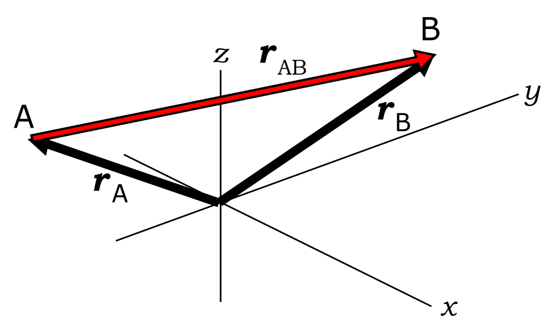

Figure 1.11. The coordinates of points A and B can be used to find the vector $\mathbf{r}_{AB}$, directed from A to B.

Solving for $\mathbf{r}_{AB}$ and expressing $\mathbf{r}_A$ and $\mathbf{r}_B$ as Cartesian vectors yield

$$\mathbf{r}_{AB} = \mathbf{r}_B - \mathbf{r}_A = (x_B\mathbf{i}+y_B\mathbf{j}) - (x_A\mathbf{i}+y_A\mathbf{j})$$

or, more conveniently,

$$\mathbf{r}_{AB} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j}. \qquad (1.12)$$

The components of the position vector, $\mathbf{r}_{AB}$ from A to B may be formed by taking the coordinates of the tail of the vector $A(x_A, y_A)$ and subtracting them from the corresponding coordinates of the head of the vector $B(x_B, y_B)$. This applies equally to vectors in 3D.

The position vectors of points A and B can also be denoted by $\mathbf{r}_{OA}$ and $\mathbf{r}_{OB}$, but the O is often omitted when the starting point of the vector is at the origin.

1.7 Directions in 3D

For vectors in 2D, the direction is given by a single angle. As mentioned before, a single angle is not sufficient for describing the direction of a vector in 3D. Although the direction of $\mathbf{a}$ in 3D may be given by *two* angles, it is often more convenient to describe the direction of a vector in 3D by using a normalized vector that points in the same direction as $\mathbf{a}$. Such a normalized vector is called a *unit* vector.

Arbitrary unit vectors

We have used the term *unit vectors* to refer to the vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. However, the term *unit vector* is also is used to describe any vector whose length is *one*.

In order to specify a certain direction in 3D, the unit vector pointing in the desired direction is the best way to capture the direction. Let $\mathbf{a}$ be a vector in 3D. We may use $\mathbf{a}$ and construct a unit vector, denoted by $\widehat{\mathbf{a}}$, pointing in the same direction as $\mathbf{a}$, as follows

$$\widehat{\mathbf{a}} = \frac{\mathbf{a}}{a}. \tag{1.13}$$

The vector $\widehat{\mathbf{a}}$ points in the same direction as $\mathbf{a}$, but is is scaled so that its length is 1. Scaling a vector so that its length is 1, is called *normalizing* the vector. In these notes the convention will be used to denote normalized vectors by the same symbol as the vector from which it was calculated, but with a hat over the symbol. **[Do Checkpoint 1.6]**

Checkpoint 1.6:

Let $\mathbf{a} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

1. Construct $\widehat{\mathbf{a}}$ and $\widehat{\mathbf{b}}$.
2. Construct a vector $\mathbf{c}$ that points in the same direction as $\mathbf{a}$ but has length 15.

Other vectors with the same direction as $\mathbf{a}$ but possibly different lengths may now be expressed easily by using $\widehat{\mathbf{a}}$. If $\mathbf{b}$ is a vector of length μ pointing in the direction of $\mathbf{a}$, then $\mathbf{b} = \mu\widehat{\mathbf{a}}$.

Direction cosines:

The three components of the unit vector $\hat{\mathbf{a}}$ are called the *direction cosines* of the vector $\mathbf{a}$. Let us briefly explain where this naming convention comes from.

