Week 6 - Topic 2: Length and Angle: The Dot Product

Math 144

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Class content

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How to study math: Math comes easier to some students and harder to others. Whatever your level of ability, even if you don't like math, there are some things you can do to improve your performance, reduce your anxiety level, and succeed in your math class.

In many ways you can approach math like sports:

- Train with a buddy. If you possibly can, get a study buddy or form a study group, and meet regularly at least two or three times a week. Very often one of you will be able to help the other one with a problem.
- Deal with cramps right away. When there's something you don't understand, you may be tempted to just put it aside and hope for the best. That strategy doesn't work at all in math! Because everything builds, if you don't understand A you will probably not understand B and C either.
- If this happens in class, ask a question right away. Don't apologize and don't worry about looking stupid; probably other people have exactly the same question.
- Outside of class, use the tutors or talk through the problem with your study buddy. Visit your instructor during office hours or make an appointment for another time.
- Warm up before the event. Before class, look back over the readings and your homework. Make sure you are ready with any questions.
- Stay in training. Review your notes after class, even rewrite them to make sure you understand everything. If there are several days between classes, review the material at least every other day to keep it fresh in your mind.
- Make sure you train enough. The College recommends 2-3 hours per classroom hour, but you may need more.
- Recognize especially that in the summer you're making a big time commitment. If your class meets five days a week for an hour and 40 minutes a day, that's two 50-minute classroom hours, which means you should expect to study 4-6 hours per night, every night.

The Dot Product

It is quite easy to reformulate the familiar geometric concepts of length distance and angle in terms of vectors. The vector versions of length, distance, and angle can all be described using the notion of the dot product of two vectors.

Definition

Let $\mathbf{u} = \begin{bmatrix} \mathbf{u_1} \\ \mathbf{u_2} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{u_n} \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} \mathbf{v_1} \\ \mathbf{v_2} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{v_n} \end{bmatrix}$ then the dot product $\mathbf{u} \cdot \mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u_1 v_1} + \mathbf{u_2 v_2} + \cdots + \mathbf{u_n v_n}.$$

The Dot Product

Remark

1. $\mathbf{u}$ and $\mathbf{v}$ must have the same number of components for the dot product to be defined;
2. the dot product is a number/scalar and not a vector ($\mathbf{u} \cdot \mathbf{v}$ sometimes called the scalar product of $\mathbf{u}$ and $\mathbf{v}$).

Exercises

Find $\mathbf{u} \cdot \mathbf{v}$, when $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$.

Solution:

$\mathbf{u} \cdot \mathbf{v} = (3)(4) + (-2)(6) = 12 - 12 = 0$

Exercises

Find $\mathbf{u} \cdot \mathbf{v}$, when $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$.

Solution:

$\mathbf{u} \cdot \mathbf{v} = (1)(2) + (2)(3) + (3)(1) = 2 + 6 + 3 = 11$

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The Dot Product

Theorem

Let **u**, **v** and **w** be vectors in $\mathbb{R}^n$ and let c be a scalar. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (commutativity);
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (distributivity);
3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$;
4. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

The Dot Product

Proof

Let **u**, **v** and **w** be vectors in $\mathbb{R}^n$ and let c be a scalar. Then

1. and 3. are done in the book.
2. Since **u**, **v** and $\mathbf{w} \in \mathbb{R}^n$. Let $\mathbf{u} = [\mathbf{u_1}, \mathbf{u_2}, \cdot, \mathbf{u_n}]$, $\mathbf{v} = [\mathbf{v_1}, \mathbf{v_2}, \cdot, \mathbf{v_n}]$ and $\mathbf{w} = [\mathbf{w_1}, \mathbf{w_2}, \cdot, \mathbf{w_n}]$.