Consider the unit vector $\mathbf{n}$ (whose length is 1) shown in Figure 1.12. Its three components are n_x, n_y and n_z, and the angles between the vector and the primary axes x, y, and z are respectively given by α, β, and γ.

By dropping a perpendicular line segment from the tip of $\mathbf{n}$ to the x-axis, we can complete the right angled triangle shown in light blue. Its hypotenuse has length 1, and the length of the side along the x-axis is n_x. The angle between these two edges is α. It is then obvious that $\cos\alpha = n_x$. A similar relationship concerning the angle between the vector and the primary axes can be derived. We therefore have that

$$\begin{aligned} n_x &= \cos\alpha \\ n_y &= \cos\beta \\ n_z &= \cos\gamma. \end{aligned}$$

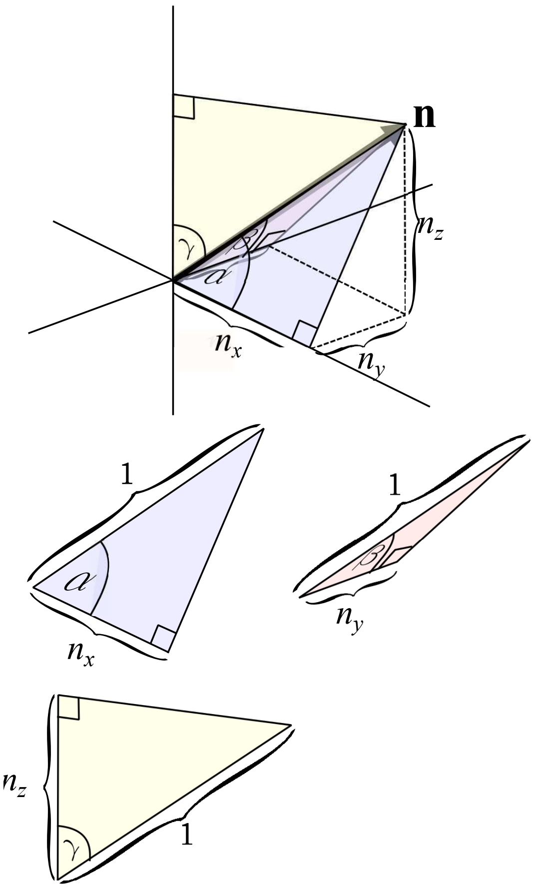

Figure 1.12. Explanation of the *direction cosines* associated with a unit vector $\mathbf{n}$.

1.8 Forces in 3D

Up to now we have used vectors to describe positions in space. Another important use of vectors is to describe *forces*. This section is included in order to illustrate how 3D unit vectors are used in real applications.

At this stage we shall treat a force as an abstract concept. It "pulls" with a certain strength in a certain direction. It is invisible, but its effects are not. It is measured in *newtons* with unit N.

In some applications of statics, cables or strings are used to apply forces to an object to keep it in place. Think for example of a telecommunications tower anchored to the ground with cables. A fundamental property of a cable is that the force in the cable always acts *along* the cable. The spatial vector along the cable therefore supplies its direction, but its magnitude is obtained in a different way. We shall explain this concept with some examples.

Example 1.4

Figure 1.13 shows a flat horizontal plank held in place by two cables. The tension in cable OA is 100 N, and in cable OB is 150 N. Find the vector form of the combined force on the peg at O as well as its direction.

<u>Solution:</u>

Denote the position vectors from O to A by $\mathbf{a}$ and from O to B by $\mathbf{b}$. Using the dimensions given in the figure,

$$\mathbf{a} = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}$$

Then the directions are respectively

$$\hat{\mathbf{a}} = \frac{1}{5}\begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{b}} = \frac{1}{\sqrt{29}}\begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}$$

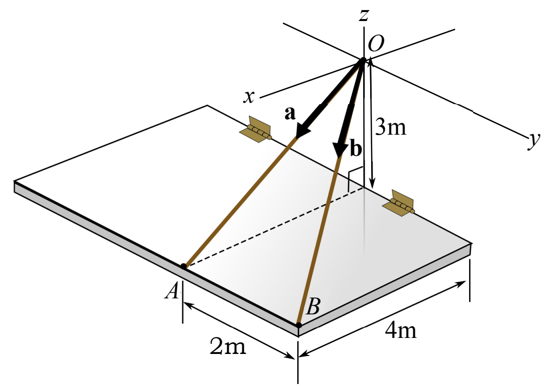

Figure 1.13. The setup for Example 1.4.

Let $\mathbf{F}_A$ be the force in cable OA and $\mathbf{F}_B$ be the force in cable OB. Then

$$\mathbf{F}_A = 100 \times \frac{1}{5} \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 80 \\ 0 \\ -60 \end{bmatrix}$$

and

$$\mathbf{F}_B = 150 \times \frac{1}{\sqrt{29}} \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 111.42 \\ 55.71 \\ -83.56 \end{bmatrix}$$

Let the combined force be $\mathbf{F}$, then

$$\mathbf{F} = \mathbf{F}_A + \mathbf{F}_B = \begin{bmatrix} 80 \\ 0 \\ -60 \end{bmatrix} + \begin{bmatrix} 111.42 \\ 55.71 \\ -83.56 \end{bmatrix} = \begin{bmatrix} 191.42 \\ 55.77 \\ -143.56 \end{bmatrix}$$

The magnitude of $\mathbf{F}$ is

$$F = \sqrt{191.42^2 + 55.77^2 + 143.56^2} = 245.69 \text{ N},$$

and the direction of $\mathbf{F}$ is

$$\frac{1}{245.69} \begin{bmatrix} 191.42 \\ 55.77 \\ -143.56 \end{bmatrix} = \begin{bmatrix} 0.78 \\ 0.23 \\ -0.58 \end{bmatrix}.$$

1.9 The cross product

The dot product of vectors was introduced in section 1.5. We shall now introduce another product between vectors that is very useful. It is called the *cross product*, and it produces a vector as result (unlike the dot product that produces a scalar). It is therefore also sometimes called the *vector product*.

Definition of the cross product:

The cross product is defined only for vectors with three components. (You may recall that the dot product is defined for vectors with any number of components). Consider the vectors $\mathbf{a}$ and $\mathbf{b}$ below.

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

The cross product is defined as follows:

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

Orthogonality of the cross product:

The vector $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ has a unique (and very useful) property: $\mathbf{c}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. One may also view it as follows: $\mathbf{c}$ is perpendicular to the plane in which both $\mathbf{a}$ and $\mathbf{b}$ lie.

This is not difficult to verify: Just confirm that the dot products $\mathbf{a} \cdot \mathbf{c}$ and $\mathbf{b} \cdot \mathbf{c}$ are both zero.

$$\begin{aligned} \mathbf{a} \cdot \mathbf{c} &= a_1c_1 + a_2c_2 + a_3c_3 \\ &= a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \mathbf{b} \cdot \mathbf{c} &= b_1c_1 + b_2c_2 + b_3c_3 \\ &= b_1(a_2b_3 - a_3b_2) + b_2(a_3b_1 - a_1b_3) + b_3(a_1b_2 - a_2b_1) \\ &= 0 \end{aligned}$$

In space we should therefore view $\mathbf{c}$ as a vector that is *normal* to the plane defined by $\mathbf{a}$ and $\mathbf{b}$. Figure 1.14 shows the configuration of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ in space.

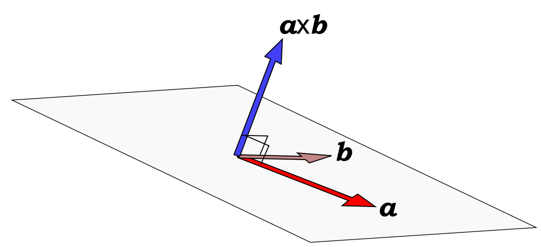

Figure 1.14.