Proof

Let **u**, **v** and **w** be vectors in $\mathbb{R}^n$ and let c be a scalar. Then

1. and 3. are done in the book.
2. Using row vectors instead,

$$\begin{aligned}
& \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) \\
= \; & [\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_n}] \cdot ([\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}] + [\mathbf{w_1}, \mathbf{w_2}, \cdots, \mathbf{w_n}]) \\
= \; & [\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_n}] \cdot [\mathbf{v_1} - \mathbf{w_1}, \mathbf{v_2} - \mathbf{w_2}, \cdots, \mathbf{v_n} - \mathbf{w_n}] \\
= \; & \mathbf{u_1}(\mathbf{v_1} + \mathbf{w_1}) + \mathbf{u_2}(\mathbf{v_2} + \mathbf{w_2}) + \cdots + \mathbf{u_n}(\mathbf{u_n} + \mathbf{w_n}) \\
= \; & \mathbf{u_1 v_1} + \mathbf{u_2 v_2} + \cdots + \mathbf{u_n v_n} \\
& + \mathbf{u_1 w_1} + \mathbf{u_2 w_2} + \cdots + \mathbf{u_n w_n} \\
= \; & \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}
\end{aligned}$$

Proof

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c be a scalar. Then

3. (a) Let $\mathbf{u} \in \mathbb{R}^n$, then

$$\begin{aligned} \mathbf{u} \cdot \mathbf{u} &= \mathbf{u_1 u_1} + \mathbf{u_2 u_2} + \cdots + \mathbf{u_n u_n} \\ &= \mathbf{u_1}^2 + \mathbf{u_2}^2 + \cdots + \mathbf{u_n}^2 \geq 0 \end{aligned}$$

since $\mathbf{u_i}^2 \geq 0$ this implies $\mathbf{u_i} = 0$, for all $i \in [[1, n]]$. Therefore $\mathbf{u} = \mathbf{0}$.
The converse is true. Indeed, when $\mathbf{u} = 0$, then $\mathbf{u} \cdot \mathbf{u} = 0$.

Remark

1. Property 2. can be read from right to left

$$\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}=\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})$$

   in other words, **u** as a common vector can be factored out from a sum of dot products. Also, the distributivity is also true when the dot product is applied on the right

$$(\mathbf{v}+\mathbf{w})\cdot\mathbf{u}=\mathbf{v}\cdot\mathbf{u}+\mathbf{w}\cdot\mathbf{u}.$$

   This is a result of the commutativity of the dot product property 1. in the Theorem and the distributivity on the right property 2. in the theorem
2. Property 3. can be extended to $\mathbf{u}\cdot(c\cdot\mathbf{v})=c(\mathbf{u}\cdot\mathbf{v})$.

Example

Prove that

$$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}$$

for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

Solution: Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then

$$\begin{aligned}(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) &= \mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) + \mathbf{v}\cdot(\mathbf{u}+\mathbf{v})\\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}\\ &= \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}\end{aligned}$$

Length

In $\mathbb{R}^2$, the length of a vector

$$v = \begin{bmatrix} a \\ b \end{bmatrix}$$

is the distance from the origin to the point (a, b). By Pythagora's Theorem, it is given by $\sqrt{a^2 + b^2}$. But $a^2 + b^2 = \mathbf{v} \cdot \mathbf{v}$.

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Length

Definition

The **length** (or **norm**) of a vector

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ . \\ . \\ . \\ \mathbf{v}_n \end{bmatrix}$$

in $\mathbb{R}^n$ is the nonnegative scalar $||\mathbf{v}||$ defined by

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{\mathbf{v_1}^2 + \mathbf{v_2}^2 + \cdots + \mathbf{v_n}^2}.$$

We have $||\mathbf{v}||^2 = \mathbf{v} \cdot \mathbf{v}$.

Length

Theorem

Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$ and c be a scalar. Then

1. $||\mathbf{v}|| = \mathbf{0}$ if and only if $\mathbf{v} = \mathbf{0}$.
2. $||c\mathbf{v}|| = |c|||\mathbf{v}||$.