The vector $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ is not the only vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ — in fact, any multiple of $\mathbf{c}$, including negative multiples, are also perpendicular to this plane. There are now two questions that arise: (1) in which direction must $\mathbf{c}$ point, up or down, and (2) what is the length of $\mathbf{c}$?

The right hand rule:

The answer to the first question is given by the so called *right hand rule*. It states that if you curl the fingers of your right hand from $\mathbf{a}$ (the first factor in the product) to $\mathbf{b}$ (the second factor in the product), then your thumb points in the direction of $\mathbf{c}$. This is illustrated in Figure 1.15

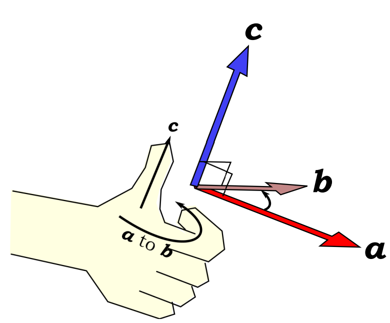

Figure 1.15.

This immediately implies a new caution that must be taken: the cross product *is not commutative*, i.e. the order in which we take $\mathbf{a} \times \mathbf{b}$ or $\mathbf{b} \times \mathbf{a}$ matters. In fact it is easy to show that

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}.$$

The length of $\mathbf{a} \times \mathbf{b}$:

It will be shown later that, if the angle between $\mathbf{a}$ and $\mathbf{b}$ is θ then the length of $\mathbf{a} \times \mathbf{b}$ is as follows:

$$\|\mathbf{a} \times \mathbf{b}\| = ab \sin\theta.$$

The presence of the $\sin\theta$ factor in this result ought to make sense. If we consider colinear vectors, i.e. if the angle between $\mathbf{a}$ and $\mathbf{b}$ is zero, then such vectors do not define a plane, and there cannot be a unique vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. The cross product for two colinear vectors therefore produces the zero vector, regardless of their lengths.

Calculating the cross product (column method):

It may be difficult to remember the specific components of the cross product – but there is a cyclic pattern to it. Here is a quick way to do the cross product: Write down $\mathbf{a} \times \mathbf{b}$ in column form:

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \quad\quad\quad\quad \\ \\ \\ \end{bmatrix}$$

Cover the first row in the cross product, so that only the next two rows are visible. Then do a 'cross-over product' as follows: multiply a_2 with b_3 and a_3 with b_2, and then subtract them. Note that the first element to pick (i.e. a_2), is the one in the leftmost column just below the cover.

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - b_2 a_3 \\ \\ \\ \end{bmatrix}$$

Then move the cover down so that it covers the second row. Start with the element immediately below the cover on the left (i.e. a_3) and do the 'cross-over product', i.e. $a_3 b_1 - a_1 b_3$.

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - b_2 a_3 \\ a_3 b_1 - a_1 b_3 \\ \\ \end{bmatrix}$$

Then move the cover down again so that it covers the third row. There is no element on the left just below the cover, but if one follows the cyclic nature of the cross product, this element is the first element on the left, i.e. a_1. Once again do the 'cross-over product', i.e. $a_1 b_2 - a_2 b_1$.

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - b_2a_3 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

Calculation of the cross product (determinant method):

Most handbooks recommend a different method for calculating the cross product. It, however, requires that you already know how to calculate the determinant of a 3×3 matrix. It is as follows: Set up a 'matrix' containing $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in the first row, write $\mathbf{a}$ (as a row) in the second row, and $\mathbf{b}$ in the third row,

$$\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}.$$

(We shall require of you to ignore the inconsistency of a 'matrix' containing both vectors and scalars as elements – consider it to be just a convenient method for establishing a recipe for the cross product.)