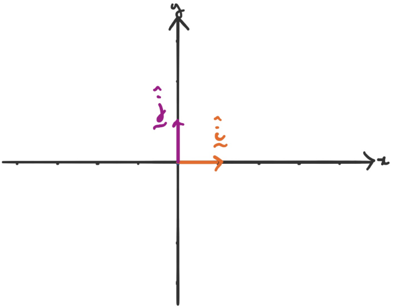
Length

Example

In $\mathbb{R}^2$, let $\mathbf{e_1} = [1,0] = \hat{\mathbf{i}}$ and $\mathbf{e_2} = [0,1] = \hat{\mathbf{j}}$. Then $\mathbf{e_1}$ and $\mathbf{e_2}$ are standard unit vectors, since $||\mathbf{e_1}|| = ||\mathbf{e_2}|| = 1$.

For any $\mathbf{v} \in \mathbb{R}^2$,

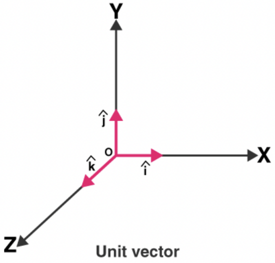
$$\mathbf{v} = [a,b] = [a,0] + [0,b] = a[1,0] + b[0,1] = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$$



Example

Similarly, in $\mathbb{R}^3$, we can construct standard unit vectors $\mathbf{e_1} = [1, 0, 0] = \hat{\mathbf{i}}$, $\mathbf{e_2} = [0, 1, 0] = \hat{\mathbf{j}}$ and $\mathbf{e_3} = [0, 0, 1] = \hat{\mathbf{k}}$.

These are the standard unit vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, they serve to locate the positive coordinate axes in $\mathbb{R}^2$ and $\mathbb{R}^3$.



Example

Find $||\mathbf{u}||$ and give a unit vector in the direction of $\mathbf{u}$ where $\mathbf{u} = [1, 2, 3]$.

Solution:

$$||\mathbf{u}|| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}.$$

If $\mathbf{v}$ is the unit vector in the same direction as $\mathbf{u}$, then

$$\mathbf{v} = \frac{1}{\sqrt{14}}[1, 2, 3].$$

Example

If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and c is a scalar, explain why the following expressions make no sense $||\mathbf{u} \cdot \mathbf{v}||$ where $|| - ||$ is the norm in $\mathbb{R}^n$, $n \geq 2$.

Solution:

The norm is a concept associated with vectors. However, $\mathbf{u} \cdot \mathbf{v}$ is not a vector, but a scalar.

Length

Theorem (Cauchy-Scharwtz)

For all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

$$|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}||||\mathbf{v}||$$

Proof: We consider the following quadratic polynomial:

$$0 \leq \sum_{i=1}^{n} (\mathbf{u_i} x + \mathbf{v_i})^2 = (\sum_{i=1}^{n} \mathbf{u_i}^2) x^2 + 2(\sum_{i=1}^{n} \mathbf{u_i v_i}) x + (\sum_{i=1}^{n} \mathbf{v_i}^2).$$

Since it is non-negative, it has at most one real root for x hence its discriminant is less or equal to 0 That is equivalent to

$$(\sum_{i=1}^{n} \mathbf{u_i v_i})^2 - (\sum_{i=1}^{n} \mathbf{u_i}^2)(\sum_{i=1}^{n} \mathbf{v_i}^2) \leq 0.$$

Thus the result.

Length

Theorem (Triangular inequality)

For all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

$$||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

Solution: Both sides of the inequality are nonnegative, so

$$\begin{aligned}
||\mathbf{u}+\mathbf{v}||^2 &\leq (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\
&= \mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) + \mathbf{v}\cdot(\mathbf{u}+\mathbf{v}) \\
&= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} \\
&= ||\mathbf{u}||^2 + 2\mathbf{u}\cdot\mathbf{v} + ||\mathbf{v}||^2 \leq ||\mathbf{u}||^2 + 2|\mathbf{u}\cdot\mathbf{v}| + ||\mathbf{v}||^2 \\
&\leq ||\mathbf{u}||^2 + 2||\mathbf{u}||||\mathbf{v}|| + ||\mathbf{v}||^2 = (||\mathbf{u}|| + ||\mathbf{v}||)^2
\end{aligned}$$

Distance

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The distance between two vectors is a direct analog of the distance between two points on the real number line or two points in the Cartesian plane.