The cross product between $\mathbf{a}$ and $\mathbf{b}$ then is the 'determinant' of this matrix:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

[Do Checkpoint 1.7]

1.10 Rules for the cross product

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

The cross product is defined as follows:

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

Rules for the cross product

Let us formally define the cross product in 3 dimensions:

Checkpoint 1.7:

Calculate the following:

(a)

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} -4 \\ 1 \\ 5 \end{bmatrix}$$

(b)

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} 2a \\ b \\ c \end{bmatrix}$$

(c)

$$\mathbf{i} \times \mathbf{j}$$

Ans: (a) [7, -17, 9], (b) $[0, ac, -ab]$, (c) $\mathbf{k}$

DEFINITION: CROSS PRODUCT		
(DEF)	**Cross product:**	$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$
	$\mathbf{a}$ and $\mathbf{b}$ are both in 3-dimensions only	

Check for yourself that the following rules are correct:

RULES: CROSS PRODUCT		
(RULE 1)	**Non-commutativity:**	$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
(RULE 2)	**Associativity with scalar multiplication:**	$\mathbf{a} \times (\lambda \mathbf{b}) = (\mathbf{a}\lambda) \times \mathbf{b}$
(RULE 3)	**Distributivity over addition:**	$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
(RULE 4)	**Relationship to θ (angle between a and b):**	$\|\mathbf{a} \times \mathbf{b}\| = ab \sin\theta$
(RULE 5)	**Vector triple product:**	$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

1.11 The cross product with Cartesian unit vectors

Using the definition of a cross product it is easy to show the following cyclic pattern:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Of course

$$\mathbf{i} \times \mathbf{i} = \mathbf{0}, \quad \mathbf{j} \times \mathbf{j} = \mathbf{0}, \quad \mathbf{k} \times \mathbf{k} = \mathbf{0},$$

since the cross product between two colinear vectors is the zero vector.

Let us summarize:

Cross products of i, j and k:

$$\begin{array}{llllll}
\mathbf{i} \times \mathbf{j} & = \mathbf{k} & \mathbf{j} \times \mathbf{i} & = -\mathbf{k} & \mathbf{i} \times \mathbf{i} & = \mathbf{0} \\
\mathbf{j} \times \mathbf{k} & = \mathbf{i} & \mathbf{k} \times \mathbf{j} & = -\mathbf{i} & \mathbf{j} \times \mathbf{j} & = \mathbf{0} \\
\mathbf{k} \times \mathbf{i} & = \mathbf{j} & \mathbf{i} \times \mathbf{k} & = -\mathbf{j} & \mathbf{k} \times \mathbf{k} & = \mathbf{0}
\end{array}$$

This result may also be used together with RULE 3 to calculate the cross product.

1.12 Derivation of the cross product

Where did the formula for the cross product come from? The following section shows one way in which it can be derived from first principles.

Consider

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

<u>*Orthogonality:*</u>

We shall first derive the formula for the cross product by assuming that (1) both **a** and **b** are perpendicular to **c**, and (2) the formula must not contain fractions of the elements of **a** and **b**.

Let us write $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ as

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Note that we consider the components as **a** and **b** as given and known, but we consider the c_1, c_2 and c_3 as unknown.

If **c** is perpendicular to both **a** and **b** then

$$\mathbf{a} \cdot \mathbf{c} = 0, \qquad \text{and} \qquad \mathbf{b} \cdot \mathbf{c} = 0.$$

Let us write out these two equations:

$$\begin{aligned} a_1c_1 + a_2c_2 + a_3c_3 \quad &= 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1) \\ b_1c_1 + b_2c_2 + b_3c_3 \quad &= 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (2) \end{aligned}$$