To calculate the length/norm of $\mathbf{a} - \mathbf{b}$,

$$||\mathbf{a} - \mathbf{b}|| = \sqrt{(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})} = \sqrt{(\mathbf{a}_1 - \mathbf{b}_1)^2 + (\mathbf{a}_2 - \mathbf{b}_2)^2}.$$

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Distance

Definition

The distance $d(\mathbf{u}, \mathbf{v})$ between two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is defined by

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||.$$

Distance

Exercises

Find the distance between $\mathbf{u}$ and $\mathbf{v}$ where $\mathbf{u} = [-1, 2]$ and $\mathbf{v} = [3, 1]$.

Solutions:

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}|| = \sqrt{(\mathbf{u}_1 - \mathbf{v}_1)^2 + (\mathbf{u}_2 - \mathbf{v}_2)^2} = \sqrt{16 + 1} = \sqrt{17}$$

Distance

Exercises

Find the distance between $\mathbf{u}$ and $\mathbf{v}$ where $\mathbf{u} = [1, 2, 3]$ and $\mathbf{v} = [2, 3, 1]$.

Solutions:

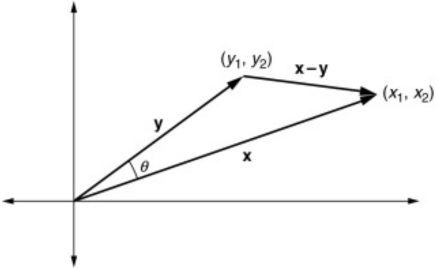
$$
\begin{aligned}
d(\mathbf{u}, \mathbf{v}) &= ||\mathbf{u} - \mathbf{v}|| = \sqrt{(\mathbf{u}_1 - \mathbf{v}_1)^2 + (\mathbf{u}_2 - \mathbf{v}_2)^2 + (\mathbf{u}_3 - \mathbf{v}_3)^2} \\
&= \sqrt{1 + 1 + 4} = \sqrt{6}
\end{aligned}
$$

Angles

The dot product can also be used to calculate the angle between a pair of vectors. In $\mathbb{R}^2$ and $\mathbb{R}^3$, the angle between the nonzero vectors **u** and **v** will refer to the angle θ determined by these vectors that satisfy $0 \leq \theta \leq \pi$.

Angles

Consider the following vectors **u** and **v** and **u** $-$ **v** where θ is the angle between **u** and **v**.



Applying the law of cosines to the triangle

$$||\mathbf{u}-\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2||\mathbf{u}||||\mathbf{v}||cos(\theta) \quad (*)$$

Angles

We also have

$$||\mathbf{u} - \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2\mathbf{u} \cdot \mathbf{v} \quad (**)$$

Equating the right hand of $(*)$ and $(**)$, we get

$$||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2||\mathbf{u}||||\mathbf{v}||cos(\theta) = |\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2\mathbf{u} \cdot \mathbf{v}$$

and

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}||cos(\theta)$$

Therefore

$$cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||}.$$

Angles

Definition

For nonzero vectors **u** and **v** in $\mathbb{R}^n$, $\theta \in [0, \pi]$,

$$cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||}.$$

Exercises

Find the angle between **u** and **v** where $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Solution:

$$\mathbf{u} \cdot \mathbf{v} = (3)(-1) + (0)(1) = -3$$

$$||\mathbf{u}|| = \sqrt{9+0} = 3$$

$$||\mathbf{v}|| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

Therefore

$$cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} = \frac{-3}{3 \times \sqrt{2}} = -\frac{1}{\sqrt{2}}$$

We have $\theta = \pi - \pi/4 = 3\pi/4$.

Remark

The proof above of the derivation of the formula for the cosine of the angle between two vectors is valid only in $\mathbb{R}^2$ and $\mathbb{R}^3$ since it depends on a geometric fact: the law of cosines. In $\mathbb{R}^n$ $n > 3$, the formula can be taken as a definition instead.

Orthogonal vectors

The concept of perpendicularity/orthogonality is fundamental in geometry in $\mathbb{R}^2$ and $\mathbb{R}^3$. We now generalize the idea of orthogonality in $\mathbb{R}^n$, where it is called orthogonality.
Let us observe the following in $\mathbb{R}^2$ and $\mathbb{R}^3$ two nonzero vector $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if the angle θ between them is $\pi/2$.
We note also that $\theta = \pi/2$ is equivalent to $cos(\pi/2) = 0$. That is $\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} = 0$. We then get $\mathbf{u} \cdot \mathbf{v} = 0$.

Orthogonal vectors

Definition

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are orthogonal to each other if $\mathbf{u} \cdot \mathbf{v} = 0$.

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Exercises

Find all the values of the scalar k for which the two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal where

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} k^2 \\ k \\ -3 \end{bmatrix}$$

Solution: Suppose $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. That is $\mathbf{u} \cdot \mathbf{v} = 0$. The latter is true if and only if

$$\begin{array}{rrcl} & (1)(k^2) + (-1)k + (2)(-3) & = & 0 \\ \Leftrightarrow & k^2 - k - 6 & = & 0 \\ \Leftrightarrow & (k+2)(k-3) & = & 0 \\ \Leftrightarrow & k = -2 \text{ or } k = 3 & & \end{array}$$

Projections

We now look at the problem of finding the distance from a point to a line in the context of vectors. This technique leads to an important concept the projection of a vector onto another.

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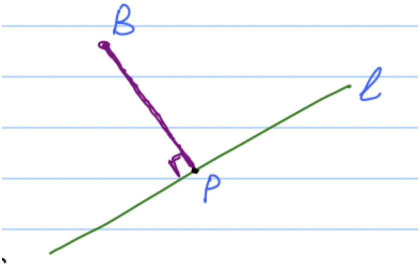
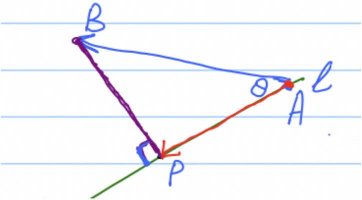
Projections

Consider the line ℓ and the point B. To find the distance between B and ℓ, we need to find the length of the perpendicular line segment $\overline{PB}$, or equivalently, the length of vector $\vec{PB}$.



Projections

Choose a point A on ℓ. We can form the right-angle triangle ΔAPB.
Vector $\vec{AP}$ is called the projection of $\vec{AB}$ onto ℓ.



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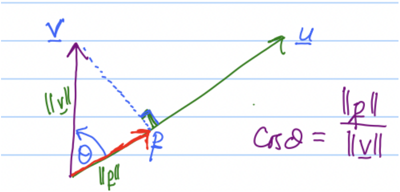
Projections

Let us derive the formula for projecting one vector onto another. Consider two non-zero obtained by dropping a perpendicular from the head of **v** onto **u** and let θ be the angle between **u** and **v**. Recall that $\frac{1}{||\mathbf{u}||}\mathbf{u}$ is the unit vector with the same direction as **u**.