This is a system of two equations in three unknowns, c_1, c_2 and c_3. Let us start by eliminating c_3 from (1) and (2). Multiply (1) by b_3 and multiply (2) by a_3, and then subtract the resulting equations:

$$\begin{aligned} b_3a_1c_1 + b_3a_2c_2 + b_3a_3c_3 \quad &= 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1') \\ a_3b_1c_1 + a_3b_2c_2 + a_3b_3c_3 \quad &= 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (2') \end{aligned}$$

$$(2') - (1') : \quad a_3b_1c_1 + a_3b_2c_2 - b_3a_1c_1 - b_3a_2c_2 = 0$$

or

$$c_2 = \frac{a_3b_1 - a_1b_3}{a_2b_3 - a_3b_2} c_1$$

One may choose c_1 to be any arbitrary nonzero number, however, if you do not want fractions in the formula, it is best to choose

$$c_1 = a_2b_3 - a_3b_2$$

and then

$$c_2 = a_3b_1 - a_1b_3.$$

Substitute these expressions for c_1 and c_2 back into equation (1) and solve for c_3:

$$a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3c_3 = 0,$$

or

$$c_3 = a_1b_2 - a_2b_1.$$

To summarize:

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

This provides a formula for a vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. We still do not know whether it satisfies the right hand rule and we do not know what its length is.

The length of $\mathbf{a} \times \mathbf{b}$*:*

Consider $\|\mathbf{c}\|^2$:

$$\begin{aligned} \|\mathbf{c}\|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\ &= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 \\ &\quad +a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 \\ &\qquad +a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \end{aligned} \tag{1.14}$$

Collecting terms gives

$$\|\mathbf{c}\|^2 = a_1^2(b_2^2+b_3^2)+a_2^2(b_1^2+b_3^2)+a_3^2(b_1^2+b_2^2)-2(a_2a_3b_2b_3+a_1a_3b_1b_3+a_1a_2b_1b_2)$$

The last term may not be well known, but it may be worth noting that it also appears in the square of the dot product:

$$\begin{aligned} (\mathbf{a} \cdot \mathbf{b})^2 &= (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ &= a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + 2(a_2a_3b_2b_3 + a_1a_3b_1b_3 + a_1a_2b_1b_2) \end{aligned}$$

therefore

$$-2(a_2a_3b_2b_3+a_1a_3b_1b_3+a_1a_2b_1b_2) = a_1^2b_1^2+a_2^2b_2^2+a_3^2b_3^2-(\mathbf{a}\cdot\mathbf{b})^2.$$

Substituting this expression into (1.14) gives

$$\begin{aligned} \|\mathbf{c}\|^2 &= a_1^2(b_2^2 + b_3^2) + a_2^2(b_1^2 + b_3^2) + a_3^2(b_1^2 + b_2^2) + a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\ &= a_1^2(b_1^2 + b_2^2 + b_3^2) + a_2^2(b_1^2 + b_2^2 + b_3^2) + a_3^2(b_1^2 + b_2^2 + b_3^2) - (\mathbf{a} \cdot \mathbf{b})^2 \\ &= a^2b^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\ &= a^2b^2 - (ab\cos\theta)^2 \\ &= a^2b^2(1 - \cos^2\theta) \\ &= a^2b^2\sin^2\theta \end{aligned}$$

and therefore

$$\|\mathbf{c}\| = ab\sin\theta.$$

This result is often also stated as follows:

$$\mathbf{a} \times \mathbf{b} = ab \sin\theta\, \hat{\mathbf{c}}$$

where $\hat{\mathbf{c}}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

Example 1.5

Consider the following two vectors. Find the angle between them twice: (a) by using the dot product, and (b) by using the cross product.

$$\mathbf{a} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}.$$

SOLUTION:

$$a = \sqrt{1^2 + 2^2 + 2^2} = 3, \quad b = \sqrt{4^2 + 2^2 + 4^2} = 6,$$

(a)

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} = 4 + 4 + 8 = 16 = 3 \times 6 \times \cos(\theta)$$

therefore

$$\theta = \arccos(\frac{16}{18}) = 27.27^\circ\ .$$

(b)

$$\mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \times \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -6 \end{bmatrix}$$

and

$$\|\mathbf{c}\| = \sqrt{4^2 + 4^2 + (-6)^2} = \sqrt{68}$$

then

$$\sqrt{68} = 3 \times 6 \times \sin\theta$$

and therefore

$$\theta = \arcsin(\frac{\sqrt{68}}{18}) = 27.27^\circ\ .$$