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Projections

So it is clear that since **p** lies on **u** that

$$\mathbf{p} = ||\mathbf{p}|| \left(\frac{1}{||\mathbf{u}||} \mathbf{u} \right) \quad (1)$$

We also have that, by elementary trigonometry

$$cos(\theta) = \frac{||\mathbf{p}||}{||\mathbf{v}||} \Rightarrow ||\mathbf{p}|| = ||\mathbf{v}|| cos(\theta) \quad (2)$$

and we know that

$$cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| ||\mathbf{v}||} \quad (3)$$

Projections

Then, from (1) we get

$$\begin{aligned}
\mathbf{p} &= ||\mathbf{p}||\left(\frac{1}{||\mathbf{u}||}\mathbf{u}\right) \text{ from (1)} \\
&= ||\mathbf{v}||cos(\theta)\left(\frac{1}{||\mathbf{u}||}\mathbf{u}\right) \text{ from (2)} \\
&= ||\mathbf{v}||\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||||\mathbf{v}||}\left(\frac{1}{||\mathbf{u}||}\mathbf{u}\right) \\
&= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||^2}\right)\mathbf{u} \\
&= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}
\end{aligned}$$

Projections

Definition

If $\mathbf{u}$ and $\mathbf{v}$ are vector in $\mathbb{R}^n$ and $\mathbf{u} \neq 0$, then the projection of $\mathbf{v}$ onto $\mathbf{u}$ is the vector $proj_{\mathbf{u}}(\mathbf{v})$ defined by

$$proj_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right)\mathbf{u}.$$

Projections

Exercise

Find the projection of $\mathbf{v}$ onto $\mathbf{u}$ where $\mathbf{u} = [1, -1, 1, -1]$ and $\mathbf{v} = [2, -3, -1, -2]$.

Solution:

$$\begin{array}{rcl} \mathbf{u} \cdot \mathbf{v} & = & (1)(2) + (-1)(-3) + (1)(-1) + (-1)(-2) \\ & = & 2 + 3 - 1 + 2 \\ & = & 7 \end{array}$$

and

$$\mathbf{u} \cdot \mathbf{u} = 1 + 1 + 1 + 1 = 4$$

We have

$$proj_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u} = \frac{6}{4}[1, -1, 1, -1] = \frac{3}{2}[1, -1, 1, -1].$$

Projections

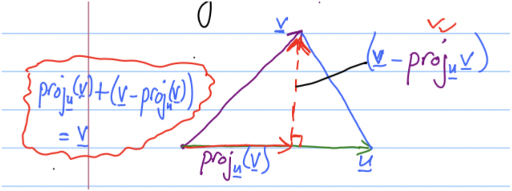
Remark

1. $proj_{\mathbf{u}}(\mathbf{v})$ is a scalar multiple of $\mathbf{u}$ and not $\mathbf{v}$.
2. In our derivation of the definition of $proj_{\mathbf{u}}(\mathbf{v})$ we require $\mathbf{v}$ as well as $\mathbf{u}$ to be non-zero. However, if $\mathbf{v}$ is the zero vector, then the projection of the zero vector onto $\mathbf{u}$ is just the zero vector.
3. If θ is obtuse then $proj_{\mathbf{u}}(\mathbf{v})$ will be in the opposite direction from $\mathbf{u}$ i.e. $proj_{\mathbf{u}}(\mathbf{v})$ will be a negative scalar multiple of $\mathbf{u}$.
4. If $\mathbf{u}$ is a unit vector then $proj_{\mathbf{u}}(\mathbf{v}) = (\mathbf{u} \cdot \mathbf{v})\mathbf{u}$

Exercise

Use vectors to determine the area of a triangle in two different ways.

Solution:

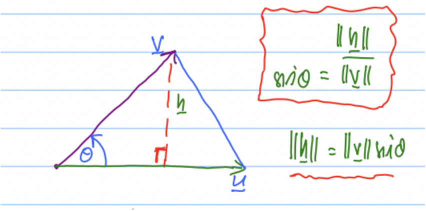


We have $A = \frac{1}{2}bh = \frac{1}{2}||\mathbf{u}||||\mathbf{v} - proj_{\mathbf{u}}(\mathbf{v})||$.

Exercise

Use vectors to determine the area of a triangle in two different ways.

Solution 2:



We have $A = \frac{1}{2} bh = \frac{1}{2} ||\mathbf{u}||||\mathbf{v}||sin(\theta)$.

Projections

Exercise

Compute the area of the triangle with the given vertices using both methods $A = (1, -1)$, $B = (2, 2)$, $C = (4, 0)$.

Solution: We have $\mathbf{v} = \vec{AB} = [1, 3]$, $||\mathbf{v}|| = \sqrt{1^2 + 3^2} = \sqrt{10}$.
$\mathbf{u} = \vec{AC} = [3, 1]$, $||\mathbf{u}|| = \sqrt{3^2 + 1^2} = \sqrt{10}$.

$$proj_{\mathbf{u}}(\mathbf{v}) = \frac{(\mathbf{u} \cdot \mathbf{v})}{||u||||v||}\mathbf{u} = \frac{(3)(1) + (1)(3)}{(3)(3) + (1)(1)}[3, 1] = \frac{3}{5}[3, 1] = [9/5, 3/5]$$

$$||\mathbf{v} - proj_{\mathbf{u}}(\mathbf{v})|| = \sqrt{(-4/5)^2 + (12/5)^2} = \frac{16/25 + 144}{25} = \sqrt{\frac{160}{25}}$$

$$cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||}$$

Projections

Exercise

Compute the area of the triangle with the given vertices using both methods $A=(1,-1)$, $B=(2,2)$, $C=(4,0)$.

Solution: We have

$$\begin{aligned}
A &= \tfrac{1}{2}bh \\
&= \tfrac{1}{2}||\mathbf{u}||\ ||\mathbf{v}-proj_{\mathbf{u}}(\mathbf{v})|| \\
&= \tfrac{1}{2}\sqrt{10}\sqrt{\tfrac{160}{25}} \\
&= \tfrac{1}{2}\sqrt{10}\sqrt{10}\sqrt{\tfrac{16}{25}} \\
&= \tfrac{1}{2}\tfrac{4}{5}10 = 4
\end{aligned}$$

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Projections

Exercise

Compute the area of the triangle with the given vertices using both methods $A = (1, -1)$, $B = (2, 2)$, $C = (4, 0)$.

Solution: We have

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}||\mathbf{u}||||\mathbf{v}||sin(\theta) \\ &= \frac{1}{2}\sqrt{10}\sqrt{10}\sqrt{1 - cos^2(\theta)} \\ &= \frac{1}{2}(10)\sqrt{1 - \frac{9}{25}} \\ &= 5\sqrt{\frac{26}{25}} \\ &= 5\frac{4}{5} = 4 \end{aligned}$$

Exercise

1. Prove that $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal in $\mathbb{R}^n$ if and only if $||\mathbf{u}|| = ||\mathbf{v}||$.

Solution: Suppose $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal, then

$$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{0}$$

$$(\mathbf{u}+\mathbf{v})\cdot\mathbf{u}+\mathbf{v}\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{0}$$

$$\mathbf{u}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{v} = \mathbf{0}$$

$$\mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{v} = \mathbf{0}$$

$$||\mathbf{u}||^2 - ||\mathbf{v}||^2 = 0$$

$$||\mathbf{u}|| = ||\mathbf{v}||.$$

Exercise

1. Prove that $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal in $\mathbb{R}^n$ if and only if $||\mathbf{u}|| = ||\mathbf{v}||$.

Solution: Conversely, suppose that $||\mathbf{u}|| = ||\mathbf{v}||$, then as before

$$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = ||\mathbf{u}||^2 - ||\mathbf{u}||^2 = 0$$

Hence, $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal.

Exercise

2. Prove that if $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{u}$ is orthogonal to $\mathbf{v} + \mathbf{w}$.

Solution: Suppose $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, then

$$\mathbf{u} \cdot \mathbf{v} = 0 \text{ and } \mathbf{u} \cdot \mathbf{w} = 0$$

Therefore,

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Hence, $\mathbf{u}$ is orthogonal to $\mathbf{v} + \mathbf{w}$.

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